KARL MARX

MATHEMATICAL MANUSCRIPTS

TOGETHER WITH A SPECIAL SUPPLEMENT

VISWAKOS PARISAD
CALCUTTA

PUBLISHED IN INDIA
BY VISWAKOS PARISAD,
73A, AMHERST ROW,
CALCUTTA - 700 009.

COMPOSED AT NEO COMPACT SYSTEMS PVT. LTD.,
161,B.K.PAUL AVENUE,
CALCUTTA - 700 005.

DATA ENTRY BY MRITYUNJOY DAS.
PAGE MAKE UP AND CORRECTION BY BIPASA CHAUDHURI AND PRADIP CHATERJEE.
PROOF READERS : SABYASACHI CHAKRABARTI AND Dr. S. KUMAR.

PRINTED AT SONA PRINTERS,
67/1/2, NIMTALLA GHAT STREET,
CALCUTTA – 700 006.

CATALOGING IN PUBLICATION DATA :

MARX, KARL.

MATHEMATICAL MANUSCRIPTS.

TRANSLATION OF : K.MARKS, MATEMATICHESKIE RUKOPISI ("NAUKA", M., 1968) EDITED BY SOFYA ALEKSANDROVNA YANOVSKAYA, TOGETHER WITH A SPECIAL SUPPLEMENT : MARX AND MATHEMATICS.

TRANSLATOR OF MARX'S MATHEMATICAL MANUSCRIPTS (M., 1968) AND EDITOR OF THE SPECIAL SUPPLEMENT : PRADIP BAKSI.

1. MARXISM. 2. MATHEMATICS (DIFFERENTIAL CALCULUS, ALGEBRA).

3. MATHEMATICS, HISTORY OF. 4. MATHEMATICS, PHILOSOPHY OF.

ISBN 81-86210-00-8

PRICE : Rs. 1000.00 (U S $ 80.00)

INTRODUCTION

The massive propaganda blitzkrieg denigrating socialism and the voluminous material brought out by imperialism and reactionaries notwithstanding, nobody so far has succeeded in challenging the basic theory that Marx placed before the world. The profundity of Marx's works lies in his scientific analysis of the evolution of society and the foresight of the path that it will charter. It is based on such a scientific analysis that Marx concluded that society will reach a stage where the state will wither away. That capitalism has failed to prove its superiority or eternity, as postulated by its protagonists, even in the wake of the so called "demise of socialism" — the continuing recession and intensified exploitation, vindicate this analysis.

Marx, unfortunately, could not complete the work that he set before himself. After the second volume of Capital had come out in print, it was left to Frederick Engels to go through the manuscripts facilitating it to be made available to the whole of mankind. In the preface to Volume III of Capital, Engels writes: "As regards the first part, the main manuscript was serviceable only with substantial limitations. The entire mathematical calculation of the relation between the rate of surplus-value and the rate of profit (which makes up our Chapter III) is introduced in the very beginning, while the subject treated in our Chapter I is considered later and as the occasion arises. Two attempts at revising, each of them eight pages in folio, were useful here. But even these did not possess the desired continuity throughout. They furnished the substance for what is now Chapter I. Chapter II is taken from the main manuscript. There was a series of uncompleted mathematical calculations for Chapter III. as well as a whole, almost complete, note-book dating from the seventies, which presents the relation of the rate of surplus-value to the rate of profit in the form of equations. My friend Samuel Moore, who has also translated the greater portion of the first volume into English, undertook to edit this note-book for me, a work for which he was far better equipped, being an old Cambridge mathematician. It was for his summary, with occasional use of the main manuscript, that I then compiled Chapter III. Nothing but the title was available for Chapter IV. But since its subject matter, the influence of turnover on the rate of profit, is of vital importance, I have written it myself, for which reason the whole chapter has been placed in brackets".

The present volume, the first complete English translation of Marx's *Matematicheskie Rukopisi*, I am sure would serve to give us more insight into his works. Marx's Mathematical Manuscripts are primarily devoted to describing and explaining the nature and history of the differential calculus. They contain many valuable ideas on the nature and role of variables, existence of functions, analytical geometry, general theory of equations etc. These in the main relate to his exploration of the history of examination of the study of mathematics.

This volume, we trust will help in understanding the depth and method of his analysis and help in drawing strategies for future investigations. In this connection I must appreciate the momentous work done by Shri Pradip Baksi in bringing out this legendary work.

Harkishan Singh Surjeet

WHY THIS PUBLICATION

With the publication of this first complete English translation of Karl Marx's *Mathematical Manuscripts* (Moscow, 1968), a so far almost unknown area of Marx's thought is being presented to our readers. Along with this a special supplement , *Marx and Mathematics,* is also being published.

Pradip Baksi, a Marx-scholar, has undertaken and completed this stupendous job at a time, when there were tremors in the socialist system, in several socialist countries including the erstwhile Soviet Union, and later on this led to their disintegration one after another.

Pradip's sincerity and firm conviction has emboldened us to rise to the occasion and publish the present volume.

Disintegration of the erstwhile socialist countries, particularly of the Soviet Union has made the imperialists and the capitalists much more aggressive these days. They are not sparing any effort to seal the fate of socialism and of the ever growing worldwide movements of the revolutionary forces. In spite of their unprecedented evil design to subjugate the developing countries and thus to control the world economy, the revolutionary and patriotic forces are facing this threat vigorously and registering political victories one after another in different countries.

It is in this context that the present volume is being published, which we firmly believe will evoke curiosity among the scholars and generate academic interaction on this subject. We are aware that in the present political scenario academic publications on Marxism are of immense value. We assure the scholars and students of Marx's thought that we would continue our endeavour to publish the unpublished works of Karl Marx and, books and monographs on allied subjects.

I sincerely acknowledge the contribution of Shri Pradip Baksi and hope this is just the beginning.

Subhas Chakraborty
Chairman, Publishing Committee, Viswakos Parisad.

MARX AND THE WORLD MATHEMATICAL YEAR 2000

It would sound somewhat banal to talk about Karl Marx as a 'living thinker', as Engels spoke at the funeral of Karl Marx. He is one of those who left an indelible legacy of thoughts, which brought in its trail a wide plethora of ideas and concepts. The enormous variety of his work triggered off rethinking and reappraisal of many facets of learning and pursuits. Needless to add that the whole gamut of Marxist interpretation or Marxist view of literature, sociology, history etc. could come up not just by practitioners of Marxism in the political arena. There are many scholars, without being wedded to Marxism, who have contributed immensely to the body of literature on Marx. It often looks, in the wake of political debacle in the erstwhile Soviet Union or in the East-European countries that, perhaps, Marx's image as an original thinker like Sigmund Freud would have remained unscathed and unsullied, had he not been drawn upon too heavily by political figures. That his profundity of thoughts and ideas still continues to have an intellectual appeal, despite what has happened and perhaps, still keep on happening by using his name and work, is borne out by the amount of contemporary literature on him and on allied matters. Looking back, one may even say that Marx's work could have gone down in the pages of history just as a thinker like the great celebrity Charles Darwin, purely on the strength of his rich output of thoughts and ideas.

The intellectual makeup of Karl Marx, whatever some of his admirers may say, was dominantly shaped by the German tradition of training and scholarship and it is no wonder that mathematics did come within the purview of his acquisition and investigation. Steeped as he was in the discourses with a philosophical content, he made some daring forays in mathematics, as is brought out in this publication. All these stemming from the German tradition would have kept him in good stead in his subsequent endeavour, even if there were no experiments undertaken by Lenin or Mao or Ho Chi Minh. Marx was, doubtless, scientific in approach which some of his adherents may not be in the styles of persuation and activity.

This publication, as its title implies, is somewhat unconventional in the parlance of mathematics or even in the historical literature on mathematics. Even though, in the course of history, mathematicians were seized of or were often involved in politics, their impact has largely been ephemeral. Indeed, Marxism or anti-Marxism has seldom been found to be a forte of mathematical celebrities. On the other hand, as remarked earlier, there is hardly any area of thought, not to speak an activity, which has kept itself fully immune from Marxism, whatever be it today in political reality. There is, perhaps, something intrinsically and perennially intellectual, in any of his endeavour dealing with society, culture, or history. There are, indeed, in recent years social historians of mathematics who look at growth and development of mathematical ideas from Marxist standpoint. There is equally a strong body of literature, particularly in the arena of mathematical education, which treats of Marxism critically without mincing logic and often, to the point of its rejection.

It is being increasingly realised that culture, excitement and history of mathematics with societal implications should be focussed more than what could be done hitherto. The World Mathematical Year (WMY) 2000 is thus being projected by the international mathematical community, with the accent, not merely on cultural facets of mathematics but also on development of historiography of the history of mathematics. The present

publication should be looked upon as a precursor to such pursuits at the international level. In keeping with its *raison d'etre*, this book seeks to dig up what Karl Marx sought to do on mathematics. Mathematical ideology may look askance at this part but the compendium of critique on mathematical thoughts and ideas, cast in Marxist or a-Marxist mould will, it is envisaged, immensly contribute to the shaping of thought processes warranted by WMY 2000. Both the parts of this treatise, taken in their entirety, ought to unleash pursuits that are free from inhibitions in the wider perspectives of the imperatives of WMY 2000. Historiography of history of mathematics will, hopefully, acquire a stimulus and new dimensions, because of a publication of this sort.

Now, a few words about Pradip Baksi, but for whose assiduity and painstaking ardour for hardwork, this publication would not have come within miles of reality. He is not a mathematician *per se*, not even a historian of mathematics in the sense we are accustomed to nowadays. He is trained in Russian language and had a stint in the Soviet Union. Pradip, obviously, is not one of the many Indians, to whom exposure to Marx's work has come via English alone and this translation of the works of Marx directly from Russian into English does not suffer, as is often said, from any constriction, if any, because of colonial tinge. His training in philosophy has definitely facilitated his endeavour. He deserves to be specially commended for bringing to fore these works by and on Marx and on related issues which would, otherwise, have remained inaccessible to many of us.

Dilip Kumar Sinha

Sir Rashbehary Ghose Professor of Applied Mathematics,
University of Calcutta.

92, Acharya Prafulla Chandra Road,
Calcutta – 700 009.

PUBLISHER'S NOTE

This volume contains the first complete English translation of K. MARKS, *MATEMATICHESKIE RUKOPISI*, Izd. "NAUKA", MOSKVA, 1968, together with a special supplement entitled *MARX AND MATHEMATICS*.

Marx's mathematical manuscripts are primarily devoted to describing and explaining the nature and history of the differential calculus. However, these will be of interest to the contemporary investigators of the symbolic calculi of mathematics and logic, of sign systems in general and, to any one interested in the history of ideas.

The special supplement to this volume contains materials, pertaining to the task of situating Marx's mathematical investigations in the history and structure of Marxism, mathematical thought and, ideas in general, as well as those, which may help to formulate strategies for future investigations.

TRANSLATOR'S NOTE AND ACKNOWLEDGEMENTS

In K.Marks, *Matematicheskie Rukopisi* ("Nauka" M.,1968), Marx's own texts have been published both in the original language (mainly German, but in places French, English or a mixture of two or more of these languages) and in Russian translation. However, the preface, editorial comments, notes and appendices are all in Russian only. Hence, Russian is the only single language through which the entirety of this volume becomes accessible. The present translation has throughout followed the texts, comments, notes and appendices in Russian. However, where Marx's own text is only in English, there that has been reproduced. I have added two notes : 98a & 111 a, and a few comments and footnotes. The preface of the 1968 edition has been variously superseded by the developments in Marx-studies and in mathematics. A new preface is due. I began writing one ; but it got out of hand. The result : a special supplement, entitled *Marx and Mathematics*. The sources of the materials included in this supplement have all been indicated at the end of each item.

I owe a great debt to the following persons ; they have helped me at various stages of the work, indicated below, culminating in the publication of the present volume.

For the work culminating in the present translation of : K. Marks, *Matematicheskie Rukopisi* (M.,1968) :

Tapan Kumar Chattopadhyay of Calcutta University, Hubert Kennedy of Providence College (U.S.A), Timir Ranjan Mukherjee and Biswarup Bhowmik of Calcutta, Dr. Dilip Banerjee of St. George's Hospital and Medical School (London), Dr. Vinay Totawar of the Central Institute of English and Foreign Languages (Hyderabad) and, Amol Padwad of Bhandara College (Maharastra).

For the work leading to the special supplement entitled *Marx and Mathematics* :

Evgeniya Mikhailovna Bykova — formerly of the Institute of Oriental Studies of the USSR Academy of Sciences, Irina Konstantinovna Antonova — formerly of the Institute of Marxisum-Leninism of the CC CPSU, Mikhail Ostapovich Ovsienko of the Pushkin Institute, Sergei Serebryany — formerly of the Maxim Gorky Institute of World Literature of the USSR Academy of Sciences and presently of the Russian State University for the Humanities (Moscow), Fabien Laurent of Poitiers (France) and, Baren Ray of New Delhi.

For generously offering to organise the publication of the present volume :

Subhas Chakroborty — Minsiter-in-charge of the Departments of Sports & Youth Services and Tourism, Government of West Bengal.

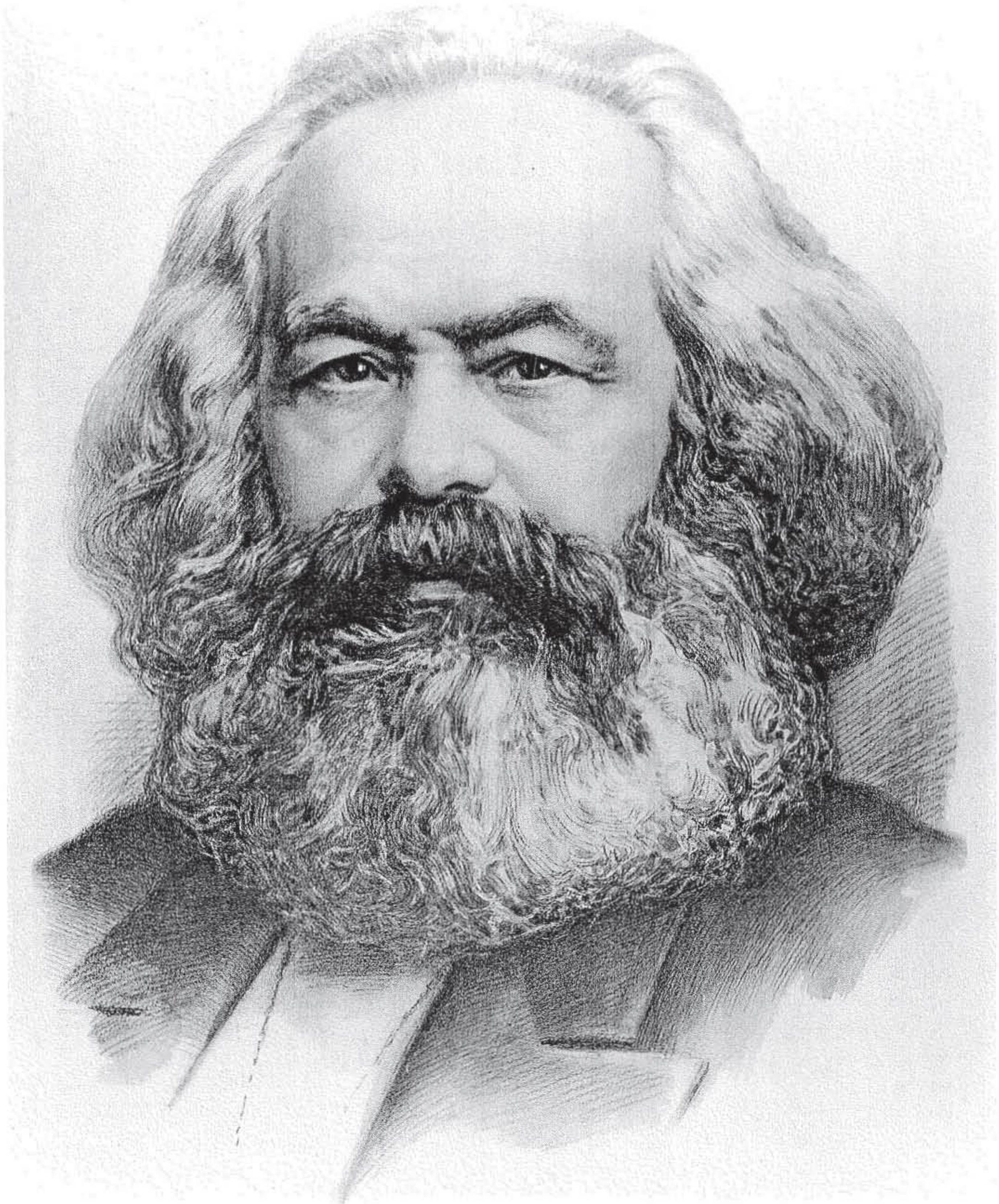
For putting me in touch with Shree Subhas Chakroborty in this connection :

Subrata Tapadar of Calcutta.

Finally, all the workers engaged in the production of this volume.

The usual disclaimers apply everywhere.

May 6,1993. Pradip Baksi.



Karl Marx

ABBREVIATIONS	
MECW(E), ..., ...	Marx-Engels : Col. Works (English ed.), Volume No., Page No.
MECW(R), ..., ...	Marx-Engels : Col. Works (Russian ed.), Volume No., Page No.
MR, ...	*Matematicheskie Rukopisi*, K. Marks ("Nauka", M., 1968), Page No.
PV, ...	Present Volume, Page No.
S. U. N.	Storage Unit Number (of the Archives of the erstwhile Institute of Marxism - Leninism of the C C CPSU).

CONTENTS

KARL MARX. MATHEMATICAL MANUSCRIPTS.

PART I
DIFFERENTIAL CALCULUS : ITS NATURE AND HISTORY

**

PART II
DESCRIPTION OF THE MATHEMATICAL MANUSCRIPTS

APPENDIX

NOTES AND INDEXES

SPECIAL SUPPLEMENT

MARX AND MATHEMATICS

PART ONE : HISTORY

ON THE HISTORY OF MARX'S MATHEMATICAL MANUSCRIPTS

PART TWO : INVESTIGATIONS

INVESTIGATIONS INSPIRED BY MARX'S MATHEMATICAL MANUSCRIPTS : A SELECTION

PART THREE : MATHEMATICSES

MATHEMATICSES : PAST, PRESENT AND FUTURE

Пролетарии всех стран, соединяйтесь!

ИНСТИТУТ МАРКСИЗМА-ЛЕНИНИЗМА при ЦК КПСС

К. МАРКС

МАТЕМАТИЧЕСКИЕ РУКОПИСИ

ИЗДАТЕЛЬСТВО «НАУКА»
ГЛАВНАЯ РЕДАКЦИЯ
ФИЗИКО-МАТЕМАТИЧЕСКОЙ ЛИТЕРАТУРЫ
МОСКВА 1968

FACSIMILE OF THE TITLE PAGE OF THE 1968 EDITION OF KARL MARX'S MATHEMATICAL MANUSCRIPTS

PREFACE TO THE 1968 EDITION

The existence of Marx's mathematical manuscripts within the body of his unpublished writings was first mentioned in 1885. F.Engels referred to them in his preface to the second edition of *Anti-Dühring*. [See : *Anti-Dühring* (English edition), Progress, Moscow, 1978, p. 18. However, the importance of Marx's mathematical investigations was indicated by Engels even earlier, in his grave-side speech in memory of Marx delivered on March 17, 1883. — Tr.] Engels considered them important and collected the same for publication.The photocopies of these (nearly 1000 pages of) manuscripts are presently being preserved in the archives of the Institute of Marxism-Leninism of the CC of the CPSU [with the dissolution of the CC CPSU, this Institute has ceased to exist —Tr.]. A part of these manuscripts and some preparatory materials were published in Russian, in 1933, on the occasion of the 50th death anniversary of Marx, in the journal "Pod Znamenem Marksizma" ("Under the Banner of Marxism") [1933, No. 1, 15-73] and in the collection of essays entitled *Markiszm i estestvoznaniya* (Marxism and the natural sciences) [1933 ; 5-61].These included the results of Marx's investigations into the nature of differential calculus.In 1881 Marx jotted them down for Engels, in two manuscripts. Till then even this part of Marx's mathematical manuscripts was not published in the original language.

In the present edition, the manuscripts, which Marx could complete in the main,or those which contain Marx's comments on some questions of mathematical significance, are all being published in their full form.

Marx's mathematical manuscripts contain various types of materials : Marx's own writings on the nature and history of differential calculus, as well as the notes and extracts he made from the sources, which he used.In consonance with this two-fold nature of the materials at hand, the present volume has been divided into two parts. The first part contains all the original writings of Marx, and the second part is devoted to detailed descriptions of the notes and extracts of mathematical significance, jotted down by him. All these are being published here [in the 1968 edition — Tr.] in the original language and in Russian translation.

It is true that the nature of Marx's original writings and that of the notes taken by him are not identical. Bulk of these notes consists of extracts from the writings of other authors, the rest being Marx's own comments. But, for a proper understanding of Marx's thoughts on mathematics, often an acquaintance with these notes of him becomes a necessity. Hence, not a part of this volume, but only its entirety, properly expresses the mathematical thinking of Marx.

While writing the *Capital*, Marx became specially interested in mathematics. In this connection, on the 11th of January 1858 he wrote to Engels : "In elaborating the PRINCIPLES of economics I have been so damnably held up by errors in calculation that in DESPAIR I have applied myself to a rapid revision of algebra. I have never felt at home with arithmetic. But by making a detour via algebra, I shall quickly get back into the way of things" [MECW(R) , 29, 210 ; MECW(E), 40, 244].

We notice the first glimpses of Marx's mathematical writings in his first note books on political economy. Some algebraic calculations have been found in some of these note books related to his researches of 1846 in their main contents. But it is possible, that these

calculations and comments were inserted by him at some later date, in the pages earlier left blank in these note books.

Some of the note books used by Marx in April-June 1858 contain preparatory materials for his *Critique of Political Economy*. In them we find : some draft sketches related to elementary geometry and some algebraic calculations concerning generalisation of the concepts of power and logarithm.

In this period Marx's mathematical studies proceeded rather disconnectedly. Often he studied mathematics, only when he was not busy with anything else. On the 23rd of November 1860 he wrote to Engels : "Writing articles is ALMOST OUT OF QUESTION for me. The only occupation that helps me maintain the necessary QUIETNESS OF MIND is mathematics" [MECW(R), 30, 88 ; MECW(E), 41, 216]. Thus, in spite of his other preoccupations, Marx's mathematical studies continued. On the 6th of July 1863 he wrote to Engles : "My spare time is now devoted to differential and integral calculus. Apropos, I have a superfluity of works on the subject and will send you one, should you wish to tackle it. I should consider it to be almost essential to your military studies. Moreover, it is a much easier branch of mathematics (so far as mere technicalities are concerned) than, say, the more advanced aspects of algebra. Save for a knowledge of the more ordinary kind of algebra and trigonometry, no preliminary study is required except a general familiarity with conic sections" [MECW(R), 30, 296 ; MECW(E), 41, 484]. Then either towards the end of 1865, or in the early parts of 1866, in an appendix to a letter to Engles, he explained the nature of differential calculus in the light of the problem of tangent to the parabola. This letter remains untraced [only the said appendix has been found — Tr.].

But even in this period Marx's mathematical studies were mainly connected with his researches in political economy. Thus in 1869, having undertaken a study of the problem of circulation of capital and that of the role of bills of exchange in the calculations relating to international trade, Marx read a very big text book on commercial arithmetic written by Feller and Odermann. He took detailed notes from this book [see : S.U.N. 2388 and 2400]. It was a principal trait of Marx's character, that when he faced a question, he never stopped before becoming fully confident about the issue, never stopped without mastering the subject to its very foundations. And this trait was revealed here also. Whenever he found that a certain mathematical technique has been used in the text book by Feller and Odermann, then , even if Marx knew about it beforehand, he considered it essential to reengage himself in its study all over again. Thus were inserted his comments of a clearly mathematical significance , in the aforementioned notes on commercial arithmetic, as well as in the notes taken afterwards [see : S.U.N. 3881, 3888 and 3931]. It is these steps, which, in their turn, took Marx closer to the higher parts of mathematics. Marx's mathematical investigations acquired an almost systematic character in the 70s of the last century, especially since 1878. In his preface to the second volume of *Capital* [in the preface to the 1968 edition of the mss this has been inadvertently put as : "In his preface to the second edition of *Capital* " — Tr.], Engels wrote about this period : "There was another intermission [in the course of Marx's writing of the *Capital* — Tr.] after 1870, due mainly to Marx's ill health. Marx employed this time in his customary way, by studying agronomy, rural relations in America, and especially, in Russia, the money market and banking, and finally the natural sciences such as geology and

physiology. Independent mathematical studies also figure prominently in the numerous excerpts-full note books of this period "[MECW(R), 24, 8 ; *Capital*, vol. II, Eng. ed., Progress Moscow, 1978, pp. 3-4].

But of course, even in this period Marx continued to remain interested in the use of mathematics in political economy. On the 31st of May 1873 Marx wrote to Engels :"I have been telling Moore about a problem with which I have been racking my brains for some time now. However, he thinks it is insoluble, at least *pro tempore*, because of the many factors involved, factors which for the most part have yet to be discovered. The problem is this : you know about those graphs in which the movements of prices, discount rates, etc., etc., over the year, etc., are shown in rising and falling zigzags. I have variously attempted to analyse crises by calculating these UPS AND DOWNS as irregular curves and I believed (and still believe it would be possible if the material were sufficiently studied) that I might be able to determine mathematically the principal laws governing crises. As I said, Moore thinks it cannot be done at present and I have resolved to give it up FOR THE TIME BEING. " [MECW(R), 33, 71-72 ; MECW (E), 44, 504]. Evidently Marx could visualise long ago, that there is scope for using mathematics in political economy.

But it does not become fully clear, even from the descriptions of the entirety of his mathematical manuscripts, presented in the second part of this volume, as to what exactly propelled Marx to proceed from the study of algebra and commercial arithmetic undertaken by him, to the study of differential calculus. In fact Marx's mathematical manuscripts were born in that period when he began to study elementary mathematics with the aim of studying differential calculus alone, i.e., when he started studying trigonometry and conic sections and wrote the aforementioned letter to Engles [dated July 6, 1863 — Tr.] indicating the necessity of such study.

Incidentally, at that time the situation in differential calculus, especially of the foundation upon which it was constructed, was in a bad shape. Engels quite graphically depicted this situation in his *Anti-Dühring* : "With the introduction of variable magnitudes and the extension of their variability to the infinitely small and the infinitely large, mathematics, usually so strictly ethical, fell from grace ; it ate of the tree of knowledge, which opened upto it a career of most colossal achievements, but at the same time a path of error. The virgin state of absolute validity and irrefutable proof of everything mathematical was gone for ever ; the realm of controversy was inaugurated, and we have reached the point where most people differentiate and integrate not because they understand what they are doing but from pure faith, because upto now it has always come out right" [MECW(R),20, 88-89 ; *Anti-Dühring*, Eng. ed. Progress, Moscow, 1978, p. 110].

Naturally Marx could not put up with this. In his words : "it becamė important for him, "here as everywhere", "to strip the veil of secrecy from science"[see : MR, 193; PV, 88]. This task became important also due to the fact, that the process of transition from elementary mathematics to the mathematics of variables of necessity attained a dialectical character, owing to its very content. And Marx and Engels considered it to be a duty of theirs' to show, how the materialist dialectics is of use not only in the social sciences, but in the natural sciences and mathematics as well. To unveil the dialectics of the process of transition to the mathematics of variables, it was necesary to conduct detailed investigations about the "mystery

which even to-day surrounds the magnitudes employed in the infinitesimal calculus, the differentials and infinitesimals of various degree" [MECW(R), 20, 582; *Anti-Dühring*, Eng. ed., Progress, Moscow, 1978, p. 444]. That is to say, Marx had before him the task of laying bare the dialectical nature of that symbolic calculus, which operates with differential symbols.

Marx was self-educated in mathematics. He could only turn to his friend Samuel Moore for relevant advice. But Moore had very little mathematical knowledge. And that is why he could not be of much help to Marx. Not only that, it appears from his remarks on the manuscripts which Marx sent to Engels in 1881, that Moore was not at all able to understand Marx's thoughts about the origin and significance of the symbolic differential calculus [see : MECW(R), 35, 93-94].

Marx began his studies of the differential calculus with the text-books which were then in use in the Cambridge University. In the 17th century Newton held the chair of mathematics in this university. And since then his tradition has been respectfully obeyed in England. It is well known, that in the 20s and 30s of the last century, the English youth assembled around the "Analytical Society"of mathematicians were forced to wage a relentless struggle against the representatives of this outdated tradition, which elevated the method of Newton into a kind of sacred and unsurpassable dogma, his distinctive expressions and the synthetic method of his "Principia" included. This tradition demanded, that all problems, to be solved with the help of the techniques of calculus, are to be solved directly from the beginnings — they are not to be subordinated to any general problem.

If the above mentioned background is kept in view, then we shall be able to understand why and under what sort of circumstances did Marx begin his study of differential calculus with the *Cours complet de mathématiques* [Paris, 1778] by Abbe Sauri. This book was written according to the method of Leibnitz, and Leibnitzian symbols were used in it. Soon after this, the very Newtonian method of "analysis through equations containing infinite number of terms", directly drew Marx's attention [see : S.U.N. 2763]. Marx became so enthused after having considered the Leibnitzian algorithms of differential calculus according to Sauri's book, that he undertook the task of explaining them (using the example of tangent to the parabola), in the special appendix of a letter to Engels [see : MR, 251-254; PV, 113-114].

After finishing the text book by Sauri, Marx studied the English translation of a newer French book : *An elementary treatise on the differential and integral calculus* [1828] by Boucharlat. In this book the ideas of d'Alembert and Lagrange were lumped together eclectically. In France alone it saw eight editions. It was translated into other languages (including Russian). This book too could not satisfy Marx. He began to study the works of other mathematicians and other text books. He studied the classical writings of Euler and MacLaurin. MacLaurin popularised the works of Newton. Marx also studied the text books by Lacroix, Hind, Hall, Hemming and others. All these books have found their place in Marx's notes and excerpts.

In these books, what at first drew Marx's attention, was the outlook of Lagrange. Lagrange attempted to tackle the characteristic difficulties of differential calculus by providing it with an "algebraic" foundation, i.e., he attempted to do it without using the concepts of *infinitesimal* and *limit*, which were till then inexactly defined and were quite vague.

However, having become acquainted with the concepts of Lagrange in detail, Marx very soon understood that no satisfactory solution of the characteristic difficulties associated with the symbolic apparatus of differential calculus was possible along these lines. And that is why he set out to elaborate the proper approach for determining the true nature of this calculus.

In the second part of this volume we become acquainted with the path which Marx traversed in his journey towards that goal. This part contains description of all the mathematical manuscripts of Marx. This description is, as far possible, chronological. Here we once again find that Marx undertook the study of algebra, with the aim of properly evaluating the point of view of Lagrange. Marx wanted to find out the algebraic roots of differential calculus. He was drawn, first of all,to the theorem about the multiple roots of an algebraic equation. Determination of these roots is connected, in its content, with the successive differentiation of the original equation. Marx, specifically and in detail, discussed this question in a series of manuscripts beginning with S.U.N. 3932 and 3933. This discussion figures under the heads "Algebra I" and "Algebra II". Theorems of Taylor and MacLaurin especially drew Marx's attention. Lagrange attempted to prove these theorems in a "purely algebraic" manner, i.e., without taking the help of differential calculus. In this connection Marx began to systematically collect the material about Newton's binomial theorem and about Taylor's and MacLaurin's theorems, from different sources. And thus were born the manuscripts contained in S.U.N. 3933,4000 and 4001.These manuscripts can no longer be considered mere notes taken from the works of other authors. And that is why here their full texts have been published. Generally speaking, in Marx's notes gradually one comes across more and more of his original comments. Especially noteworthy in this connection, are his comments on the concept of function and on the substitution of the symbol *0/0* by the symbol *dy/dx*.

His original comments are also to be found in a series of other manuscripts [see : S.U.N. 2763, 3888, 3932 and 4302]. Having become convinced that Lagrange's "purely algebraic" method is incapable of solving the problems of the foundations of differential calculus, and even after arriving at his own view point about the content and method of this calculus, Marx, all the same, went on collecting materials about the various ways of differentiating, from the sources at his disposal [see : S.U.N. 4038 and 4040]. And only after that did he begin writing his own opinions about the "algebraic" method of differentiating (a certain class of functions). Subsequently he began drafting the fundamental ideas expressing his characteristic point of view. These ideas have found expression in the articles included in the first part of this volume, as well as in their different drafts. Now, let us turn to the contents of these writings.

The greater part of Marx's original mathematical writings were born in the seventies of the last century. At that time, in Europe, modern classical analysis was being constructed, with its characteristic theories of real numbers and limit (above all in the works of K. Weierstrass, R. Dedekind and G.Cantor).

This direction of the works of European mathematicians was at that time virtually unknown in the English universities. The famous English mathematician Hardy wrote his "Course of Pure Mathematics" many years later (in 1917).[This information is incorrect. The first edition of G.H.Hadry's *A Course of Pure Mathematics* was published not in 1917, but

in 1908. — Tr.] In the preface of the 1937 edition of this text book Hardy wrote with justification: "It [this book] was written when analysis was neglected in Cambridge, and with an emphasis and enthusiasm which seem rather ridiculous now. If I were to rewrite it now I should not write (to use Prof. Littlewood's similie) like a missionary talking to cannibals". And Hardy noted lower down there, that "even in England, there is now [i.e., in 1937. — Ed.] no lack" of manuals on analysis.

That is why it is not surprising, that the relatively modern problematic of the then nascent continental mathematical analysis remains unmentioned and undiscussed in Marx's mathematical manuscripts. But still, even to-day, his ideas about the content of symbolic differential calculus are of interest.

The concepts of the "differential", of the different orders of the "infinitesimal"etc., and the symbols dx, dy, d^2y, d^3y,..., dy/dx, d^2y/dx^2, d^3y/dx^3... etc. are characteristic of the differential calculus. In the text books of differential calculus published in the 19th century and, available to Marx, some special magnitudes were mentioned over and above the aforementioned concepts and symbols. These special magnitudes were considered to be different from the usual mathematical numbers and functions. And it was considered imperative that mathematical analysis be conducted with the help of these special magnitudes. To-day the situation is different : there are no special magnitudes in analysis. But the symbols and the terminology have been retained. These have been found to be very useful. How come ? If the corresponding concepts turned out to be meaningless, then how could the words and symbols be retained ? Marx's mathematical manuscripts provide the best possible answer to this question. In addition, these manuscripts also provide that answer, which helps us to determine the contents of all types of symbolic calculi. It may be noted that only recently the general theory of symbolic calculi has been constructed in modern mathematical logic.

The main thing here is the operational role of the symbols of calculus. If one and the same computational process is repeatedly used to solve very different kinds of problems, then it is useful to choose a special symbol for this entire process. This symbol designates in brief, what Marx has called, the "operational strategy" of its formulator. Here, first of all comes the very process. Marx has called this process "real", to distinguish it from the symbolic designation introduced for it.

But why is it so worthwhile to introduce a new symbol here ? Marx's answer to this question, consists of the following point : owing to this we need not complete the entire process everytime; and by utilising the fact, that we have already completed this process in certain cases, we can reduce the task of completing it in more complex cases, into its completion in the simpler ones. For this, only the regularity of the process under consideration is required to be studied, and then some general rules for operating with the new symbols are to be determined. These rules will permit the aforementioned reduction. But in that case we also obtain a calculus, already operating with new symbols. Thus, in the words of Marx, we enter into its "ground proper". And Marx explained in detail the dialectics of that "inversion of method", which is connected with this transition to the symbolic calculus ; conversely, its rules do not permit a transition from the "real" process to the symbol, but rather, these rules allow us to seek the "real"process which corresponds to a symbol. By prescribing a "strategy of operations", these rules make the symbol operational.

All this Marx investigated in his two fundamental articles written in 1881 and sent to Engels. These are : *On the concept of the derived function* [see : MR, 29; PV, 19] and *On the differential* [see : MR, 47; PV, 26]. In the first article Marx considered the "real" process (the algorithm) of searching the derived functions and differentials of a certain class of functions and introduced the corresponding symbols for such processes (he called it the process of "algebraic" differentiation). In the second he depicted the "inversion of method" and went over to the "ground proper" of the differential calculus. For this, first of all he utilised the theorem about the derivative of product. This theorem permits the search for the derivative of product to be reduced into a search for the derivatives of factors. In Marx's own words, thus the "symbolic differential co-efficient has become an *independent point of departure*, only its real equivalent must be found out ... But with this, the differential calculus too appears as a specific kind of calculus, already operating independently upon its own ground, since its points of departure *du/dx, dz/dx* are mathematical magnitudes which belong only to this calculus and characterize it " [MR, 55 & 57; PV, 30-31]. They are thereby "at once transformed into *operational symbols,* into the symbols of processes, which are to be carried out ... for finding out ... [the] "derivatives". Having initially emerged as the symbolic expression of the "derivatives", i.e., of the operations of differentiation already carried out, the symbolic differential co-efficient now plays the role of the symbol of those operations of differentiation, which remain to be carried out" [MR, 57; PV, 31]. Marx was not aware of the strict definitions of the fundamental concepts which characterize modern analysis. That is why, the contents of his manuscripts appear to be dated at first sight. It seems that these are confined within the framework of what was known to Lagrange, i.e., confined to what was known at the end of the 18th century. However, in reality the fundamental characteristic tendencies of Marx's manuscripts are of considerable significance even to-day. It is true that Marx was not acquainted with the modern definitions of the concepts of real number, limit and continuity. But it appears, that even if he had an acquaintance with these definitions, he would not have been satisfied with them.

The situation is as follows : Marx was in search of the "real" process of finding out the derived function, i.e., of an algorithm, which will permit, firstly, to answer the question — whether there exists a derivative for a given function, and secondly — if it exists then how to find it out effectively. It is well known that the concept of limit is not algorithmic, and that is why such problems are solvable only for a certain class of functions. One of those classes is the class of analytical functions, i.e., the class of such functions, which are decomposable into powered serieses and — to use an expression of Marx — are objects of "algebraic" differentiation. As a matter of fact only such functions were investigated by Marx. At present the class of such functions — for which the two aforementioned questions can be answered — may be significantly extended, and operations with them may be so constructed, as to satisfy all the modern demands of strictness and exactitude. However, from Marx's point of view it is essential, that all passages to limit be investigated in the light of their effective completion ; in other words,it is essential to construct mathematical analysis basing it upon the theory of algorithms, as we would put it now.

The following statement made by Engels in his *Dialectics of Nature*, is well known to us : "The turning point in mathematics was Descartes' *variable magnitude*. With it came

motion and hence *dialectics* in mathematics, and *at once, too, of necessity the differential and integral calculus,* which moreover immediately begins, and which on the whole was completed by Newton and Leibnitz, not discovered by them " [MECW(R), 20, 573; *Dialectics of Nature,* Eng. ed., Progress, Moscow, 1976, p. 258].

But what is a "variable"? Generally speaking, what is a "variable" in mathematics? The famous English philosopher Bertrand Russell said in this connection, that "it is of course, one of those concepts which are the most difficult to understand". And the mathematician Karl Menger listed at least six entirely different meanings of this concept. For explaining the concept of variable, in other words, that of the function in general in mathematics, Marx's mathematical manuscripts are of considerable value, even to-day. Marx directly raised the question of the different meanings of the concept of function: functions "of x" and functions "in x", and specifically dwelt upon how the changes of variables are depicted in mathematics, upon the dialectics of this change. Marx attached special significance to the question of the means of representing the changes of variables, since, the characteristics of that method of "algebraic" differentiation, which belongs to him, are at issue here.

The point is this : Marx decisively came out against representing all changes in the value of the variable in the form of an addition (or a subtraction) of some predetermined value of the increment (of its absolute value). Sufficient idealisation of the real changes of the values of any magnitude already assumes, that we can fixate *all* its values with exactitude. But in reality all such values can only be approximately fixated. So the assumption upon which differential calculus is built, must be such that, to obtain the expression $f'(x)$, for a function derived from a given function $f(x)$, informations regarding the exact value of any variable are not demanded, but the expression $f(x)$ for the function, is deemed sufficient. Here one is merely required to know, that the value of the variable x in fact so changes, that any neighbourhood (however small) of every value of the variable x (from the domain of its values under consideration) has the value x_1 ; this value is different from x_1 but not greater than it : "in fact x_1 remains as indeterminate as x" [MR, 159; PV, 74].

Here it is understandable, that when x changes into x_1, then the difference $x_1 - x$ is formed. This difference is also designated by Δx, so that, as a result, x_1 appears as equal to $x + \Delta x$. However, Marx stressed, that this takes place only as a *result* of the transformation of the value of x into the value of x_1 and it does not precede this change, and that the representation of x_1 as definable by the expression $x + \Delta x$, signifies, as it were, an insertion, thereby, of distortive assumptions into the representation of motion (and of all change in general). These assumptions are distortive, because in that case : "Though in $x + \Delta x$, as a magnitude Δx is as indeterminate, as is the indeterminate variable x itself, nevertheless, Δx is determinate as distinct form the particular magnitude x, as the foetus beside its own mother, before she became pregnant" [MR, 159; PV, 74].

In consonance with these observations, Marx begins his definition of the function $f'(x)$, derived from the function $f(x)$ with the statement that x changes into x_1. As a result of this $f(x)$ changes into $f(x_1)$. And the differences $x_1 - x$ and $f(x_1) - f(x)$ are formed. The first of these differences is wittingly different from zero, since $x_1 \neq x$. "Here the increased x, i.e., x_1 is

distinct *from itself*, from what it was prior to the increase, i.e., from x, but x_1 does not appear as x increased by Δx; that is why, in fact x_1 remains as indeterminate as x" [MR, 159; PV, 74].

According to Marx the real secret of the differential calculus is as follows : to determine the value of the derived function at a point x (where the derivative exists), it is necessary not only to find out the point x_1 different from x in the neighbourhood of this point, and to form the ratio of differences $f(x_1) - f(x)$ and $x_1 - x$, i.e., the expression $\frac{f(x_1) - f(x)}{x_1 - x}$, but also to return there after back to that same point x; however, this is no direct return, but rather, a return in some special manner, connected with the concrete determination of the function $f(x)$, since the simple assumption of $x_1 = x$ in the expression $[f(x_1) - f(x)]/[x_1 - x]$ turns it into $[f(x) - f(x)]/[x - x]$, i.e., into *0/0* , in other words into absurdity.

This character of the definition of the derivative, which consists of the formation of a difference $x_1 - x$ other than zero, and then — after establishing the ratio

$$[f(x_1) - f(x)] / [x_1 - x]$$

— the dialectical "sublation" of that difference, is retained also in the modern definition of the derivative, where the removal of the difference $x_1 - x$ is effected with the help of the passage to the limit : from x_1 to x.

In his work under the title *APPENDIX TO THE MANUSCRIPT "ON THE HISTORY OF DIFFERENTIAL CALCULUS". ANALYSIS OF D'ALEMBERT'S METHOD* Marx too spoke of the "derivative", in essence as the limiting value of the ratio $[f(x_1) - f(x)]/[x_1 - x]$ though in this connection he used a different terminology. The confusion connected with the terms "limit" and "limiting value" led Marx to comment, that "perhaps the concept of limiting value has been incorrectly interpreted" [MR, 217; PV, 98]. This muddle prompted him to replace the term "limit" in the definition of the derivative, by the term "absolutely minimal expression". However, he did not insist upon this substitution, as he foresaw that a more precise concept of limit, with which he got acquainted from Lacroix's big *Treatise on differential and integral calculus*, a book which satisfied Marx considerably more than the other text books, may render the introduction of a new term unnecessary. In fact [elsewhere] Marx wrote about the concept of limit, that : "This category which has found wide use in [mathematical] analysis mainly in that of Lacroix, acquires an important significance as a substitute for the category of minimal expression" [MR, 129; PV, 62].

Thus in essence Marx has also explained the dialectics connected with the definition of derivative in modern mathematical analysis. Here we have a dialectical, and not formal, contradiction. It will be shown below, that the presence of the latter made the differential calculus of Newton and Leibnitz "mystical". Here it is necessary to keep only the following in view : Marx by no means forbids the presentation of every change in the value of a variable in the form of an addition of some "increment" to its already existing value. On the contrary, when the question of evaluating the *result* of an already accomplished change arises, then the talk about an increase in the value of the variable (for example, about the dependence of the increment of a function upon the increment of the corresponding independent variable), and what Marx calls "the point of view of summation" ($x_1 = x + \Delta x$ or $x_1 = x + h$), become entirely

justified. In his last work entitled the *Taylor's Theorem*, Marx especially dwells upon this transition from the "algebraic" to the "differential" method. But unfortunately this work remained unfinished. And that is why only a part of it has been included in Part I of the present volume. [However, a very detailed description of this manuscript, almost its, entire text, has been included in the second part of this book; see : MR, 498-562; PV, 264-301]. Here Marx stressed, that when in the "algebraic" method $x_1 - x$ exists for use only in the form of a difference, and not as $x_1 - x = h$ and, that is why, not as $x_1 = x + h$, then in the transition to the "differential" method we may consider h "as the *increment* (positive or negative) of x. We have the right to do this also as : $x_1 - x = \Delta x$, and this Δx itself, instead of serving, as in our mode, as a simple symbol or a simple sign for the difference of x-s, i.e., for $x_1 - x$, may also be treated as the magnitude of the difference $x_1 - x$, [itself] as indeterminate as $x_1 - x$ and, as changing as it (this magnitude). Thus, $x_1 - x = \Delta x$ or = the indeterminate magnitude h. Hence it follows that $x_1 = x + h$, and $f(x_1)$ or y, turns into $f(x + h)$" [MR, 522; PV, 278-279].

Thus, it will be extremely unfair to depict the standpoint of Marx as the rejection of all other methods used in differential calculus. If these methods turn out to be successful, then Marx sets before himself the task of explaining the secret of their success. And when he succeeds in this, i.e., after the method under consideration turns out to be well grounded and the conditions of its applicability are fulfilled, then Marx considers a transition to that method not only entirely justified, but also expedient.

After his manuscripts of 1881, containing the fundamental results of his reflections on the nature of differential calculus, Marx intended to send a third work to Engels, related to the history of the methods of differential calculus. At first he wanted to outline this history in the light of concrete examples of the different methods of proving the theorem about the derivative of a product. But afterwards he renounced this intention and proceeded to outline the general characteristics of the principal periods in the history of the methods of differential calculus.

Marx could not give a proper shape to this third work. Only the indications regarding the fact that he intended to write it and the rough copy of this manuscript, have been preserved. From these we come to know : how Marx formulated and changed his plan of the historical essay on this theme. In Part I of the present volume this rough copy is being reproduced in its full form [see : MR, 137-189; PV, 64-86]. Therein all the instructions of Marx, regarding the necessity of inserting this or that passage from the other manuscripts into the text [of the historical essay], have been fully taken into consideration. This manuscript gives us an opportunity to elucidate the standpoint of Marx on the history of the principal methods of differential calculus. These are :

1) the "mystical differential calculus" of Newton and Leibnitz,

2) the "rational differential calculus" of Euler and d'Alembert,

3) the "purely algebraic calculus" of Lagrange.

According to Marx the characteristic trait of the methods of Newton and Leibnitz was this : their creators did not see the "algebraic" roots of the differential calculus, they started working directly with its operational formulae. That is why, their origin and meaning

remained obscure and even mysterious. And the calculus itself appeared "as an independent means of computation, distinct from the ordinary algebra" [MR, 153; PV, 72], as a quite special mathematical discipline, just discovered, which is "as far away as the stars in heaven, by way of ordinary algebra" [MR, 199; PV, 90].

To the question as to "How were the starting points for the differential symbols as operational formulae obtained", Marx answered that this was done "with the help of either secret or evident metaphysical presuppositions, which in their turn lead to metaphysical non-mathematical consequences : what happens is a forcible destruction of certain magnitudes, blocking the path of deduction, [which], however, were generated by those very presuppositions" [MR, 123; PV, 59].

Elsewhere Marx wrote about these very methods of Newton and Leibnitz: "$x_1 = x + \Delta x$ at once turns into $x_1 = x + dx$ ..., where dx is postulated by a metaphysical *explanation.* At first it exists, and is explained only subsequently". "From this arbitrary postulation it follows, that ... terms ... must be *removed by jugglery,* so that a correct result may be obtained" [MR, 165; PV, 77].

In other words, so long as the means of *introducing* differential symbols into mathematics remain unelucidated, and what is more, remain in general incorrect — so long as the differentials dx and dy are simply identified with the increments Δx and Δy — they turn out to be unfounded, they are presented as the means of "forcible" abolition "by jugglery", and as the means of their own *removal,* it becomes necessary to metaphyscially invent certain actually infinitely small magnitudes. These are treated, *at the same time*, both as ordinary magnitudes other than zero (as what are now called "Archimedean" magnitudes), and as "vanishing" magnitudes (turning in zero) as distinct from the finite or infinitesimal magnitudes of even lower order (i.e., as "non -Archimedean"magnitudes); simply put : as both zero and not zero at the same time. In this connection Marx said : "there remains nothing else to do, but to present the increments of h as infinitely small [magnitudes] and to register them as such, as *independent beings,* for example, in the symbols ... dx, dy [etc]. But infinitely small magnitudes are also magnitudes, as are the infinitely big(the word infinitely (small) signifies only the fact that it is indefinitely small) ; that is why, these dy, dx, ... also figure in the computation as ordinary algebraic magnitudes, and in the equation ...

$$\ldots k = 2x\,dx + dx\,dx$$

the term $dx\,dx$ has as much right to exist as has $2x\,dx$". That is why, "most astonishing is that argument, by which this term is forcibly cast away" [MR, 151 and 153; PV, 71].

According to Marx, the presence of such actually infinitely small magnitudes — i.e., of formally contradictory objects, which are not introduced with the help of successive mathematically grounded operations, but are postulated upon the foundations of metaphysical "explanations", and are removed thereafter by a "sleight of hand" — is what makes the calculus of Newton and Leibnitz *mystical,* notwithstanding the series of advantages obtained, owing to the fact that this calculus straight off beings with the operational formulae.

At the same time Marx did of course highly appreciate the *historic* significance of the methods of Newton and Leibnitz. He wrote : "Thus, they themselves believed in the mysterious character of the newly discovered calculus, which provided correct (and

moreover in the geometrical applications, really astonishing) results by a positively incorrect mathematical procedure. They were thus self-mystified, valued the new discovery all the higher, enraged the crowd of old orthodox mathematicians all the more, and thus called forth the cry of opposition; it aroused an echo even in the lay world, and that is necessary for paving the path for something new" [MR, 169; PV, 78].

According to Marx, the next stage in the development of the methods of differential calculus appears to be the "rational differential calculus" of d'Alembert and Euler. Here the mathematically incorrect methods of Newton and Leibnitz stand corrected, but the point of departure remains the same. "d'Alembert starts directly from *the starting point of Newton and Leibnitz* : $x_1 = x + dx$. But he at once makes a fundamental correction : $x_1 = x + \Delta x$, i.e., $x+$ an *indeterminate,* but first of all a *finite increment.* [In the literature at Marx's disposal "a finite increment" meant: a finite increment other than zero.] This he calls h. With him the transformation of this h or Δx into dx ... takes place only as the last result of the development or at least just at the eleventh hour, while with the mystics and the initiators of calculus it appears as the starting point" [MR, 169 & 171; PV, 79]. And Marx stressed, that here the removal of the differential symbols from the final result takes place through "a correct mathematical operation. Hence, now they are removed without a trick" [MR, 173; PV, 80].

That is why, Marx made a high evaluation of the historical significance of d'Alembert's methods. He wrote : "d'Alembert tore off the shroud of mystery from the differential calculus and thereby took a great step forward" [MR, 175; PV, 80].

However, since, for d'Alembert the starting point remains that very representation of a change of x as the sum : $x+$ the increment Δx, which exists beforehand and independently of the change of x — here the dialectics proper of the process of differentiation is not yet revealed. And in criticism of d'Alembert Marx makes the remark : "d'Alembert starts from *(x+ dx)* but corrects this expression changing it into $(x + \Delta x)$, and correspondingly into $(x+ h)$; now a development becomes indispensable, with the help of which Δx or h turns into dx, but the entire development, which actually takes place, is reduced to this" [MR, 221; PV, 100].

It is well known, that to obtain the derivative dy/dx from the ratio of finite differences $\Delta y/\Delta x$, d'Alembert took recourse to the "transition to limit". In the manuals used by Marx this passage to the limit came before the expansion of the expression $f(x+ h)$ into a series of ascending integral powers of h, wherein the coefficient of h in its first power, was the "ready made" derivative $f'(x)$. That is why, the task was reduced to "freeing" it from the multiplier h and from the other terms of the series. However, this could have been done in a more natural way, by simply defining the derivative as the coefficient of h in its first power, in the expansion of $f(x+ h)$ into a series of powers of h.

Actually, in "the first method 1), as well as in the rational 2), the unknown real coefficient is manufactured in a ready-made form by the [use of] binomial theorem, and is met with already as the second term of the expanded series, thus, in the term necessarily containing h^1. Consequently, as in 1), so also in 2), the entire further course of differentiation is a luxury. This is why, let us cast aside this useless ballast" [MR, 177; PV, 81].

Lagrange the founder of the next stage in the development of the methods of differential calculus, of the "purely algebraic" calculus according to Marx's periodisation, did exactly the same thing. At first Marx very much appreciated the method of Lagrange, "whose theory of derived functions provided a new basis for the differential calculus" [MR, 193; PV, 88]. Usually $f(x+h)$ is expanded into a powered series of h, with the help of Taylor's theorem. Historically, this theorem appeared as the completed construction of the entire differential calculus, and thereby turned into its starting point, connecting it directly with [the existing] mathematics, which comes before the [differential]calculus (it does not use the specific symbolism of this calculus). In this context Marx commented : "The real, and accordingly the simplest, interconnection between the new and the old, is always discovered only when this new itself already attains its final form, and it may be said, that in the differential calculus this return (taking) backwards was carried out by the theorems of Taylor and MacLaurin. [The theorem of MacLaurin may be viewed as a particular instance of Taylor's theorem, and Marx too did the same ; see : MR, 195 & 197; (PV, 88-89).] That is why the idea of leading the differential calculus on to a strictly algebraic foundation was conceived only by Lagrange" [MR, 199; PV, 90]

However, Marx soon found out that Lagrange failed to effect this reduction.

As is well known, Lagrange attempted to prove, that "generally speaking", i.e., save "certain particular instances" where the differential calculus is "not applicable", the expression $f(x+h)$ is decomposable into the series $f(x) + ph + qh^2 + rh^3 + \ldots$, where $p, q, r, \ldots$ — the coefficients of the powers of h, are the new functions of x independent of h and are "derived " from $f(x)$.

Largrange proposed a proof of this. In essence, this proof did not have sufficiently exact mathematical sense. Naturally, it did not come off. Marx wrote about it : "This leap from the *ordinary algebra,* and besides *with the help of ordinary algebra,* into the *algebra of variables* [i.e., into the general theory of functions, which reflects movement and change in general — Ed.], is accepted as an *accomplished fact ;* it is not proved and, first of all, it *contradicts all the rules* of ordinary algebra" [MR, 207; PV, 93].

And about the "initial equation" of Lagrange Marx concludes, that it is not only unproved, but also the very "deduction of this equation from algebra rests upon a fraud" [MR, 207; PV, 94].

In the concluding part of Marx's incomplete manuscript [on Taylor's Theorem] Lagrange's method appears as the completion of the methods of Newton and Leibnitz, corrected by d'Alembert. It is the "algebricisation" of Taylor's formula, carried out with the help of this method itself. "Thus did Fichte side with Kant, Schelling — with Fichte and, Hegel — with Schelling, wherein neither Fichte, nor Schelling, nor Hegel did investigate the general basis of Kant, of idealism in general ; or else they could not have developed it further"[MR, 209; PV, 94].

We find that in the historical essay Marx gave us a graphic example, from his point of view, of the application of the methods of materialist dialectics in the science of history of mathematics.

*   *
*

Preparation of the present edition of Marx's *Mathematical Manuscripts* entailed a lot of work.Texts of the manuscripts had to be deciphered in their entirety.The work connected with the dating of the manuscripts had to be carried out. Marx's own statements were separated from the extracts and notes taken by him. Storage Units had to be formed as unbroken pieces of manuscripts, on the basis of an analysis of the mathematical contents of the manuscripts (in fact many manuscripts are not contained in copy books, but are separate sheets — often quite disorderly). In the overwhelming majority of instances the sources, from which Marx took notes or extracts, were ascertained. All original comments of Marx contained in these notes were singled out, by comparing the notes with their sources. All independent works and comments of Marx were translated into Russian.

The task of separating the original comments of Marx from the notes and extracts, involved a series of difficulties. Marx wrote the notes for himself, to have the necessary material on hand. As usual, he used a large number of most diverse sources. But if the source did not deserve a special mention, if, for example, it was simply a compiled text book, quite widely used in England at that time, then often Marx did not mention it as the source of an extract. The task was further complicated by the fact, that most of the books used by Marx are now bibliographical rarities. Finally, this entire work could be completed at first hand, only in England, where the stocks of corresponding literature were inspected and studied in detail, for solving this problem. This was done in the following libraries: the British Museum, the libraries of the London and Cambridge Universities, those of the University College of London, of the Trinity and St. James Colleges of Cambridge, of the Royal Society of London, as well as in the personal libraries of the eminent English scientists of 19th century — De Morgan and Graves. Inquiries were made also in the other libraries, for example, at the St. Catharine's College and elsewhere. There are some manuscripts, for which it was natural to assume that their sources were German. For these, the German historian of mathematics Vuccing inspected the library stocks of G.D.R at the request of the Institute.

A few missing pages of the manuscripts were obligingly supplied, in photocopies, by the Institute of Scoial History of Amsterdam, where the originals of Karl Marx's mathematical manuscripts are being preserved.

Since these manuscripts are of a draft character, omissions and even computational errors are to be found in them. In this edition the corresponding insertions or corrections have been placed within square brackets. In this connection all the square brackets of Marx had to be changed into double square brackets. Marx wrote some words in an abbreviated form. We have given their full form. But the text remains in the main unaltered. In places even the old spellings have been retained.

The principal language of these manuscripts is German. But if the source was in French or English, then Marx's often wrote the entire text of the corresponding manuscript in French or English. In a number of instances Marx's text turns out to be so mixed up, that it becomes difficult to state, in which language that manuscript was originally written.

The task of dating the manuscripts with exactitude also entailed a lot a difficulties. These difficulties have been mentioned in detail in the description of each separate manuscript. These descriptions have been provided in accordance with the archival number of the manuscript

and the title conferred upon it, characterising its source and content. Where the title or the sub-title belongs to Marx himself, it has been put within quotation marks, in the original language, as well as in the Russian translation. In the first part of the present volume, the titles which do not belong to Marx are accompanied by an asterisk mark.

Descriptions of the manuscripts have been given following the order of the archival sheets. In these descriptions, while indicating the archival sheets, Marx's own numerations of the pages have also been given, including those in letters. The published texts of Marx are everywhere accompanied by numbers of the archival sheets, indicating where they are to be found. [All these manuscripts are related to Stock 1, Inventory 1 of the Archives of the erstwhile Institute of Marxism-Leninism of the Central Committee of the Communist Party of Soviet Union — Tr.]

In many cases the language of Marx's mathematical manuscripts is different from the language which we now use, and to understand the ideas of Marx it becomes necessary to turn to the sources used by him and to explain the meaning of the terms used in them.Such explanations have been given in the notes at the end of this book, so as not to interrupt Marx's texts. Where more detailed informations about the contents of the sources used by Marx, were deemed necessary, the same has been provided in the Appendix. All such notes and insertions are of a purely factual character.

The texts of Marx contain a large number of underlines with which he stressed the places which appeared to be especially important to him. All such stresses have been reproduced here by using special type faces.

* *
*

This volume has been prepared by late Professor S.A. Yanovskaya of the M.V.Lomonosov State University of Moscow. The Preface, Description of the Mathematical Manuscripts (put together with the help of A.Z Ryvkin), Appendix and Notes belong to her. Professor K.A. Rybnikov took part in the preparation of this volume. That apart, he conducted the huge work of bringing to light the sources Karl Marx used, while he worked on his *Mathematical Manuscripts*. Comments and advices of academicians A.N. Kolmogorov and I.G. Petrovsky have been taken into account while preparing the present edition.

A. Z. Ryvkin (of the Main Editorial Office of Physico-Mathematical Literature of the "Nauka" Publishers) and O.K. Senekina (of the Institute of Marxism-Leninism of the C C CPSU) conducted the entire editorial work connected with this volume, the preparation for its printing and proof-reading.

This volume carries an index of the literature quoted and mentioned, as well as a name index. In the indexes the references to the pages of Marx's text have been indicated in italics.

Institute of Marxism-Leninism

C C CPSU

KARL MARX

MATHEMATICAL MANUSCRIPTS

PART I

DIFFERENTIAL CALCULUS : ITS NATURE AND HISTORY

TWO MANUSCRIPTS
ON
THE DIFFERENTIAL CALCULUS

I

* ON THE CONCEPT OF THE DERIVED FUNCTION [1]

I

Let the independent variable x increase to x_1 and then the dependent variable y increases to y_1 [2].

Here, sub I), the simplest case is being investigated. Here x appears only in its first power.

1) $y = ax$; if x increases to x_1, then $y_1 = ax_1$ and $y_1 - y = a(x_1 - x)$. If now we carry out the *differential operation,* i.e., allow x_1 to decrease to x, then we would get

$$x_1 = x \ ; \ x_1 - x = 0,$$

hence,

$$a(x_1 - x) = a \cdot 0 = 0.$$

Further, since y increased to y_1 only owing to the fact that, x increased to x_1, we would also have

$$y_1 = y \,; y_1 - y = 0.$$

Thus,

$y_1 - y = a(x_1 - x)$ would turn into $0 = 0$.

Thus, at first the postulation of a difference, and then its inverse removal will lead literally, to *nothing.* The entire difficulty in understanding the differential operation (as in that of any *negation of negation* whatever) lies precisely in seeing *how* it differs from such a simple procedure, and thus leads to valid results.

If we divide $a(x_1 - x)$, and also, correspondingly, the left hand side of the equation, by the factor $x_1 - x$, then we shall get

$$\frac{y_1 - y}{x_1 - x} = a.$$

Since y is a *dependent variable,* it can by no means accomplish any independent movement. That is why [here, as $y = ax$] y_1 cannot became equal to y, and so too $y_1 - y$ cannot become equal to 0, unless earlier x_1 became equal to x.

On the other hand, we saw that x_1 could not become equal to x in the function $a(x_1 - x)$, unless the latter turned into 0. That is why, the factor $x_1 - x$ was *necessarily a finite difference*[3] at that moment, when we divided by it, both the sides of the equation. Thus, at the moment of constituting the ratio $\frac{y_1 - y}{x_1 - x}$, $x_1 - x$ always presents itself as a finite difference, and hence, $\frac{y_1 - y}{x_1 - x}$ is a *ratio of finite differences* ; accordingly

$$\frac{y_1 - y}{x_1 - x} = \frac{\Delta y}{\Delta x}.$$

And thus,

$$\frac{y_1 - y}{x_1 - x} \text{ or}^4 \ \frac{\Delta y}{\Delta x} = a,$$

where the constant a figures as the *limit*[5] of the ratio of finite differences of the variables.

Since a is a constant, neither it, nor, consequently, *the right hand side* of the equation, reduced to it, may undergo any change. In that case, the *differential process* runs its course in the left hand side :

$$\frac{y_1 - y}{x_1 - x} \text{ or } \frac{\Delta y}{\Delta x},$$

and this is a characteristic of such simple functions as ax.

Let, in the denominator of this ratio, [the variable] x_1 decrease, approaching x; the limit of its decrease is attained, as soon as x_1 turns into x. With this the difference $x_1 - x$ will become equal to $x_1 - x = 0$, and hence also $y_1 - y = y - y = 0$.

We thus obtain, $\frac{0}{0} = a$.

Since in the expression $\frac{0}{0}$ every trace of its origin and its meaning has been wiped out, we change it into $\frac{dy}{dx}$, where the finite differences $x_1 - x$ or Δx and $y_1 - y$ or Δy appear symbolized, as *removed* or *vanished* differences, so that $\frac{\Delta y}{\Delta x}$ turns into $\frac{dy}{dx}$. Thus,

$$\frac{dy}{dx} = a.$$

The closely held consolation of some rationalizing mathematicians, that the quantities dy and dx are in fact only infinitely small and [that their ratio] merely approaches $\frac{0}{0}$, is a chimera, as will be shown tangibly, further, sub II). It requires to be mentioned further, as a characteristic of the instance under consideration, that as $\frac{\Delta y}{\Delta x} = a$, likewise $\frac{dy}{dx} = a$ i.e., the limit [of the ratio] of finite differences is at the same time also the limit [of the ratio] of the differentials.

2) The following may serve as the second example of the same case :

$$\left.\begin{aligned} y &= x \\ y_1 &= x_1 \end{aligned}\right. \;;\; y_1 - y = x_1 - x \;;$$

$$\frac{y_1 - y}{x_1 - x} \text{ or } \frac{\Delta y}{\Delta x} = 1 \;;\; \frac{0}{0} \text{ or } \frac{dy}{dx} = 1.$$

II

When $y = f(x)$, wherein at the right hand side of the equation the function x is situated in its *expanded algebraic expression*[6], then we call it the expression for the *initial function* of x; its first modification, obtained through the postulation of a difference, [is called] the

preliminary "derivative " of the function x ; and the final form which it takes as a result of the *differential process* [is called] the *"derived " function* of x [7].

1) $y = ax^3 + bx^2 + cx - e$.

If x increases to x_1, then

$$y_1 = ax_1^3 + bx_1^2 + cx_1 - e,$$

$$y_1 - y = a(x_1^3 - x^3) + b(x_1^2 - x^2) + c(x_1 - x) =$$
$$= a(x_1 - x)(x_1^2 + x_1x + x^2) + b(x_1 - x)(x_1 + x) + c(x_1 - x).$$

Hence,

$$\frac{y_1 - y}{x_1 - x} \text{ or } \frac{\Delta y}{\Delta x} = a(x_1^2 + x_1 + x^2) + b(x_1 + x) + c.$$

The preliminary "derivative"

$$a(x_1^2 + x_1x + x^2) + b(x_1 + x) + c$$

is here the *limit* of *the ratio* of finite differences, i.e., however small we may take these differences to be , the value of $\frac{\Delta y}{\Delta x}$ will be given by this "derivative". However, it does not coincide, as sub I), with the limit of the ratio of differentials*.

If in the function

$$a(x_1^2 + x_1x + x^2) + b(x_1 + x) + c$$

the variable x_1 decreases, till it attains the *boundary* of its decrease; i.e., till it becomes *equal* to x, then x_1^2 turns into x^2, $x_1 x$ into x^2 and $x_1 + x$ into $2x$ and we get the *"derived" function* of x :

$$3ax^2 + 2bx + c.$$

Here it is clear, that :

Firstly, for obtaining the "derivative" it is essential to assume that $x_1 = x$, hence in the *strict mathematical sense* $x_1 - x = 0$, without any subterfuge concerning merely infinite approximation.

Secondly, the assumption that $x_1 = x$, and hence, $x_1 - x = 0$, does not introduce anything symbolic into the "derivative"**.

The magnitude x_1, introduced initially through the change of x, does not vanish, it is merely *carried to* its minimal limit $= x$, and remains once again an element introduced in the initial function x, which in combinations partly with itself, and partly with the initial

* In the rough draft of this work (S.U.N. 4146, s.4), after this phrase it is said :"On the other hand, now the differential process takes place in the preliminary "derivative" of the function x (right hand side) , while , at the left hand side, the same process necessarily accompanies this movement". —Ed.

** In place of this in the draft it is said : "b) The search for the "derivative" from the initial function x so proceeded, that at first we undertook a few *finite differentiations* [assumptions of finite differences]; the latter gave us the preliminary "derivative", which is the limit of $\frac{\Delta y}{\Delta x}$. The differential process to which we pass over after this, *takes* this *limit* to its *minimal value*. The magnitude x_1, introduced in the first differentiation does not vanish ... ". — Ed.

function, gives us the final "derivative", i.e., the preliminary *"derivative", carried to its minimal value.*

Reduction of x_1 into x inside the first (preliminary) "derived" function turns $\frac{\Delta y}{\Delta x}$ of the left hand side into $\frac{0}{0}$ or $\frac{dy}{dx}$, that is to say

$$\frac{0}{0} \text{ or } \frac{dy}{dx} = 3ax^2 + 2bx + c,$$

so that the *derivative* appears as the *limit* of the ratio of differentials.

The transcendental or symbolic mishap occurs only in the left hand side, but it has already lost its horrifying form, since, now it occurs only as an expression of a process, the real content of which has already been revealed, in the right hand side of the equation.

In the "derivative"

$$3ax^2 + 2bx + c$$

the variable x is situated under conditions entirely different from those obtained in the initial function x (namely, in $ax^3 + bx^2 + cx - e$). That is why it [this derivative] in its turn can appear as an initial function and, with the help of a renewed differential process, can become the mother of some other "derivative". This may be repeated till the variable x finally becomes bereft of all "derivatives", hence, it may continue endlessly for functions of x, presented only as infinite serises, as is often the case. The symbols $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ etc. merely indicate the genealogical "derivative" in respect of the initial function of x, given at first. They become mysterious, when they are treated as the *starting points* of movement, and not simply as *expressions of successively deduced functions of* x. Then it really seems astonishing, that the ratio of vanishing quantities must again pass through higher orders of vanishing, while no one wonders, for example, about the fact that $3x^2$ may pass through the process of differentiation, as successfully, as did its grandmother x^3. It is possible to proceed from $3x^2$, just as from the initial function of x.

However, *notabene.* $\frac{\Delta y}{\Delta x}$ appears as the starting point of the *differential process*, in fact, only in those equations, which we had sub I), where x entered only into its first power. But then , as shown sub I), as a result, we get

$$\frac{\Delta y}{\Delta x} = a = \frac{dy}{dx}.$$

Thus, here, with the help of the differential process, through which $\frac{\Delta y}{\Delta x}$ passes, in fact *no new limit* is found. This [search for a new limit] is possible, only in so far as the preliminary "derivative" contains the variable x, i.e., in so far as $\frac{dy}{dx}$ remains a symbol of some real

process*. Of course, this does not in any way prevent the symbols $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc., and their combinations, in differential calculus, from entering also into the right hand side of the equation. But then, we also know, that such purely symbolic equations merely indicate those *operations,* which must subsequently be carried out, upon the real functions of the variables.

2) $y = ax^m$.

If x turns into x_1, then $y_1 = ax_1{}^m$ and $y_1 - y = a(x_1{}^m - x^m) =$

$= a(x_1 - x)(x_1{}^{m-1} + x_1{}^{m-2}x + x_1{}^{m-3}x^2 +$ etc. upto the term $x_1{}^{m-m}x^{m-1})$.

Hence,

$$\frac{y_1 - y}{x_1 - x} \text{ or } \frac{\Delta y}{\Delta x} = a(x_1^{m-1} + x_1^{m-2}x + x_1^{m-3}x^2 + ... + x_1^{m-m}x^{m-1}).$$

If we now apply the differential process to this "*preliminary derivative*", so that x_1 becomes equal to x or $x_1 - x$ becomes equal to 0, then

x_1^{m-1}	turns	into	x^{m-1};
$x_1^{m-2}x$	"	"	$x^{m-2}x = x^{m-2+1} = x^{m-1}$;
$x_1^{m-3}x^2$	"	"	$x^{m-3}x^2 = x^{m-3+2} = x^{m-1}$;

and finally,

$x_1^{m-m}x^{m-1}$	"	"	$x^{m-m}x^{m-1} = x^{0+m-1} = x^{m-1}$.

We thus obtain, m times, the function x^{m-1}, and that is why the "derivative" is max^{m-1}. Owing to the equalisation $x_1 = x$ inside the "preliminary derivative"**, on the left hand side $\frac{\Delta y}{\Delta x}$ turns into $\frac{0}{0}$ or $\frac{dy}{dx}$, hence ,

$$\frac{dy}{dx} = max^{m-1}.$$

It is possible to set forth all the operations of differential calculus in this manner, but that would be awfully useless pedantry. All the same, we shall cite here one more example, since in the foregoing, the difference $x_1 - x$ entered into the function x *only once*, and that is why, while constituting the expression

$$\frac{y_1 - y}{x_1 - x} \text{ or } \frac{\Delta y}{\Delta x}$$

it vanished from the right hand side.

This does not happen in the following instance :

3) $y = a^x$;

---

* The corresponding sentence in the draft reads thus (sheet 7) : "It may be obtained only there, where the preliminary "derivative" contains the variable x, for its movement too, may form some genuinely new value, such that $\frac{dy}{dx}$ is the symbol of some real process".—Ed

** That is in the right hand side. —Ed.

if x turns into x_1, then

$$y_1 = a^{x_1}.$$

Hence,

$$y_1 - y = a^{x_1} - a^x = a^x(a^{x_1 - x} - 1)$$

[But]

$$a^{x_1 - x} = \{1 + (a-1)\}^{x_1 - x}$$

and $\{1 + (a-1)\}^{x_1 - x} = 1 + (x_1 - x)(a-1) + (x_1 - x)\dfrac{(x_1 - x - 1)}{1 \cdot 2}(a-1)^2 +$ etc.[8]

Hence,

$$y_1 - y = a^x(a^{x_1 - x} - 1) =$$

$$= a^x\left\{(x_1 - x)(a-1) + \frac{(x_1 - x)(x_1 - x - 1)}{1 \cdot 2}(a-1)^2 + \right.$$

$$\left. + \frac{(x_1 - x)(x_1 - x - 1)(x_1 - x - 2)}{1 \cdot 2 \cdot 3}(a-1)^3 + \text{etc.}\right\}.$$

$\therefore$ [9] $\dfrac{y_1 - y}{x_1 - x}$ or $\dfrac{\Delta y}{\Delta x} = a^x\left\{(a-1) + \dfrac{(x_1 - x - 1)}{1 \cdot 2}(a-1)^2 + \right.$

$$\left. + \frac{(x_1 - x - 1)(x_1 - x - 2)}{1 \cdot 2 \cdot 3}(a-1)^3 + \text{etc.}\right\}.$$

If now x_1 becomes equal to x, hence $x_1 - x$ becomes equal to zero, then we obtain as the "*derivative*"

$$a^x\left\{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \text{etc.}\right\}.$$

Thus,

$$\frac{dy}{dx} = a^x\left\{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \text{etc.}\right\}.$$

If we now designate the sum of constants within the brackets by A, then

$$\frac{dy}{dx} = A\,a^x\ ;$$

here, however A = the Napierian logarithm of the number a, so

$\dfrac{dy}{dx}$ or, if in place of y we put its value, then $\dfrac{da^x}{dx} = \log a \cdot a^x$, and $da^x = \log a \cdot a^x dx$

Additionally[10].

We have :

1) considered the case, where the factor $(x_1 - x)$ is contained only once in [the expression leading to] the "*preliminary derivative*", i.e., in the finite difference equation[11], owing to which, when both the parts are divided by $(x_1 - x)$, [an expression] is formed [for]

$$\frac{y_1 - y}{x_1 - x} \text{ or } \frac{\Delta y}{\Delta x}$$

[which does not contain entries of the difference $(x_1 - x)$], i.e., this factor is excluded from the function x;

2) considered the case (in the example : $d\,(a^x)$), where the factor $(x_1 - x)$ remains in the function x after [the ratio] $\frac{\Delta y}{\Delta x}$ is formed[12].

3) Still remains to be considered the case, where the factor $(x_1 - x)$ is *not immediately* excluded from the first difference equation [leading to] ("the preliminary derivative")

$$y = \sqrt{a^2 + x^2},$$
$$y_1 = \sqrt{a^2 + x_1^2},$$
$$y_1 - y = \sqrt{a^2 + x_1^2} - \sqrt{a^2 + x^2};$$

We shall divide this function of x— consequently , also the left hand side — by $x_1 - x$. Then

$$\frac{y_1 - y}{x_1 - x} \quad (\text{or } \frac{\Delta y}{\Delta x}) \quad = \frac{\sqrt{a^2 + x_1^2} - \sqrt{a^2 + x^2}}{x_1 - x}$$

In order to free the numerator of irrationality we multiply both the numerator and denominator by $\sqrt{a^2 + x_1^2} + \sqrt{a^2 + x^2}$ and get

$$\frac{\Delta y}{\Delta x} = \frac{a^2 + x_1^2 - (a^2 + x^2)}{(x_1 - x)(\sqrt{a^2 + x_1^2} + \sqrt{a^2 + x^2})} =$$
$$= \frac{x_1^2 - x^2}{(x_1 - x)(\sqrt{a^2 + x_1^2} + \sqrt{a^2 + x^2})}.$$

But

$$\frac{x_1^2 - x^2}{(x_1 - x)(\sqrt{a^2 + x_1^2} + \sqrt{a^2 + x^2})} =$$
$$= \frac{(x_1 - x)(x_1 + x)}{(x_1 - x)(\sqrt{a^2 + x_1^2} + \sqrt{a^2 + x^2})}.$$

Hence,

$$\frac{\Delta y}{\Delta x} = \frac{x_1 + x}{\sqrt{a^2 + x_1^2} + \sqrt{a^2 + x^2}}$$

If now x_1 becomes equal to x, or $x_1 - x$ becomes equal to 0, then

$$\frac{dy}{dx} = \frac{2x}{2\sqrt{a^2 + x^2}} = \frac{x}{\sqrt{a^2 + x^2}}.$$

Hence,

$$dy \text{ or } d\sqrt{a^2 + x^2} = \frac{x\,dx}{\sqrt{a^2 + x^2}}.$$

---

*ON THE DIFFERENTIAL [13]

I

1) Suppose the function $f(x)$ or $y=uz$ is to be differentiated, where u and z are both functions dependent upon the independent variable x ; they are independent variables relative to the function y dependent upon them, which, thus depends also upon x.

$$y_1 = u_1 z_1,$$

$$y_1 - y = u_1 z_1 - uz = z_1(u_1 - u) + u(z_1 - z),$$

$$\frac{y_1 - y}{x_1 - x} \text{ or } \frac{\Delta y}{\Delta x} = z_1 \frac{u_1 - u}{x_1 - x} + u\,\frac{z_1 - z}{x_1 - x} = \frac{z_1 \Delta u}{\Delta x} + \frac{u\,\Delta z}{\Delta x} \text{ *}.$$

Now, if in the right hand side x_1 becomes equal to x, consequently $x_1 - x = 0$, then $u_1 - u = 0$, $z_1 - z = 0$, hence, [in the expression] $z_1 \frac{u_1 - u}{x_1 - x}$ the factor z_1 also turns into z and finally, in the left hand side $y_1 - y = 0$. Thus

A) $\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$.

This equation, being multiplied by the denominator dx common to all terms, turns into

B) dy or $d(uz) = zdu + udz$[14].

2) Let us, at first consider the equation A) :

$$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}.$$

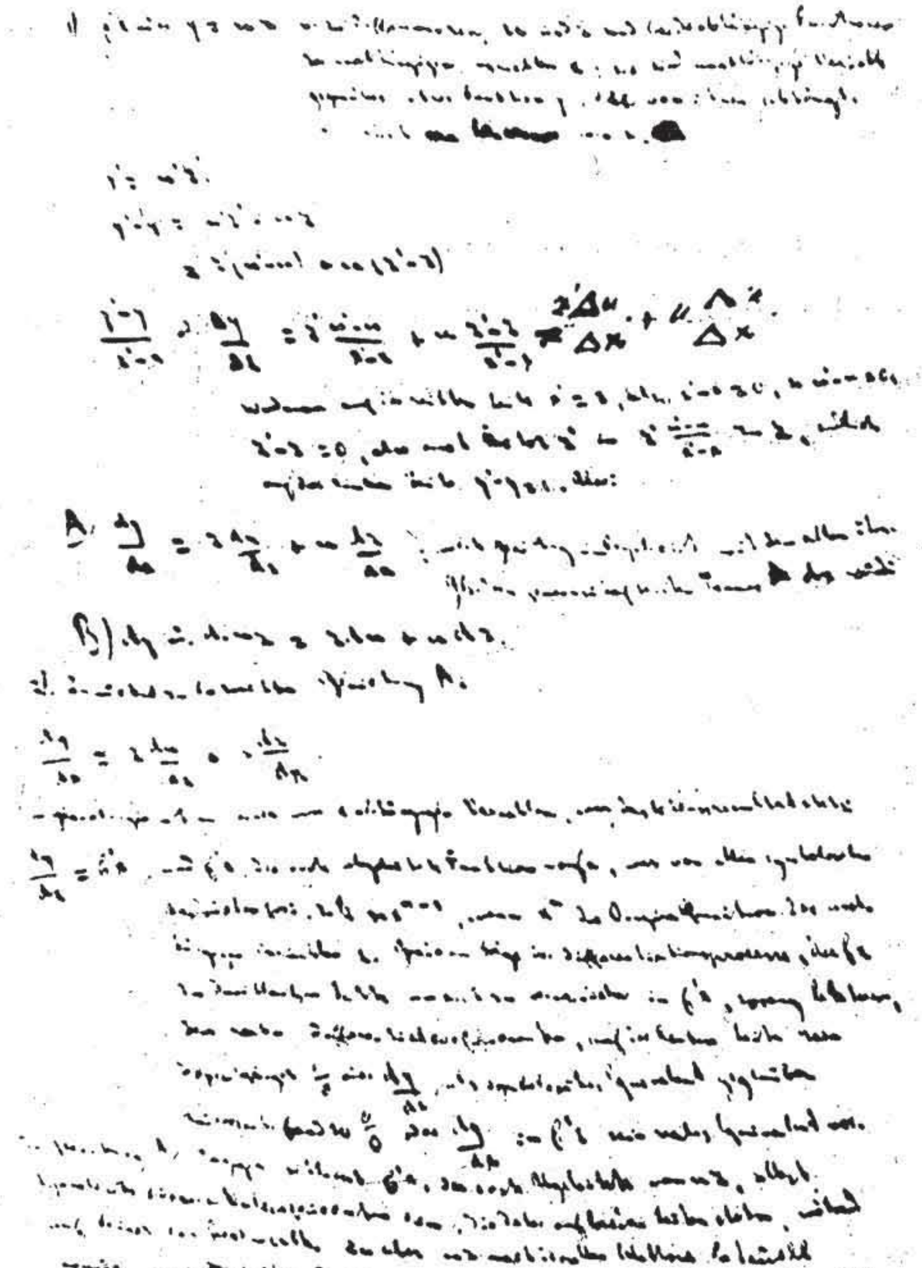
In the equations with only one variable dependent upon x the final result was always

$$\frac{dy}{dx} = f'(x),$$

where $f'(x)$, the first derived function of $f(x)$, was free from all symbolic expressions[15], like, for example, mx^{m-1}, in the instance where x^m is the initial function of the independent variable x. Precisely owing to the processes of differentiation, through which the function $f(x)$ had to pass, in order to turn into $f'(x)$, there appeared, opposite the latter, i.e., opposite the real differential coefficient[16], on the left hand side, as the symbolic equivalent of its double, $\frac{0}{0}$ or $\frac{dy}{dx}$. On the other hand, $\frac{0}{0}$ or $\frac{dy}{dx}$ thus found its real equivalent in $f'(x)$.

In the equation A), the first derivative of uz, $f'(x)$ itself, contained within it the symbolic differential co-efficients, which, that is why, stood on both sides, while on neither side they had any real value. But since, in our treatment of uz we followed the same method as earlier, when we operated upon the functions of x with only one independent variable, this contrast in result is obviously conditioned by the specific character of the initial function itself, i.e., of uz. On this in greater detail, sub 3).

* The latter part of this equality was completed by Engels. — Ed.



PHOTOCOPY OF THE FIRST PAGE OF THE MANUSCRIPT " ON THE DIFFERENTIAL".

However, preliminarily, let us consider further, whether or not there are any snags in the deduction of equation A).

In its right hand part

$$\frac{u_1 - u}{x_1 - x} \text{ or } \frac{\Delta u}{\Delta x} \text{ and } \frac{z_1 - z}{x_1 - x} \text{ or } \frac{\Delta z}{\Delta x}$$

turned into $\frac{0}{0}, \frac{0}{0}$, since x_1 became equal to x, hence, $x_1 - x = 0$. But, in place of $\frac{0}{0}, \frac{0}{0}$ we wrote $\frac{du}{dx}, \frac{dz}{dx}$, without much of a reflection. Was this permissible under the circumstances, that these $\frac{0}{0}$s figure here as *multipliers of the variables* u and z respectively, whereas in the instances involving one dependent variable the sole symbolic differential co-efficient obtained therein — $\frac{0}{0}$ or $\frac{dy}{dx}$ — did not have any multiplier, save the constant 1?

If on the right hand side we substitute for the expressions $\frac{du}{dx}, \frac{dz}{dx}$, their initial main form, then it turns into $z\frac{0}{0} + u\frac{0}{0}$. Had we multiplied further z and u by the numerator of [the expression] $\frac{0}{0}$ accompanying each of them, then we would have got $\frac{0}{0} + \frac{0}{0}$, and since the variables z and u themselves became equal to zero[17], their derivatives too are equal to zero; hence, in the end

$\frac{0}{0} = 0$, and not $z\frac{du}{dx} + u\frac{dz}{dx}$ But this procedure is mathematically wrong.

Take for example,

$$\frac{u_1 - u}{x_1 - x} = \frac{\Delta u}{\Delta x}.$$

The numerator turned out to be $= 0$ not because we started with the equalisation $u_1 - u = 0$; the numerator became equal to zero, or $u_1 - u = 0$ only because, the denominator, i.e., difference of the independent variable x or $x_1 - x$ *became equal to zero.*

Thus opposite the variables u and z stands not 0, but $\frac{0}{0}$, the *numerator of which is, in this form, inseparable from its denominator.* That is why $\frac{0}{0}$ as a multiplier can turn its co-efficients into zero, only when and in so far as $\frac{0}{0} = 0$.

Even in the ordinary algebra, had the product $P \cdot \frac{m}{n}$ appeared in the form $P \cdot \frac{0}{0}$, it would have been wrong, to draw the conclusion straight off from there, that it *must be* equal to zero,

though here it *can be* always assumed to be equal to zero, so long as we can arbitrarily start the nullification either from the numerator or from the denominator[18].

For example, $P \cdot \frac{x^2 - a^2}{x - a}$. If we assume [that $x = a$, wherefrom] $x^2 = a^2$, *i.e.*, $x^2 - a^2 = 0$, then we shall get $P \cdot \frac{0}{0} = \frac{0}{0}$. And the latter may always be assumed to be equal to zero, since $\frac{0}{0}$ may as well be zero as also it can be any other [number].

If we expand $x^2 - a^2$ into its factors, then we shall get

$$P \cdot \frac{x-a}{x-a} \cdot (x + a) = P\,(x + a)$$

and, since $x = a$ [19], $= 2Pa$.

Successive differentiation —for example, of [the function] x^3, where $\frac{0}{0}$ becomes $= 0$ only for the fourth derivative, since in the third the variable x has vanished and has been substituted by a constant —shows that, only under completely determined conditions $\frac{0}{0}$ becomes $= 0$.

However, in our instance, where the origin of these $\frac{0}{0}$, $\frac{0}{0}$ as the differential expressions proper for $\frac{\Delta z}{\Delta x}$, $\frac{\Delta u}{\Delta x}$ are well known, the dress-coat of $\frac{dz}{dx}$, $\frac{du}{dx}$ fits them at once.

3) In the equations considered earlier, like $y = x^m$, $y = a^x$ etc. *some initial function of* x stands opposite the y, which is "dependent "on it.

In $y = uz$ both sides are occupied by "dependents ". Here, if y is immediately "dependent" upon u and z, then u and z are, in their turn, dependent upon x. This specific character of the initial function uz inevitably leaves its mark upon its "derivative".

That u is a function of x, and that z is some other function of x, may be expressed as follows :

$$u = f(x), \qquad u_1 - u = f(x_1) - f(x);$$

that is why

$$z = \varphi(x), \qquad z_1 - z = \varphi(x_1) - \varphi(x).$$

But neither for $f(x)$, nor for $\varphi(x)$, does the initial equation provide the primary functions of x, i.e., the determinate values* in x. As a result of this u and z figure only as names, as symbols of functions dependent on x; that is why, only the *general forms of this ratio of dependence* :

$$\frac{u_1 - u}{x_1 - x} = \frac{f(x_1) - f(x)}{x_1 - x}, \quad \frac{z_1 - z}{x_1 - x} = \frac{\varphi(x_1) - \varphi(x)}{x_1 - x},$$

* Here in the sense of "determinate expressions". — Ed.

immediately give the process of arriving at the derivative of uz. When the process attains the point, where it is assumed that $x_1 = x$, i.e., $x_1 - x = 0$, then these general forms turn into

$$\frac{du}{dx} = \frac{df(x)}{dx}, \quad \frac{dz}{dx} = \frac{d\varphi(x)}{dx},$$

and the symbolic differential co-efficients as such appear inside the "derivative". But in the equations with only one dependent variable, $\frac{dy}{dx}$ certainly does not have any other content, apart from that of $\frac{du}{dx}, \frac{dz}{dx}$ here. It too is only the symbolic differential expression for

$$\frac{y_1 - y}{x_1 - x} = \frac{f(x_1) - f(x)}{x_1 - x}.$$[20]

Though the nature of $\frac{du}{dx}, \frac{dz}{dx}$, i.e., generally, that of the symbolic differential co-efficients does not change a bit, if they appear *inside the very derivative*, i.e., also in the right hand side of the differential equation, thereby, however, changes their role as well as the character of the equation.

If we represent the initial function of uz in the general form, by $f(x)$ and, hence, its first "derivative" by $f'(x)$, then

$$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$$

turns into :

$$\frac{dy}{dx} = f'(x).$$

We obtain this general form, for equations with only one dependent variable. In both the cases the initial forms of $\frac{dy}{dx}$ emerge from the processes of deduction, which transform $f(x)$ into $f'(x)$. That is why, as soon as $f(x)$ got transformed into $f'(x)$, the latter also stood opposite $\frac{dy}{dx}$ as its symbolic expression proper, as its double or symbolic equivalent.

This is why in both the cases $\frac{dy}{dx}$ plays an *identical role.*

The case is somewhat different with $\frac{du}{dx}$ and $\frac{dz}{dx}$. Along with the other elements of [the derivative] $f'(x)$, wherein they are contained, they find in $\frac{dy}{dx}$ their symbolic expression, their symbolic equivalent, but they themselves, on their part, do not stand opposite any $f'(x)$, $\varphi'(x)$, for which they, in their turn, would be symbolic doubles. They have come into the world one-sidedly, as shadows without the bodies which have cast them, symbolic differential co-efficients without the real differential co-efficients, i.e., without the corresponding equivalent "derivatives". Thus, the symbolic differential co-efficient has become an *independent point of departure*, only its real equivalent must be found out. Thus, the initiative

has shifted from the right algebraic pole, to the left symbolic one. But with this, the differential calculus too appears as a specific kind of calculus, already operating independently upon its own ground, since its points of departure $\frac{du}{dx}, \frac{dz}{dx}$ are mathematical magnitudes, which belong only to this calculus and characterize it. And this reversal of the method resulted here from the algebraic differentiation of *uz*. Thus, the algebraic method, by itself, turns into its opposite, the differential method *. But, what are the ''derivatives'' corresponding to the symbolic differential co-efficients $\frac{du}{dx}, \frac{dz}{dx}$? The intial equation $y = uz$ does not give us any information to help answer, this question. However, it may be answered, if, in place of u and z, one may assume some arbitrary initial functions of x, for example :

$$u = x^4 , z = x^3 + ax^2.$$

But thereby itself, the symbolic differential coefficients are at once transformed into *operational symbols*, into the symbols of processes, which are to be carried out over x^4 and $x^3 + ax^2$, for finding out their "derivatives". Having initially emerged as the symbolic expression of the "derivatives ", i.e., of the operations of the algebraic differentiation already carried out, the symbolic differential co-efficient now plays the role of the symbol of those operations of differentiation, which remain to the carried out.

At the same time, with this the equation

$$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$$

— which is purely symbolic from the very beginning, since no side of it is free from symbols — turns into a general symbolic operational equation.

We note further that**from the beginning of the 18th century upto the present time, the general task of differential calculus is usually formulated thus : how to find out the real equivalent for the symbolic differential coefficient.

4)

A) $\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$.

---

* In a draft of the article "On The Differential" this paragraph is set forth as follows [S.U.N. 4148,s.16-17] : "Conversely with $\frac{du}{dx}, \frac{dz}{dx}$. Born inside the derivative, they, together with the remaining elements of the latter, find in $\frac{dy}{dx}$ their symbolic expression proper, consequently, their symbolic equivalent. But they themselves exist without equivalents, without proper differential co-efficients, i.e., without the derivatives $f'(x)$, $\varphi'(x)$ of which, in their part, they would be symbolic expressions. They appear before us as ready-made differential symbols, whose real values resemble shadows, the bodies corresponding to which are to be sought out. Thus the task at hand literally took a turn. The symbolic diffferential co-efficients became *starting points* in the full sense of the words. Their equivalents – the differential co-efficients proper or, the corresponding derived functions – are still to be found out. Thereby the intiative has shifted from the right hand pole to the left. Since this inversion of method emerged from the algebraic movement of the function *uz*, its basis is thereby algebraic". —Ed..

**In the draft : "save a few exceptions". — Ed.

Evidently, this is not the simplest expression of the equation A), since all of its terms contain the common denominator dx. Discarding this common denominator, we get

B) $d\,(uz)$ or $dy = zdu + udz$.

In B) all the traces of its origin from A) has vanished. That is why, it is correct, in that situation when u and z are dependent upon x, as well as then, when they are only interdependent[21] – independent of any relation with x whatsoever. From the very beginning it is a symbolic equation and can, from the very beginning, serve as a symbolic operational equation. Finally, it asserts that, if

$$y = zu \text{ etc;}$$

i.e., $y =$ the product of any number of variables, then $dy =$ the sum of those products, in each of which, one of the multipliers is by turns considered to be a variable, others – constants.

However, for our purposes – namely, for further investigations into the differential of y generally – the form B) won't do. That is why, If we put

$$u = x^4 \ , \ z = x^3 + ax^2,$$

then [we can operate further, as follows :]

$$du = 4x^3\,dx\,,\, dz = (3x^2 + 2ax)\,dx,$$

as it has been shown earlier for equations with only one dependent variable. Let us put these values of du and dz in the equation A). Then :

A) $\dfrac{dy}{dx} = (x^3 + ax^2)\ \dfrac{4x^3dx}{dx} + x^4\,\dfrac{(3x^2 + 2ax)\,dx}{dx}$;

consequently,

$\dfrac{dy}{dx} = (x^3 + ax^2)\,4x^3 + x^4\,(3x^2 + 2ax)$;

that is why

$dy = \{(x^3 + ax^2)4x^3 + x^4\,(3x^2 + 2ax)\}dx.$

The expression within brackets is the first derivative of uz; but since $uz = f(x)$, its derivative $= f'(x)$. Putting the latter in place of the algebraic function, we get

$$dy = f'(x)\,dx.$$

We have already obtained the same result from an arbitrary equation with only one independent variable, for example:

$y = x^m$,

$\dfrac{dy}{dx} = mx^{m-1} = f'(x)$,

$dy = f'(x)dx$.

Generally speaking, we have : if $y = f(x)$, then, irrespective of the fact whether this function of x is certain initial function in x or whether it contains dependent variables, [it is] always [the case that] $dy = df(x)$, *and* $df(x) = f'(x)dx$, such that

B) $dy = f'(x)dx$ is the universal form of the differential of y. The same could have been shown, at once, also in the instance when $f(x)$ has the form $f(x, z)$, i.e., when it is a function *of two variables, independent of each other*. But that is unnecessary for our purposes.

II

1) The differential

$$dy = f'(x)dx$$

at first looks more suspicious than the differential coefficient

$$\frac{dy}{dx} = f'(x),$$

from which it is deduced.

In $\frac{dy}{dx} = \frac{0}{0}$ the numerator and the denominator are inseparably connected with each other; in $dy = f'(x)\,dx$ they look separated, so that the conclusion suggests itself, that this is only a disguised expression for

$0 = f'(x) \cdot 0$ or $0 = 0$

with which "there is nothing to be done".

A French mathematician of the first third of the 19th century – Boucharlat, who, like the "elegant" Frenchman[22] known [to you], but in an entirely different manner, has clearly connected the differential method with the algebraic method of Lagrange, says :

If, for example, $\frac{dy}{dx} = 3x^2$, then " $\frac{dy}{dx}$, in other words $\frac{0}{0}$ or, rather, its value $3x^2$ is the differential coefficient of the function y. Since $\frac{dy}{dx}$ is thus a symbol, representing the limit $3x^2$, dx should *always have stood* * *under* dy. But, *to facilitate the algebraic operations* we consider $\frac{dy}{dx}$ to be an ordinary fraction and $\frac{dy}{dx} = 3x^2$ to be an ordinary equation; freeing it from the denominator we get as the result : $dy = 3x^2dx$, which is called the differential of y"[23].

Thus, in order to "facilitate algebraic operations" a false formula is deliberately introduced, christening, it as the "differential ".

In reality the case is not so fraudulent.. In $\frac{0}{0}$ **the numerator is inseparable from the denominator. But why? Because only in the unseparated form do they express a relation, in this case the ratio

$$\frac{y_1 - y}{x_1 - x} = \frac{f(x_1) - f(x)}{x_1 - x},$$

reduced to its absolute minimum[24], where the numerator has become zero, because the denominator has. Separated, they are both zeroes and that is why they lose their symbolic meaning, their sense. But as soon as $x_1 - x = 0$ obtains in dx a form which unalterably

* In the draft : "remained". —Ed.

** In the draft : "In the form $\frac{0}{0}$ ". —Ed.

presents it as a vanished difference of the independent variable x, and consequently, also dy as a vanished difference of the function of x or, of the dependent variable y, such a separation becomes an entirely permissible operation. Wherever dx now stands, such a change of position leaves the relation of dy to it, untouched. Thus $dy = f'(x)dx$ appears to us as another form of

$$\frac{dy}{dx} = f'(x),$$

and is always replaceable by the latter[25].

2) The differential $dy = f'(x)dx$ was obtained by direct algebraic deduction from A) (see I, 4). But algebraic deduction of equation A) has already shown that, the differential symbols, in the given instance the symbolic differential coefficients – initially originating only as the symbolic expressions for algebraically carried out processes of differentiation – necessarily turn into independent starting points, into symbols of operations, which still remain to be carried out, or into operational symbols. In consequence, the symbolic equations which emerged along algebraic lines, also turn into symbolic operational equations.

Thus we have a double right to consider $dy = f'(x)dx$ to be a symbolic operational equation. Moreover, we now know *a priori*, that if in

$$y = f(x) \quad [\text{and}] \quad dy = df(x)$$

the differential operation indicated by $df(x)$ is to be performed upon $f(x)$, then the result will be $dy = f'(x)dx$ and that hence, we finally obtain

$$\frac{dy}{dx} = f'(x).$$

But [it happens] only from that moment, when the differential begins to function as the starting point of calculus, when the inversion of the algebraic method of differentiation is completed, and hence the differential calculus itself appears as an altogether special mode, a specific way, of reckoning with the variable quantities.

To make it more graphic, I shall put forward the sum total of the algebraic method applied by me, substituting moreover, only the determinate algebraic expressions in x by the expression $f(x)$ *and designating the "preliminary derivative" (see the first mss*) by* $f^1(x)$, *as distinct from the final "derivative"* $f'(x)$

Now, if $f(x) = y$, $f(x_1) = y_1$, [then]

$$f(x_1) - f(x) = y_1 - y \text{ or } \Delta y,$$

$$f^1(x)(x_1 - x) = y_1 - y \text{ or } \Delta y.$$

The preliminary derivative $f^1(x)$, just like its multiplier $x_1 - x$, *must*** contain expressions in x_1 and x, *save the sole exception*, where $f(x)$ is an initial function of *first power*

$$f^1(x) = \frac{y_1 - y}{x_1 - x} \text{ or } \frac{\Delta y}{\Delta x}.$$

Now, putting

$$x_1 = x, \text{ i.e., } x_1 - x = 0 \text{ in}$$

* See :" On The Concept Of The Derived Function" [PV,19]. — Ed.

**In the draft; "*must as a rule* ".– Ed.

$f^1(x)$, we get :

$$f'(x) = \frac{0}{0} \quad or \quad \frac{dy}{dx},$$

and finally :

$$f'(x)\,dx = dy \quad \text{or} \quad dy = f'(x)\,dx.$$

The differential of y is thus the final point of algebraic development : it becomes the starting point of differential calculus, now moving upon its own ground. Here dy, considered in an isolated manner, i.e., without its equivalent the differential part [26] of y – at once plays the same role, which Δy played in the algebraic method, and dx – the differential part of x – plays the same role, which Δx played there.

Had we freed

$$\frac{\Delta y}{\Delta x} = f^1(x)$$

from its denominator, [we would have got]:

I) $\Delta y = f^1(x)\,\Delta x$.

Conversely, starting from the differential calculus as a ready-made, separate means of computation – and such a starting point was, in its turn , derived algebraically – we at once begin with the differential expression of the equation I), namely with :

II) $dy = f'(x)\,dx$.

3) Since the symbolic equation of the differential appears already in the algebraic treatment of the most elementary functions with only one dependent variable, it may seem that, the inversion of method too, could have been carried through in a manner, which is much easier than what took place in the case of

$$y = uz.$$

The most elementary functions are functions of the first power :

a) $y = x$, which gives the differential coefficient $\frac{dy}{dx} = 1$, hence the differential $dy = dx$.

b) $y = x \pm ab$, which gives the differential coefficient $\frac{dy}{dx} = 1$, hence again the differential $dy = dx$.

c) $y = ax$, which gives the differential coefficient $\frac{dy}{dx} = a$, hence the differential $dy = adx$.

Let us consider the simplest instance [sub a)].

There : $y = x$, $y_1 = x_1$;

$y_1 - y$ or $\Delta y = x_1 - x$ or Δx.

I) $\frac{y_1 - y}{x_1 - x}$ or $\frac{\Delta y}{\Delta x} = 1$; hence also $\Delta y = \Delta x$.

If, now, in $\frac{\Delta y}{\Delta x}$ we put $x_1 = x$ or $x_1 - x = 0$, then:

II) $\frac{0}{0}$ *or* $\frac{dy}{dx} = 1$; hence $dy = dx$.

From the outset, as soon as we obtain I), i.e., $\frac{\Delta y}{\Delta x} = 1$, we are forced to operate further on the left hand side of the equation, since the right hand side is occupied by the constant 1. But with this the *inversion of method*, which throws the initiative from the right hand side to the left, appears, as if from the very beginning, proved once and for all, in fact, as the first word of the [new]algebraic method itself. Let us take a closer look at the issue.

The actual result was:

I) $\frac{\Delta y}{\Delta x} = 1$

II) $\frac{0}{0}$ *or* $\frac{dy}{dx} = 1.$

Since both I) and II) leads to one and the same result , we can choose between them. The assumption $x_1 - x = 0$ is unnecessary in all cases, and is hence also an arbitrary operation. That apart, operating further upon II), starting from its left hand side, since on the right hand side there is "nothing to do", we get :

$$\frac{0}{0} \ or \ \frac{d^2y}{dx^2} = 0.$$

The final conclusion would be $\frac{0}{0} = 0$, i.e., the method by which $\frac{0}{0}$ was obtained, was wrong. At the first step it does not give us anything new, and at the second leads already to nothing[27].

Finally, we know from algebra, that if the right hand sides of two equations are identical, then their left hand sides also must be identical. Hence it follows that,

$$\frac{dy}{dy} = \frac{\Delta y}{\Delta x} \ .$$

But since x, and depending on it y, are both variable magnitudes, so Δx, while remaining a finite difference, may, however, decrease infinitely; in other words, it may *approach* zero as closely as one wishes, i.e., may become *infinitely small* ; so may Δy, which depends upon it. From $\frac{dy}{dx} = \frac{\Delta y}{\Delta x}$ it follows that $\frac{dy}{dx}$ does not in fact designate the extravagant $\frac{0}{0}$, but conversely, it is the ceremonial dress-coat for $\frac{\Delta y}{\Delta x}$, when the latter functions as the ratio of infinitely small differences, i.e., [functions] differently from the usual way of calculating differences.

But the differential $dy = dx$ is, in its turn, bereft of all sense, or rather, has exactly only that much sense, as much we have discovered in both the differential parts, by analysing $\frac{dy}{dx}$. If we take the latter only in the value attached to it [28], then it is possible to perform wonderful operations with the differential, as is shown, for example, by the role of adx in the definition

of the sub-tangent to the parabola, for which an actual entry into the nature of dx and dy is not at all necessary.

4) Before passing over to section III), where a very brief draft outline of the historic course of development of the differential calculus will be given, let us examine one more example of the algebraic method, applied so far. For a clear cut characterization of it, I shall place the concrete function on the left hand side, which is always the side of initiative, since, we write from left to right; hence also the general equation :

$x^m + Px^{m-1} + etc. + Tx + U = 0,$

and not

$0 = x^m + Px^{m-1} + etc. + Tx + U.$

Suppose, that the function y and the independent variable x have been separated and that they are situated in two equations, of which the first presents y as the function of the variable u, and the second – *u as the function of x,* and suppose, that *the symbolic differential coefficient common to both the equations, is required to be found* [29]. Let :

1) $3u^2 = y,$ $\qquad$ $3u_1{}^2 = y_1;$

then

2) $x^3 + ax^2 = u,$ $\qquad$ $x_1{}^3 + ax_1{}^2 = u_1.$

At first, from equation 1) we get :

$$3u_1{}^2 - 3u^2 = y_1 - y,$$
$$3\,(u_1{}^2 - u^2) = y_1 - y.$$
$$3\,(u_1 + u)\,(u_1 - u) = y_1 - y,$$
$$3\,(u_1 + u) = \frac{y_1 - y}{u_1 - u} \;\; or \;\; \frac{\Delta y}{\Delta u}.$$

If now we put in the left hand side $u_1 = u$, so that $u_1 - u = 0$, then

$$3\,(u + u) = \frac{dy}{du},$$
$$3\,(2u) = \frac{dy}{du},$$
$$6u = \frac{dy}{du}.$$

Let us insert in place of u its value $x^3 + ax^2$, then :

3) $6\,(x^3 + ax^2) = \dfrac{dy}{du}.$

Now let us turn to equation 2), then :

$x_1^3 + ax_1^2 - x^3 - ax^2 = u_1 - u,$

$(x_1^3 - x^3) + a\,(x_1^2 - x^2) = u_1 - u,$

$(x_1 - x)\,(x_1^2 + x_1x + x^2) + a\,(x_1 - x)\,(x_1 + x) = u_1 - u,$

$$(x_1^2 + x_1 x + x^2) + a\,(x_1 + x) = \frac{u_1 - u}{x_1 - x} \text{ or } \frac{\Delta u}{\Delta x}.$$

Suppose on the left hand side of the equation $x_1 = x$, then $x_1 - x = 0$, and hence

$$(x^2 + xx + x^2) + a\,(x + x) = \frac{du}{dx}.$$

4) $3x^2 + 2ax = \dfrac{du}{dx}.$

Now if we multiply the equations 3) and 4), then:

5) $6\,(x^3 + ax^2)\,(3x^2 + 2ax) = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{dy}{dx}$[30].

Thus, the operational formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

has been found algebraically. It is at times suitable also for the equations with two independent variables.

It will be obvious from the following example, that now no sleight of hand is necessary for giving a general shape to the developments observed in the concrete functions.

Let:

1) $y = f(u)$, $\quad y_1 = f(u_1)$, $\quad y_1 - y = f(u_1) - f(u)$,

then hence

2) $u = \varphi(x)$, $\quad u_1 = \varphi(x_1)$, $\quad u_1 - u = \varphi(x_1) - \varphi(x)$.

From the difference sub 1) it follows that

$$\frac{y_1 - y}{u_1 - u} = \frac{f(u_1) - f(u)}{u_1 - u}, \quad \frac{dy}{du} = \frac{df(u)}{du}.$$

But since $df(u) = f'(u)\,du$, therefore

$$\frac{dy}{du} = \frac{f'(u)\,du}{du};$$

hence:

3) $\dfrac{dy}{du} = f'(u)$.

From the difference sub 2) it follows that

$$\frac{u_1 - u}{x_1 - x} = \frac{\varphi(x_1) - \varphi(x)}{x_1 - x}, \quad \frac{du}{dx} = \frac{d\,\varphi(x)}{dx},$$

but since $d\varphi(x) = \varphi'(x)\,dx$, therefore

$$\frac{du}{dx} = \frac{\varphi'(x)\,dx}{dx},$$

hence :

4) $\frac{du}{dx} = \varphi'(x).$

Let us multiply the equations 3) and 4), then :

5) $\frac{dy}{du} \cdot \frac{du}{dx}$ *or* $\frac{dy}{dx}$[31] $= f'(u) . \varphi'(x)$, and

this is what was required to be demonstrated.

N.III The end of this second installment will follow after I look over John Landen in the Museum[32].

THE DRAFTS OF AND ADDITIONS TO "ON THE DIFFERENTIAL"[33]

*FIRST DRAFT[34].

As soon as we set about differentiating $f(u, z)$ [$= uz$], where the variables u and z are both functions of x, we get — as distinct from the earlier instances, where there was only one *dependent variable*, namely y — the differential expression on both sides, namely :

in the first instance :

$$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx} ;$$

in the second, briefly :

$$dy = zdu + udz .$$

The latter has not yet attained that form, which is obtained in the case with one dependent variable ; for example, in $dy = max^{m-1}dx$. Since here $\frac{dy}{dx}$ at once gives us $f'(x) = max^{m-1}$, which is free from differential symbols. Such a form has no place in $dy = zdu + udz$. In the equations with one dependent variable we have seen once and for all, how the functions derived from [the functions in] x — as in the above mentioned instance of x^m — are obtained through actual differentiation [supposition of difference] and its subsequent removal and, how along with this emerges the symbolic equivalent for the derived function $\frac{0}{0} = \frac{dy}{dx}$. Here the substitution of $\frac{dy}{dx}$ in place of $\frac{0}{0}$ is not only permissible, but also inevitable, since in its virgin primitive form proper $\frac{0}{0}$ is equal to any magnitude, in so far as $\frac{0}{0} = X$ must always give $0 = 0$. However, here $\frac{0}{0}$ appears as equal to certain entirely defined particular value, as equal to mx^{m-1}, and by itself, it is the symbolic result of those operations, by means of which this value is deduced from x^m. And it is presented in $\frac{dy}{dx}$ in the capacity of such a result. Consequently, here $\frac{dy}{dx}(= \frac{0}{0})$ has been shown in its emergence as the symbolic value or, as the differential expression of the already reduced [derived] $f'(x)$, and not conversely, not as $f'(x)$ obtained through the symbol $\frac{dy}{dx}$.

But at the same time having once obtained this result, and having thus found oneself already on the grounds of differential calculus, conversely, if for example,

$$x^m = f(x) = y$$

is required to be differentiated, then we know beforehand that

$$dy = mx^{m-1}dx$$

or,

$$\frac{dy}{dx} = mx^{m-1}.$$

Consequently, here we proceed from the symbol; it appears as nothing more than the result of a deduction from the function of x, but it is already a *symbolic expression*[35], indicating which operations are to be carried out upon $f(x)$ for obtaining the real value of $\frac{dy}{dx}$, i.e., of $f'(x)$. In the first instance, $\frac{0}{0}$ or $\frac{dy}{dx}$ is obtained as the symbolic equivalent of $f'(x)$, and in order to reveal the emergence of $\frac{dy}{dx}$, it is essential to begin with this; in the second instance, $f'(x)$ is obtained as the real value of the symbol $\frac{dy}{dx}$. However, since the symbols $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc. became operational formulas of the differential calculus[36], they can also appear in the *right hand side of the equation*, in the capacity of such formulae, as has already happened in the simplest instance of $dy = f'(x)\,dx$. If, as distinct from the fact that such an equation appears in the given instance in its final from, it does not at once give us $\frac{dy}{dx} = f'(x)$ etc., then this signifies that, it is an equation which only symbolically expresses : which operations are yet to be carried out upon *definite functions*.

This is just what takes place also in the simplest instance — $d\,(uz)$, where u and z are both variables, but, at the same time, these [u and z] are functions of one and the same third variable, for example of x[37].

Let us assume, that $f(x)$ or $y = uz$ is to be differentiated, where u and z *are both variables dependent on x.*

Then

$$y_1 = u_1\,z_1$$

and,

$$y_1 - y = u_1\,z_1 - u\,z.$$

Hence,

$$\frac{y_1 - y}{x_1 - x} = \frac{u_1\,z_1}{x_1 - x} - \frac{uz}{x_1 - x},$$

or,

$$\frac{\Delta y}{\Delta x} = \frac{u_1\,z_1 - u\,z}{x_1 - x}.$$

But

$$u_1\,z_1 - u\,z = z_1\,(u_1 - u) + u\,(z_1 - z),$$

since this is equal to

$$z_1 u_1 - z_1 u + u z_1 - u z = z_1 u_1 - u z \ .$$

Thus

$$\frac{u_1 z_1 - u z}{x_1 - x} = z_1 \frac{u_1 - u}{x_1 - x} + u \frac{z_1 - z}{x_1 - x} \ .$$

If, on the right hand side,

$x_1 - x$ becomes $= 0$, or x_1 becomes $= x$, then

$u_1 - u = 0$, i.e., $u_1 = u$ and, $z_1 - z = 0$,

i.e., $z_1 = z$; from this we get

$$\frac{dy}{dx} = z \frac{du}{dx} + u \frac{dz}{dx}$$

and hence

$$d\,(uz) \text{ or } dy = zdu + udz.$$

In respect of this differentiation of uz it is necessary to note that, as distinct from our earlier instances, where we had *only one dependent variable*, here the differential symbols are found, at once on both the sides of the equation, namely ;

in the first instance :

$$\frac{dy}{dx} = z \frac{du}{dx} + u \frac{dz}{dx} ;$$

in the second :

$$d\,(uz) \text{ or } dy = zdu + udz,$$

which also has a form different from the one obtained in the case when there was one independent variable, as, for example, when we had $dy = f'(x)\,dx$; since here division by dx at once gives us $\frac{dy}{dx} = f'(x)$: as a special expression for the derivative of the function of x, this $f'(x)$ is free from symbolic co-efficients ; it has no place in $dy = zdu + udz$.

It has been shown for the functions *with only one dependent variable* : how from a certain function of x , for example, from $f(x) = x^m$, a second function of x, $f'(x)$ or, in the given instance mx^{m-1}, is deduced, *through authentic differentiation and its subsequent removal* and how from this very process, at the same time emerges the symbolic equivalent $\frac{0}{0} = \frac{dy}{dx}$, in the left hand side of the equation, for the derived function.

Further, here the supposition of $\frac{0}{0} = \frac{dy}{dx}$ was not only permissible, but mathematically necessary, since in its virgin primitive from proper $\frac{0}{0}$ can assume any numerical value, since $\frac{0}{0} = X$ always necessarily gives $0 = 0$. Here $\frac{0}{0}$ appears as the symbolic equivalent of some

entirely determinate real value, for example, of mx^{m-1} above ; and it itself appears only as the result of the operations by means of which this value was deduced from x^m; as such a result, it is consolidated in the form $\frac{dy}{dx}$.

Consequently, here, where $\frac{dy}{dx}\left(=\frac{0}{0}\right)$ has been shown in its emergence, $f'(x)$ is not obtained through the symbol $\frac{dy}{dx}$, but, conversely, the differential expression $\frac{dy}{dx}$ is obtained as the symbolic equivalent of the already derived function of x.

But as soon as this result is obtained, we can operate in the reverse direction. If some $f(x)$, for example x^m, is required to be differentiated, then first of all we seek the value of dy and find that $dy = mx^{m-1}\,dx$, whence $\frac{dy}{dx} = mx^{m-1}$.

Here the symbolic expression figures as the starting point and [we] are already operating on the proper grounds of differential calculus, in other words, $\frac{dy}{dx}$ etc. already serve us as *formulae*, indicating :

to which of the differential operations, already known to us the function of x is to be subjected.

In the first instance $\frac{dy}{dx}\left(=\frac{0}{0}\right)$ was obtained as the symbolic equivalent for $f'(x)$, in the second — $f'(x)$ is sought and found as the real value of the symbols $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc.

But if these symbols already serve as operational formulae of the differential calculus, then, as such, they can also appear in the right hand side of the equation, as it has already happened in the simplest instance of $dy = f'(x)\,dx$. If a similar equation, in its final form, can not at once be reduced into $\frac{dy}{dx} = f'(x)$ etc., i.e., into some real value, as in the simplest instance, then this signifies that the given equation only symbolically expresses : which operations are to be carried out, when *determinate functions* occupy the place of the indeterminate [signs of the functions].

The simplest instance, where this happens, is $d\,(uz)$, where u and z are both variables, but both are at the same time functions of one and the same third variable, for example, of x.

If the process of differentiation at once leads us to

$$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx},$$

(see its origin in note book I, repeated in p.10 of this note book) [PV,43], then it should not be forgotten, that here u and z are both *variables dependent on x*, as is y, which depends on x

only because it depends on z and u. In the case of *one* dependent variable we have the latter in the symbolic side. But now we have two variables u and z in the right hand side, both independent of y, but both *dependent* on x, and their character, [as] variables dependent on x, stands out in the symbolic co-efficients corresponding to them $\frac{du}{dx}, \frac{dz}{dx}$.

If dependent variables appear also in the right hand side then, that is why, they must also of necessity stand out as symbolic differential co-efficients in that side.

From the equation

$$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$$

it follows that :

$$d\,(uz) \text{ or } dy = zdu + udz.$$

But this equation only indicates the operations which are required to be carried out, as soon as u and z are given as determinate functions of x.

For example, had the simplest instance been

$$u = ax,\ z = bx,$$

then

$d\,(uz)$ or $dy = bx{\cdot}adx + ax{\cdot}bdx.$

Dividing both sides by dx we get :

$$\frac{dy}{dx} = abx + bax = 2abx$$

and

$$\frac{d^2y}{dx^2} = ab + ba = 2ab.$$

Had we taken from the very beginning the product

y or $uz = ax{\cdot}bx = abx^2$

then

uz or $y = abx^2$, $\frac{dy}{dx} = 2abx$, $\frac{d^2y}{dx^2} = 2ab.$

As soon as a formula, for example like $[w =]\, z_1 \frac{du}{dx}$ is obtained, it becomes clear, that this equation — which may be called a general operational equation — is the symbolic expression of the differential operations to be carried out. If, for example, we take the expression $y\frac{dx}{dy}$, where y is the ordinate, and x is the abscissa, then this is the general symbolic expression for the subtangent to any curve (just as $d\,(uz) = zdu + udz$ is the same for the differentiation of the product of any two variables, dependent upon one and the same third variable).

But so long as we keep this expression as it is, it does not give us anything more, though we may visualize, that dx is the differential of abscissa, and dy — the differential of ordinate.

In order to obtain some positive result we must first of all take an equation of some determinate curve, which would give us a determinate value for y in x, and that is why also for dx, such as, for example, the equation of ordinary parabola : $y^2 = ax$. Differentiating the latter, we get $2ydy = adx$;

hence, $dx = \frac{2ydy}{a}$. Substituting this determinate value of dx in the general formula of subtangent $y\frac{dx}{dy}$ we get

$$\frac{y\frac{2ydy}{a}}{dy} = \frac{y \cdot 2ydy}{ady} = \frac{2y^2}{a}.$$

and since $y^2 = ax$ [this]

$$= \frac{2ax}{a} = 2x,$$

which is the value of the *subtangent* to the ordinary porabola. Thus, this value is equal to twice that of the abscissa.

But if we designate the subtangent by τ, *then the general equation* $y\frac{dx}{dy} = \tau$ *merely gives us* $ydx = \tau dy$. *That is why from the point of view of differential calculus(with the exception of Lagrange) most often the question was : [how] to find out the real value of* $\frac{dy}{dx}$.

It may be shown, that this difficulty already reveals itself, if we substitute in place of $\frac{dy}{dx}$ etc. their primary from $\frac{0}{0}$; then

$$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$$

takes the form of

$$\frac{0}{0} = z \cdot \frac{0}{0} + u \cdot \frac{0}{0}.$$

This is a correct equation. But it leads us nowhere further than [telling] that, these three $\frac{0}{0}$s emerged from different differential co-efficients, and of the differences of their emergence nothing remained. However, it should be remembered that :

1) in the instance of one independent variable, already in the first exposition, at first we obtained

$$\frac{0}{0} \text{ or } \frac{dy}{dx} = f'(x), \text{ hence, } dy = f'(x)\,dx.$$

But, since

$$\frac{dy}{dx} = \frac{0}{0}, \; dy = 0 \text{ and } dx = 0,$$

hence,

$$0 = 0.$$

However, by transforming $\frac{dy}{dx}$ conversely into its indeterminate expression $\frac{0}{0}$, here we commit a positive mistake, since here $\frac{0}{0}$ is found merely as the symbolic equivalent of the real value of $f'(x)$ and is as such consolidated in the expression $\frac{dy}{dx}$, hence, also in $dy = f'(x)\,dx$.

2) $\frac{u_1 - u}{x_1 - x}$ turns into $\frac{du}{dx}$ or into $\frac{0}{0}$, because x_1 becomes equal to x, or $x_1 - x = 0$, thus, for $\frac{u_1 - u}{x_1 - x}$ what we at once get is not 0, but $\frac{0}{0}$. But generally speaking we know, that $\frac{0}{0}$ may assume any value and that in determinate instances it attains a special value, obtainable if in place of u some determinate function of x is put; that is to say, we not only have the right to transform $\frac{0}{0}$ through $\frac{du}{dx}$, but this must be done, since both $\frac{du}{dx}$ and $\frac{dz}{dx}$ figure in the given instance only as symbols of the differential operations to be carried out. [But] so long as we do not proceed further than the result

$$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx},$$

hence,

$$dy = zdu + udz,$$

these $\frac{du}{dx}, \frac{dz}{dx}$, du and dz remain as inderterminate as is the $\frac{0}{0}$, which can assume any value.

3) Even in ordinary algebra $\frac{0}{0}$ can appear as a form of the expressions having a certain real value, just because $\frac{0}{0}$ can be the symbol of any magnitude. For example, assume that $\frac{x^2 - a^2}{x - a}$ is given. Let us put, $x = a$, then $x - a = 0$ and $x^2 = a^2$, that is why $x^2 - a^2 = 0$. Hence, we get,

$$\frac{x^2 - a^2}{x - a} = \frac{0}{0};$$

so far, the result is correct, but though $\frac{0}{0}$ can also have any value, it would be wrong to assert on that ground that, $\frac{x^2 - a^2}{x - a}$ has no real value.

Having factorised $x^2 - a^2$ into its factors, we get $(x + a)(x - a)$; that is

$$\frac{x^2 - a^2}{x - a} = (x + a) \cdot \frac{x - a}{x - a} = x + a \ ;$$

hence, if $x - a = 0$, then $x = a$, and that is why $x + a = a + a = 2a$ [38].

If we had a term of the form $P(x - a)$ in an ordinary algebraic equation, then, when $x = a$, i.e., $x - a = 0$, we would necessarily have $P(x - a) = P \cdot 0 = 0$; and also subject to the same presuppositions $P(x^2 - a^2) = 0$. Factorisation of $x^2 - a^2$ into the factors $(x + a)(x - a)$ would not introduce any change into this, for

$$P(x + a)(x - a) = P(x + a) \cdot 0 = 0$$

However, from this it does not at all follow that, if by supposing $x = a$ we obtain a term of the type $P \cdot \left(\frac{0}{0}\right)$, then its value is necessarily equal to zero.

$\frac{0}{0}$ can assume any value, since $\frac{0}{0} = X$ always gives us $0 = X \cdot 0 = 0$; but just because $\frac{0}{0}$ can take any value, it should not necessarily be equal to zero, and if we are aware of its emergence then as soon as some real value hides behind it, the latter can also be found out.

Thus, for example in $P \cdot \frac{x^2 - a^2}{x - a}$; if $x = a$, $x - a = 0$ and, hence, also $x^2 = a^2$, $x^2 - a^2 = 0$, then

$$P \cdot \frac{x^2 - a^2}{x - a} = P \cdot \frac{0}{0} .$$

Though this result too was obtained in a manner which is mathematically fully correct, it would, however, be mathematically no less wrong to assume without further [preconditions] that, $P \cdot \frac{0}{0} = 0$, since from this supposition it would follow, that $\frac{0}{0}$ is unconditionally incapable of assuming any value other than zero, and that, hence,

$$P \cdot \frac{0}{0} = P \cdot 0.$$

Further, it would be necessary to investigate , whether or not any other result may be obtained by factorising $x^2 - a^2$ into its factors $(x+a)$, $(x - a)$; actually this factorisation turns the given expression into

$$P \cdot (x + a) \cdot \frac{x - a}{x - a} = P \cdot (x + a) \cdot 1,$$

and [if] $x = a$, then also into $P \cdot 2a$ or into $2Pa$. More so, when we operate with the variables[39] ; then it is not only right, but also absolutely necessary to consolidate the emergence of $\frac{0}{0}$ through the differential symbols $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc., for we have proved beforehand that, they emerge as symbolic equivalents of the functions derived from the variables subjected to the determinate processes of differentiation. Thus, if primarily they are results of the elapsed processes of differentiation, then, just by virtue of that, they can also play an *inverse* role — that of the symbols of those operations, to which the variables are but supposed to be

subjected, i.e., [the role] of *operational symbols*, figuring by now not as results, but as points of departure — and herein lies their essential role in the differential calculus. In the capacity of similar operational symbols they themselves can become the content of the equations involving the different variables (in the case of implicit functions, from the very beginning, at the right hand side [of the equation] stands 0, and all the dependent and independent variables, along with their co-efficients, are situated on the left).

That is how the matter stands in the equation

$$\frac{d\,(uz)}{dx} \text{ or } \frac{dy}{dx} = \frac{zdu}{dx} + \frac{udz}{dx}.$$

When disengaged from what has been said earlier, the functions z and u, dependent upon x, themselves appear here again as invariables, but each of them is endowed with the capacity of being the multiplier of the symbolic differential co-efficient of the other.

Hence this equation has only the significance of some general equation, indicating through symbols : which operations are to be carried out, if u and z are correspondingly given as dependent variables of two determinate functions of x.

Only when u and z are [some] determinate functions of [x] may the expressions $\frac{du}{dx}\left(=\frac{0}{0}\right)$ and $\frac{dz}{dx}\left(=\frac{0}{0}\right)$ and, hence, also $\frac{dy}{dx}\left(=\frac{0}{0}\right)$ turn into 0, i.e., the value $\frac{0}{0}=0$ is not anticipated beforehand but must itself appear as the consequence of determinate equations expressing functional dependence.

If, for example, $u = x^3 + ax^2$, then

$$\frac{0}{0} = \frac{du}{dx} = 3x^2 + 2ax, \qquad \left(\frac{0}{0}\right)_2 = \frac{d^3u}{dx^3} = 6,$$

$$\left(\frac{0}{0}\right)_1 = \frac{d^2y}{dx^2} = 6x + 2a, \qquad \left(\frac{0}{0}\right)_3 = \frac{d^4u}{dx^4} = 0,$$

i.e., in this last instance $\frac{0}{0} = 0$.

The essence of this entire history consists of the following : here, through the very differentiation we get the *differential co-efficients in their symbolic form as results*, as values [of the symbol] $\frac{dy}{dx}$ in the differential equation, namely, in the equation :

$$\frac{d\,(uz)}{dx} \text{ or } \frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}.$$

But we know that u = some determinate function of x, *for example* $f(x)$. That is why $\frac{u_1 - u}{x_1 - x}$, in its differential symbol $\frac{du}{dx}$, is equal to $f'(x)$, i.e., the first derived function of $f(x)$. Exactly in

the same way $z = \varphi(x)$, and that is why likewise $\frac{dz}{dx} = \varphi'(x)$, i.e., the first derived function of $\varphi(x)$. But the initial equation itself gives us neither u, nor z in the from of a determinate function of x, as, for example, would be, $u = x^m$, $z = \sqrt{x}$.

It gives u and z only as general expressions for any two functions of x, whose product is to be differentiated.

This equation says that if the product of any two functions of x represented by the expression uz is required to be differentiated, then at first the corresponding real value of the symbolic differential co-efficient $\frac{du}{dx}$, i.e., let us say, the first derived function of $f'(x)$ is to be found out, and this value is to be multiplied by $\varphi(x) = z$, there upon in the same way, the real value of $\frac{dz}{dx}$ is to be found out and multiplied by $f(x) = u$; finally, the two products so obtained are to be added. Here the operations of differential calculus are assumed to be already known.

Thus, the given equation only symbolically indicates the operations to be carried out, and along with this the symbolic differential co-efficients $\frac{du}{dx}, \frac{dz}{dx}$ become here the symbols of those differential operations, which remain still to be carried out in each concrete instance, whereas initially they themselves were deduced as symbolic formulae of the differential operations already carried out.

As soon as they attain such a character, they themselves can become the content of differential equations, as for example in *Taylor's Theorem* :

$$y_1 = y + \frac{dy}{dx} h + \text{etc.}$$

But even in that case, it is, anyway, also merely a general symbolic operational equation. Differentiation of uz is of interest, because it is the simplest instance, wherein — as distinct from the development of such instances, where the independent variable x has only one dependent variable y — the very application of the original method leads to the emergence of differential symbols, also in the right hand side of the equation (its developing expression), that is why here they appear at the same time as operational symbols and as such become the content of the equation itself.

This role, in which they indicate the operations to be carried out and that is why serve as the starting point, is their characteristic role, already operative in the ground proper of differential calculus; but there is no doubt that, none of the mathematicians paid any attention to this turning point, to this role inversion, and what is more, none of them proved its necessity in any absolutely elementary differential equation. Only this is mentioned as a fact, that while the inventors of differential calculus and the majority of their followers make differential symbols the point of departure of calculus, Lagrange conversely takes algebraic deduction of the real[40] functions of the independent variables as his starting point, and makes the differential symbols purely symbolic expressions of the derived functions.

Returning once more to $d\,(uz)$ we get at first, as a result of the supposition $x_1 - x = 0$, as a result of the differential operation itself:

$$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}.$$

Since here the denominators are one and the same, we get

$$dy = zdu + udz$$

as the reduced expression. This corresponds to the fact that, in the case of only one dependent variable on the left hand side we had $\frac{dy}{dx}$, and as its symbolic expression we had $\frac{dy}{dx} = f'(x)$, in the capacity of the symbolic expression for the derived function of x, i.e., for $f'(x)$ (for example, for max^{m-1}, which is the $f'(x)$, if $ax^m = f(x)$) and hence, only as the result [we had]

$$dy = f'(x)\,dx$$

(for example, $\frac{dy}{dx} = max^{m-1}$, $dy = max^{m-1}\,dx$, which is the differential of the function y) (the latter, we can at once conversely turn into $\frac{dy}{dx} = max^{m-1}$). But the instance

$$dy = zdu + udz$$

is different further owing to the fact that, the differentials du, dz here stand on the right hand side, as operational symbols, and dy is determined only after the operations indicated by them are completed.

If

$$u = f(x) \text{ and } z = \varphi(x),$$

then we know that for du we get

$$du = f'(x)\,dx$$

and [for dz]

$$dz = \varphi'(x)\,dx.$$

Hence,

$$dy = \varphi(x)\,f'(x)\,dx + f(x)\,\varphi'(x)\,dx$$

and

$$\frac{dy}{dx} = \varphi(x)\,f'(x) + f(x)\,\varphi'(x).$$

Thus, in the first instance, at first the differential co-efficient

$$\frac{dy}{dx} = f'(x) \text{ was obtained,}$$

and there upon the differential

$$dy = f'(x)\,dx.$$

In the second — at first the differential, and there upon the differential co-efficient $\frac{dy}{dx}$. In the first instance, where the differential symbols themselves appear only out of the

operations carried out upon $f(x)$, there at first the derived function — the real differential co-efficient, is to be found out, so that opposite it appeared $\frac{dy}{dx}$, as its symbolic expression, and only after it has been found out, can the differential

$$dy = f'(x)\,dx$$

be deduced.

Conversely for $dy = zdu + udz$.

Since here du, dz figure as operational symbols, and besides indicate those operations which we have already learnt to carry out in the differential calculus, so, for searching the real value of $\frac{dy}{dx}$ we must first of all change u and z into their values in x, in every concrete instance, in order to find out

$$dy = \varphi(x)f'(x)\,dx + f(x)\varphi'(x)\,dx,$$

and only further division by dx gives us the real value of

$$\frac{dy}{dx} = \varphi(x)\,f'(x) + f(x)\,\varphi'(x)\,.$$

The same holds for

$$\frac{du}{dx},\ \frac{dz}{dx},\ \frac{dy}{dx},\ \frac{d^2y}{dx^2}$$

etc. and for all the more complex formulae, where the *differential symbols* themselves appear as the content of the general symbolic operational equations.

*SECOND DRAFT [41]

[I]

. . . .

We began with the algebraic deduction of $f'(x)$, to reveal the emergence of its symbolic differential expression $\frac{0}{0}$ *or* $\frac{dy}{dx}$, and to thus lay bare its meaning at the same time. Conversely, now we must proceed from the symbolic differential co-efficients $\frac{du}{dx}$ *or* $\frac{dz}{dx}$ taken as given formulae, in order to find out their corresponding real equivalents $f'(x)$, $\varphi'(x)$. And besides these different modes of interpreting the differential calculus, emanating from different poles and, producing two different historical schools, do not emerge here from changes in our subjective method, but are produced by the nature of the function uz under consideration. We treated it just the same as we did the functions of x with only one independent variable, when we proceeded from the right hand pole, and operated upon it algebraically. I don't think that any mathematician has proved or even noted the necessity of this transition from the first (historically, the second) algebraic method. They were too absorbed with the material of the calculus, to do this.

In fact we see that in the equation

$$\frac{0}{0} \text{ or } \frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx},$$

$\frac{dy}{dx}$ emerged from the process of deduction occuring upon uz in the right hand side, absolutely in the same way as it happened earlier, for the functions of x with only one dependent variable; but on the other hand, the differential symbols $\frac{du}{dx}$, $\frac{dz}{dx}$, appear, in their turn, to be included in $f'(x)$ itself or in the first derivative of uz. Owing to this they appear as elements of the equivalent of $\frac{dy}{dx}$. Thus, the symbolic differential co-efficients themselves, in their turn, already became the *subject matter* or *content* of the differential operation, instead of figuring, as before, merely as its symbolic results.

Here we have two moments. *Firstly*, along with the variables themselves, the symbolic differential co-efficients in their turn become contentful elements of the deduction, [they become] the *objects* of differential operations. *Secondly*, formulation of the question so turns round, that instead of seeking the symbolic expression for the real differential co-efficients (for $f'(x)$), the real differential co-efficient is sought for its symbolic expression. Along with these two moments, a third also is given at the same time : namely, the symbolic differential co-efficients no longer appear as symbolic results of the differential operations carried out upon real functions of x, but, conversely [they] now play the role of symbols indicating those differential operations, which must be carried out over the real functions of x, i.e. [they] thus become operational symbols.

In our case, where

$$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx} ,$$

we could have operated further, had we known not only that z and u are both functions of x, but also if, as in the case of $y = x^m$, the real values of u and z were given in x as, for example,

$u = \sqrt{x}$, $z = x^3 + 2ax^2$.

Thus $\frac{du}{dx}$, $\frac{dz}{dx}$ in fact play the role of indicators of the operations, the mode of carrying out which is supposedly well known for all such functions of x, which are substituted in place of u and z.

c) The equation obtained is not simply a symbolic operational equation, but is only a preparatory symbolic operational equation. Since in

[I)] $$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$$

the denominator dx is present in every term in both the sides, the reduced expression for this equation will be :

II) $$dy \text{ or } d(uz) = zdu + udz.$$

This equation immediately says that, if the product of two arbitrary variables are to be differentiated (in further application it may be generalised for the product of any number of variables), then each of the factors is to be multiplied by the differential of the other multiplier and the two products so obtained are to be added.

Thus the first operational equation

$$\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$$

becomes superfluous as a preparatory equation — if the product of two arbitrary variables is to be differentiated — for it has fulfilled its role, namely : to provide the general symbolic operational formula, leading directly to the goal.

And here it should be noted that, the method of initial algebraic deduction again turns into its opposite. There, first of all we obtained $\Delta y = y_1 - y$ as a symbol corresponding to $f(x_1) - f(x)$, where both [$f(x_1)$ and $f(x)$] are ordinary algebraic expressions (since $f(x)$ and $f(x_1)$ were given in the form of determinate algebraic functions of x). Further $\frac{f(x_1) - f(x)}{x_1 - x}$ was presented in the form of $\frac{\Delta y}{\Delta x}$, and subsequently $f'(x)$ *(the first derived function of $f(x)$)* — in the form of $\frac{dy}{dx}$, and only from the final equation for the differential co-efficient $\frac{dy}{dx} = f'(x)$ we obtained the differential

$$dy = f'(x)\,dx.$$

Conversely, the equation obtained above gives us the differentials dy, dz, du as starting points. Namely, if we substitute for u and z some determinate algebraic function of x, which we shall designate only as

$$u = f(x) \quad \text{and} \quad z = \varphi(x),$$

then we get

$$dy = \varphi(x)\, df(x) + f(x)\, d\varphi(x),$$

and these d symbols indicate only the differentiation which is yet to be carried out. Result of this differentiation assumes the general form :

$$df(x) = f'(x)\, dx$$

and

$$d\varphi(x) = \varphi'(x)\, dx.$$

Thus ,

$$dy = \varphi(x) f'(x)\, dx + f(x)\varphi'(x)\, dx.$$

Finally ,

$$\frac{dy}{dx} = \varphi(x) f'(x) + f(x)\varphi'(x).$$

Here, where the differential already plays the role of a ready-made operational symbol, we deduce the differential co-efficients from it, while in the initial algebraic development, conversely, the differential was obtained from the equation for the differential co-efficients.

Let us consider the very *differential*, as obtained in its simplest form, namely, as obtained from a function of the first power ;

$$y = ax, \quad \frac{dy}{dx} = a\,;$$

whence the differential

$$dy = adx.$$

This equation, connecting these two differentials appears to be much more dubious, than the equation for the differential co-efficient

$$\frac{0}{0} \text{ or } \frac{dy}{dx} = a,$$

from which it was deduced.

Since $dy = 0$ and $dx = 0$, $dy = adx$ is identical with $0 = 0$.

But, nevertheless, we have the full right to use dy and dx in place of the extinct — but fixated in their act of extinction with the help of these symbols — differences $y_1 - y$, $x_1 - x$.

So long as we do not proceed further than the expression

$$dy = adx$$

or, generally,

$$dy = f'(x)\, dx,$$

it is nothing but a mere record in another form of the fact that

$$\frac{dy}{dx} = f'(x),$$

which $= a$ in our case, owing to which we always have the opportunity, to again transform it into this latter form. But this possibility of transformation already makes it an operational symbol. We see at once that, if we find $dy = f'(x)\, dx$ as a result of the processes of differentiation, then we should only divide both the sides by dx to find out $\frac{dy}{dx} = f'(x)$, i.e., the differential co-efficients.

Thus, for example, in $y^2 = ax$,

$$d(y^2) = d(ax), \qquad 2ydy = adx.$$

This latter equation for the differentials gives us two equations for differential co-efficients namely :

$$\frac{dy}{dx} = \frac{a}{2y} \quad \text{and} \quad \frac{dx}{dy} = \frac{2y}{a}.$$

But $2ydy = adx$ gives us, and also, *immediately*, the value $\frac{2y\,dy}{a}$ for dx, which, for example, having been put into the general formula for subtangent $y\,\frac{dx}{dy}$, helps us lastly to obtain $2x$, the abscissa doubled, as the value of the subtangent to the ordinary parabola.

II

Now let us take an example in which the symbolic expressions serve as ready-made operational formulae of the calculus from the very beginning and, consequently, the real value of the symbolic differential co-efficient is sought; and after that we shall give the opposite elementary algebraic exposition.

1) Let the dependent function y and the independent variable x be connected not by the one and the only equation, but let them be so connected that y figures immediately as a function of the variable u in some first equation, and u immediately figures as a function of the variable x in some second equation. The task is : to *find out the real value of the symbolic differential co-efficient* $\frac{dy}{dx}$.

Let

a) $\quad y = f(u), \qquad$ b) $\quad u = \varphi(x).$

At first 1) $y = f(u)$ gives :

$$\frac{dy}{du} = \frac{df(u)}{du} = \frac{f'(u)\,du}{du} = f'(u).$$

2) $$\frac{du}{dx} = \frac{d\varphi(x)}{dx} = \frac{\varphi'(x)\,dx}{dx} = \varphi'(x).$$

Consequently ,

$$\frac{dy}{du}\cdot\frac{du}{dx} = f'(u)\cdot\varphi'(x).$$

But

$$\frac{dy}{du}\cdot\frac{du}{dx} = \frac{dy}{dx};$$

consequently

$$\frac{dy}{dx} = f'(u)\cdot\varphi'(x).$$

Example. If a) $y = 3u^2$, $\qquad$ b) $u = x^3 + ax^2$,

then according to the formula

$$\frac{dy}{du} = \frac{d\,(3u^2)}{du} = 6u\,(= f'(u)\ );$$

but the equation b) gives [the value] $u = x^3 + ax^2$. If we put this value of u in $6u$, then

$$\frac{dy}{du} = 6(x^3 + ax^2)\ \ (= f'(u)\,).$$

Further,

$$\frac{du}{dx} = 3x^2 + 2ax\,(= \varphi\,'(x)\,).$$

Consequently,

$$\frac{dy}{du}\cdot\frac{du}{dx}\ \text{or}\ \frac{dy}{dx} = 6(x^3 + ax^2)\ \ (\,3x^2 + 2ax\,)\,(= f'(u)\cdot\varphi\,'(x)\,).$$

2) Now let us take as our initial equations, those which are contained in the last example, for developing them according to the first, algebraic mode.

a) $y = 3u^2$, b) $u = x^3 + ax^2$.

Since $y = 3u^2$, $y_1 = 3u_1{}^2$ and $y_1 - y = 3(u_1{}^2 - u^2) = 3\,(u_1 - u)\,(u_1 + u)$.

Hence,

$$\frac{y_1 - y}{u_1 - u} = 3\,(u_1 + u).$$

If now it is supposed that $u_1 - u = 0$ and consequently, $u_1 = u$, then $3\,(u_1 + u)$ turns into $3\,(u + u) = 6u$.

Let us put in place of u its value from the equation b) ; then

$$\frac{dy}{du} = 6(x^3 + ax^2).$$

Further, since

$$u = x^3 + ax^2,\ \ u_1 = x_1{}^3 + ax_1{}^2\,;$$

consequently,

$$u_1 - u = (x_1{}^3 + ax_1{}^2) - (x^3 + ax^2) = (x_1{}^3 - x^3) - a(x_1{}^2 - x^2),$$

$$u_1 - u = (x_1 - x)\,(x_1{}^2 + x_1 x + x^2) + a(x_1 - x)\,(x_1 + x);$$

hence,

$$\frac{u_1 - u}{x_1 - x} = (x_1{}^2 + x_1 x + x^2) + a(x_1 + x)\,.$$

If now it is supposed that $x_1 - x = 0$, consequently, $x_1 = x$, then $x_1{}^2 + x_1 x + x^2 = 3x^2$

and,

$$a(x_1 + x) = 2ax\,.$$

Consequently, $\dfrac{du}{dx} = 3x^2 + 2ax.$

Now multiplying both the functions of the right hand side, we shall get

$6(x^3 + ax^2)(3x^2 + 2ax)$,

to which corresponds on the left hand side

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx},$$

i.e., the same as before.

We shall put the determinate functions of the variable on the left hand side, and the functions dependent upon it on the right, so that the difference in the modes of deduction comes out more clearly, since, thanks to the general equation, where on the right hand side stands only zero, we have become accustomed to consider that the initiative lies on the left hand side . Consequently

a) $3u^2 = y$, b) $x^3 + ax^2 = u$.

Since

$$3u^2 = y \quad , \qquad 3u_1{}^2 = y_1 ,$$

then,

$$3(u_1{}^2 - u^2) = y_1 - y ,$$

or,

$$3(u_1 - u)(u_1 + u) = y_1 - y ,$$

hence,

$$3(u_1 + u) = \frac{y_1 - y}{(u_1 - u)} .$$

If now $u_1 = u$ and, consequently $u_1 - u = 0$, then we get

$$3(u + u) \text{ or } 6u = \frac{dy}{du} .$$

Let us put in 6u the value of u from the equation b), then

$$6(x^3 + ax^2) = \frac{dy}{du}$$

Further if

$$x^3 + ax^2 = u,$$

then

$$x_1{}^3 + ax_1{}^2 = u_1,$$

and

$$x_1{}^3 + ax_1{}^2 - x^3 - ax^2 = u_1 - u ;$$

hence

$$(x_1{}^3 - x^3) + a(x_1{}^2 - x^2) = u_1 - u .$$

Expanding into factors :

$$(x_1 - x)(x_1{}^2 + x_1x + x^{2)} + a(x_1 - x)(x_1 + x) = u_1 - u .$$

Hence,

$$(x_1{}^2 + x_1x + x^2) + a(x_1 + x) = \frac{u_1 - u}{x_1 - x},$$

when $x_1 = x$ and consequently, $x_1 - x = 0$ then

$$3x^2 + 2ax = \frac{du}{dx}.$$

Multiplying the two derived functions we shall get :

$$6(x^3 + ax^2)\,(3x^2 + 2ax) = \frac{dy}{dx},$$

and, putting it in the usual order ,

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx} = 6(x^3 + ax^2)\,(3x^2 + 2ax).$$

It stands to reason all by itself, that owing to its cumberousness, and often also due to [its] difficulty, the expansion of the first difference $f(x_1) - f(x)$ into such terms , each of which contains the factor $x_1 - x$,the latter method can not be compared (as an instrument of computation), with the one historically formed long ago.

But on the other hand, in the latter, one may proceed from dy , dx , $\frac{dy}{dx}$ as given operational formulae, whereas in the former their emergence is visible, which is clearly algebraic. I am not asserting anything more than this. How were the starting points for the differential symbols as operational formulae obtained in the [historically] first method ? With the help of secret or evident metaphysical presuppositions, which in their turn lead to metaphysical, non-mathematical consequences : what happens is a forcible destruction of certain magnitudes, blocking the path of deduction, [which], however, where generated by those very presuppositions. In order to show the difference of the methods emanating from opposite poles, in the light of a historical example, I have compared — as instantiated above — the situation of $d(uz)$, as per Newton and Leibnitz on the one hand, with that according to Lagrange, on the other.

1) *Newton* .

First of all, we are told, that if the variables grow, then $\dot{x}, \dot{y}$ etc. designate the speed of their flow, in other words, designate the corresponding growth of the [variables] x,y etc. Further, since the numerical magnitudes of all possible quantities may be represented by straight lines, the generated *moments* or *infinitely small quantities* are equal to the *product* of the speeds $\dot{x}, \dot{y}$ etc. and an infinitely small part of time τ , in course of which they last, consequently, [they are]= $\dot{u}\tau$, $\dot{x}\tau$, $\dot{y}\tau$ [42].

*THIRD DRAFT

If now we consider the differential of y in its general form $dy = f'(x)dx$, then, here we already have in front of us the pure symbolic operational equation even in that case, when $f'(x)$ is a constant from the very beginning as in $dy = d(ax) = adx$. This baby [expression] $\frac{0}{0}$ *or* $\frac{dy}{dx} = f'(x)$ looks more suspicious, than its mother. For in $\frac{dy}{dx} = \frac{0}{0}$ *the denominator and the numerator are inseparably connected* ; in $dy = f'(x)dx$ they are visibly separated, so that the following conclusion thrusts itself : $dy = f'(x)\,dx$ is only a masked expression for $0 = f'(x) \cdot 0$, consequently $0 = 0$, and with this "nothing can be done". More subtle analysts belonging to our century, for example, like the Frenchman Boucharlat, also smelt that here something was wrong. He says : "For example in $\frac{dy}{dx} = 3x^2$, $\frac{0}{0}$, i.e., $\frac{dy}{dx}$ or more rightly, its value $3x^2$ is the differential co-efficient of the function y. Since $\frac{dy}{dx}$ is thus the symbol representing the limit $3x^2$, dx must always stand below dy. But in order to *facilitate algebraic operations*, we consider $\frac{dy}{dx}$ to be an ordinary fraction , and $\frac{dy}{dx} = 3x^2$ — to be an ordinary equation. Freeing it from the denominator dx, we get as the result $dy = 3x^2dx$ — and this expression is called the *differential* of y "[43].

Thus, in order to "facilitate algebraic operations", we introduce a false formula. In reality the case is not like that. In $\frac{0}{0}$ $\left(\text{strictly speaking, we should write } \left(\frac{0}{0}\right)\right)$ the ratio of the minimal expression for $y_1 - y$, i.e., for $f(x_1)-f(x)$ or , for the increment of $f(x)$, and of the minimal expression for $x_1 - x$, i.e., for the increment of the independent variable x, assumes such a form, wherein the numerator is inseparable from the denominator. But why ? So that $\frac{0}{0}$ is preserved as the *ratio* of extinct differences. But as soon as $x_1 - x = 0$ obtains in dx a form, which shows it to be the extinct difference of [the variable] x, and so $y_1 - y = 0$ also comes forth as dy, the separation of the numerator from the denominator becomes an entirely permissible operation. Now wherever dx may be situated, its connection with dy is not affected by such change of place. Consequently, $dy = df(x) = f'(x)\,dx$ is only another expression for $\frac{dy}{dx} = f'(x)$], which must appear at the end, so that a free [from the multiplier dx] $f'(x)$ could be obtained. Incidentally, the following example shows, how far useful this formula $dy = df(x)$ becomes at once, as an *operational formula* :

$$y^2 = ax,$$

$$d\,(y^2) = d\,(ax)\,, \qquad 2y\,dy = a\,dx,$$

hence,

$$dx = \frac{2y\,dy}{a}.$$

Putting this value of dx in the general formula of the subtangent $y\frac{dx}{dy}$, we get

$$\frac{y\,\frac{2y\,dy}{a}}{dy} = \frac{2y^2dy}{a\,dy} = \frac{2y^2}{a}$$

and since

$y^2 = ax$,

$$\left[\text{ so } \frac{2y^2}{a}\right] = \frac{2ax}{a} = 2x\,;$$

thus $2x$, i.e., double the abscissa of the ordinary parabola, is the value of its subtangent. However, if $dy = df(x)$ is considered to be the first starting point, where from only subsequently $\frac{dy}{dx}$ itself is deduced, then, so that this differential of y has some sense, it has *to be supposed* that the differential particles dy, dx are symbols having a definite purport. Had such a presupposition not been generated by mathematical metaphysics, but, for example, if it had to be deduced immediately from some function of the first power like $y = ax$, then, as we saw earlier, it would have led us, to $\frac{y_1 - y}{x_1 - x} = a$, which turns into $\frac{dy}{dx} = a$. But from here nothing definite can be extracted *a priori*. Since $\frac{\Delta y}{\Delta x} = a$, exactly in the same way as $\frac{dy}{dx} = a$, and since it is true that Δx, Δy are terminal differences or increments, but terminal differences or increments with limitless capacity to decrease, so with equal success they may present themselves as dy, dx and as infinitely small magnitudes which are as proximate to zero as possible and, as magnitudes emerging as a result of actual equalisation of zero and $x_1 - x$ and, consequently, of $y_1 - y$ [and zero]. In both the cases, the result on the right hand side remains one and the same, so long as it is not assumed in this side that $x_1 = x$, that is to say, $x_1 - x = 0$. That is why, on the other hand, this equalisation with zero would appear to be as arbitrary an hypothesis, as is the assumption that dx, dy are infinitely small magnitudes. I shall briefly show the historical course of development sub IV), in the light of the example $d(uz)$. However, before that, sub III)[44] I shall give one more example, which will at first be investigated on the basis of a symbolic calculus, with the help of some ready-made operational formulae, and after that it will be presented algebraically. We have already shown upto sub II) that, even when applied to such elementary functions as the product of two variables, the latter method, with the help of its proper results, perforce gives the starting point for a method, which operates from the opposite pole.

Ad IV.

Finally (*according to Lagrange*), it should be noted further that, the *limit* or the *limiting value*, which is already met with sometimes in place of the differential co-efficient of Newton, is still deduced by him from purely geometric representations, and it plays a prominent role till date. Do the symbolic expressions figure therein as the limits of $f'(x)$, or, conversely, does $f'(x)$ figure as the limit of the symbol, or [whatever], do both figure as limits ? This category, which has found wide use in [mathematical] analysis, mainly in that of Lacroix,

acquires an important significance as a substitute for the category of "minimal expression" — either of the derivative in contrast to the "primary derivative", or of the ratio $\frac{y_1 - y}{x_1 - x}$, as soon as the application of calculus to the curves is at issue. It is easier to *represent* it geometrically, and that is why it is to be met with already in [the writings of] the *ancient geometers.*

For some of our contemporaries the limit is still hidden, owing to the fact that the differential particles and differential co-efficients express only approximate values[45].

---

*SOME ADDITIONS [46]

A) additionally on the differentiation of uz [47].

1) In the last manuscript, while developing $d(uz)$, it was essential for me to show in application to the equation

A) $\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$,

that the algebraic method here applied itself turns into the differential method, owing to the fact that it develops inside the derivative, i.e., on the right hand side [we have] the *symbolic differential co-efficients* without the corresponding equivalent real co-efficients ; along with this, these symbols, as such, become *independent starting points*, and are given in the ready-made form of *operational formulae.*

For this purpose the form of the equation A) turned out to be even more suitable, as it permits a comparison of [the expressions] $\frac{du}{dx}$, $\frac{dz}{dx}$ obtained within the derivative $f'(x)$, with $\frac{dy}{dx}$ standing opposite [them] on the left hand side, which is the symbolic differential co-efficient for $f'(x)$, and that is why constitutes its symbolic, equivalent.

Concerning the character of $\frac{du}{dx}$, $\frac{dz}{dx}$ as operational formulae, I have limited myself to indicating, that for these symbolic differential co-efficients any "derivative" may be found out as their real value, if in place of u any $f(x)$, for example, $3x^2$, is put, and in place of z — any $\varphi(x)$, for example , $x^3 + ax^2$.

However, I could have also indicated the geometric application of these operational formulae, since, for example, $y\frac{dx}{dy}$ serves as the *general formula for subtangents to curves,* which is, in form, entirely identical with $z\frac{du}{dx}$, $u\frac{dz}{dx}$, since all of them are products of a variable and a symbolic differential co-efficient.

And finally, it may be noted further, that $y = uz$ is the *simplest elementary function* (here y is y^1 and uz is the simplest form of the second power), upon which our theme may be developed.

A) DIFFERENTIATION OF $\frac{u}{z}$ [48]

3) Since the situation of $d\frac{u}{z}$ is converse in relation to $d(uz)$ — here multiplication, there division — it seems natural for the *algebraically* found operational formula

$$d\,(uz) = zdu + udz$$

to be utilized immediately for finding out $d\frac{u}{z}$. That is what I shall do, so that the difference between the method of immediate deduction and the simple application of some result of differentiation obtained earlier — which is, in its turn, an operational formula — comes out clearly.

a) $y = \frac{u}{z}$;

b) $u = yz$.

since

$$y = \frac{u}{z},$$

$$yz = \frac{u}{z} \cdot z = u.$$

Thus, u only formally masquerades as the product of two multipliers. However, thereby the problem is in actuality already solved, for the problem of differentiating a fraction has turned into the problem of differentiating a product, for which we have a magic formula in our pocket. According to this formula :

c) $du = zdy + ydz$.

We at once observe that the form of the first term on the right hand side, namely, of zdy, is such, that it must *quietly remain* in its post till the last minute, since that task is to find out the differential of $y\left(= \frac{u}{z}\right)$, i.e., to express it in [terms of] the differentials of u and z. On the other hand, for this very reason ydz should be *shifted* to the left hand side. That is why :

d) $du - ydz = zdy$.

If now we put the value of y namely $\frac{u}{z}$, in ydz, then

$$du - \frac{u}{z}dz = zdy;$$

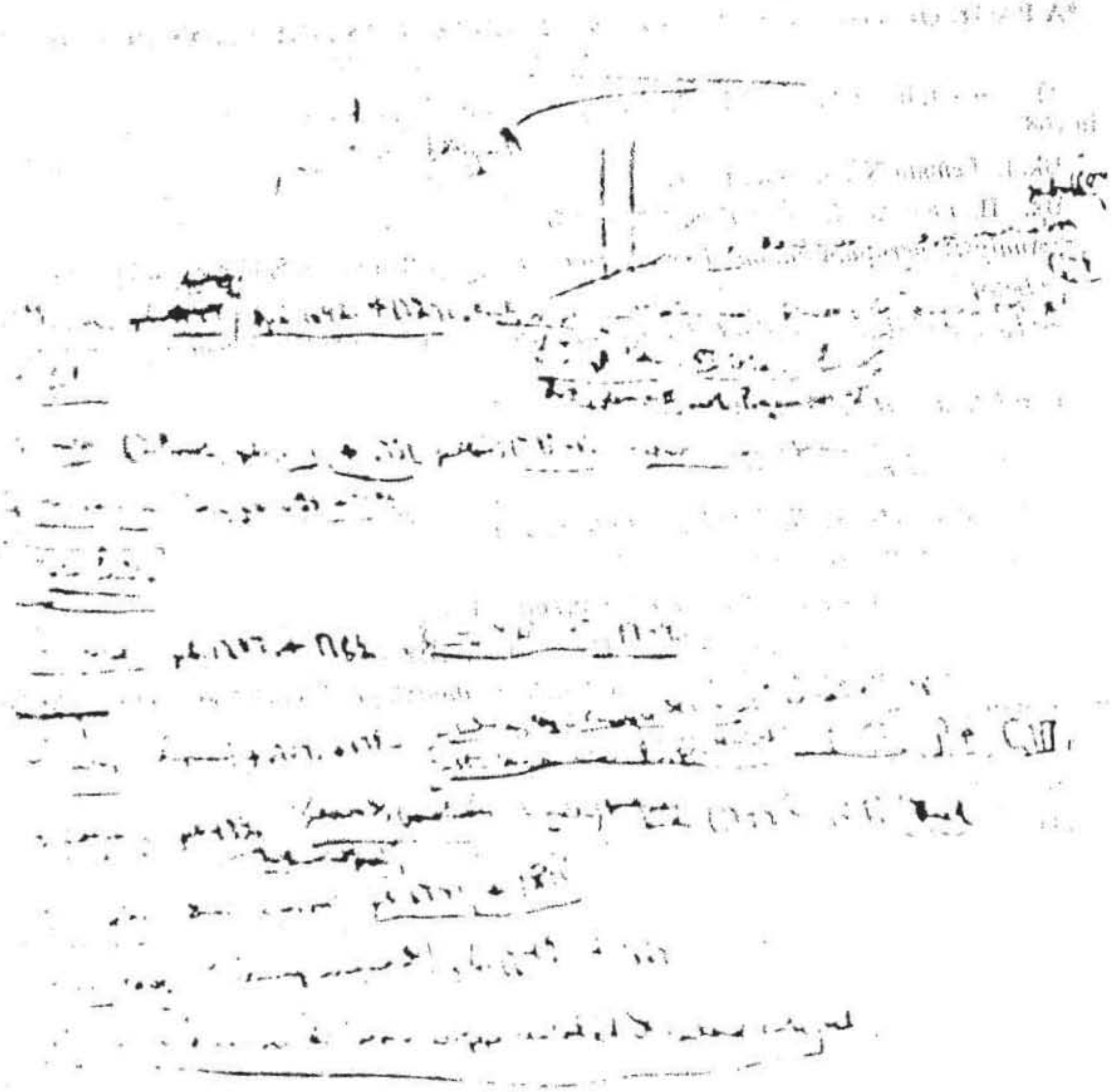
that is why

$$\frac{zdu - udz}{z} = zdy.$$

Now the moment of freeing dy from its sleeping partner z has arrived, and we get

$$\frac{zdu - udz}{z^2} = dy = d\,\frac{u}{z}\cdot$$

ON THE HISTORY OF DIFFERENTIAL CALCULUS [49]



PHOTOCOPY OF A PAGE OF THE NOTE BOOK ENTITLED "B (CONTINUATION OF A). II".

*A PAGE OF THE NOTE BOOK ENTITLED "B (CONTINUATION OF A) II"[50]

1) *Newton*, b. 1642, † 1727. "*Philosophiae naturalis principia mathematica*", published in 1687.

Bk. I., *Lemma* XI, *Scholia*. Bk. II .

Bk. II. *Lemma* II, *after Proposition* VII[51] .

"*Analysis per quantitatum series, fluxiones etc.*", written in 1665, published in 1711[52].

2) *Leibnitz.*

3) *Taylor* (J.Brook), b. 1685, † 1731, published in 1715-17 : "*Methodus incrementorum etc.*".

4) *MacLaurin (Colin)*, b. 1698, † 1746.

$\boxed{\text{5) \textit{John Landen.}}}$

6) *d'Alembert,* b. 1717, † 1783. "*Traité de fluides*" 1744[53].

7) *Euler* (Leonhard), [b.] 1707, † 1783.

"*Introductio in analysin infinitorum*", Lausanne, 1748.

"*Institutiones calculi differentialis*", 1755 (P. I, ch. III)[54].

8) *Lagrange*, b. 1736. "*Theorié des fonctions analytiques*" (1797 and 1813) (*See : Introduction*) .

9) *Poisson* (Denis, Simeon), b. 1781, † 1840.

10) *Laplace* (P. Simon, Marquis de), b. 1749, † 1827.

11) *Moigno*, "*Leçons de calcul différentiel et de calcul intégral*"[55].

*I. THE FIRST DRAFTS

Newton : b. 1642, † 1727 (at the age of 85). "*Philosophiae naturalis principia mathematica*" (first published in 1687, see *Lemma* I and *Lemma* XI, *Scholia*). Then especially : "*Analysis per quantitatum series, fluxiones etc.*" first published in 1711, but written in 1665, whereas Leibnitz made similar discoveries for the first time in 1676.

Leibnitz : b.1646, † 1716 (at the age of 70).

Lagrange : b. 1736, was already dead during the rule of emperor (Napoleon I), inventor of the *method of variations*. "*Theorié des fonctions analytiques*" (1797 and 1813).

d'Alembert : b. 1717, † 1783 (at the age of 66). "*Traité de fluides*", 1744.

1) *Newton*. Speeds, or fluxions, for example *of the variables x,y* etc. are designated by $\dot{x}$, $\dot{y}$ etc. If, for example u and x and are *interrelated magnitudes (fluents), generated by continuous motion*, then $\dot{u}$ and $\dot{x}$ designate the speeds of their increment and, consequently, $\frac{\dot{u}}{\dot{x}}$ is the *ratio* of the speeds, with which their increment is generated.

Since the numerical magnitudes of all possible quantities may be represented by straight lines, so the *moments*, or infinitely small parts of the generated magnitudes = the products of their speeds and of the infinitely small parts of time, in course of which these speeds last[56], such that, if τ designates this infinitely small part of time, then the moments of the magnitudes x and y are correspondingly represented by $\tau\dot{x}$ and $\tau\dot{y}$. For example $y = uz$; if we designate the speeds of increment of the magnitudes correspondingly by $\dot{y}$, $\dot{z}$, $\dot{u}$ then their *moments* will be $\tau\dot{y}$, $\tau\dot{z}$, $\tau\dot{u}$, and we shall get :

$y = uz$, $y + \tau\dot{y} = (u + \tau\dot{u})(z + \tau\dot{z}) = uz + u\tau\dot{z} + z\tau\dot{u} + \tau^2\dot{u}\dot{z}$;

hence,

$$\tau\dot{y} = u\tau\dot{z} + z\tau\dot{u} + \tau^2\dot{u}\dot{z} .$$

Since τ is infinitely small, so it vanishes all by itself, and even more completely vanishes $\tau^2\dot{u}\dot{z}$, as the product, corresponding not to the infinitely small segment of time τ, but to its second power. (If, for example,

$$\tau = \frac{1}{\text{million}}, \quad \text{then} \quad \tau^2 = \frac{1}{1 \text{ million} \times 1 \text{ million}} .)$$

Thus, we get

$$\dot{y} = \dot{u}z + \dot{z}u ,$$

or the fluxia of $y = uz$ is $\dot{u}z + \dot{z}u$ [57].

2) *Leibnitz*. Suppose the differential of uz is required to be found out.

u turns into $u + du$, z — into $z+dz$.

Thus,

$uz + d(uz) = (u + du)(z + dz) = uz + udz + zdu + dudz$.

If from this the given magnitude uz is subtracted then $udz + zdu + dudz$ will remain as the increment, $du\,dz$ — the product of infinitely small du and infinitely small dz — is an

infinitesimal of second order and vanishes as compared to the infinitesimals of first order *udz* and *zdu*. That is why

$$d(uz) = udz + zdu$$ [58].

[3)] *d'Alembert.* States the task in a general from as follows : Let

$$y = f(x),$$
$$y_1 = f(x+h);$$

the value of $\frac{y_1 - y}{h}$, when the magnitude h vanishes, i.e., the value of $\frac{0}{0}$ [59], is to be determined.

Newton and Leibnitz as well as the majority of their successors, operate upon the ground of differential calculus from the very beginning, and that is why the differential expressions at once serve them as operational formulae for searching the real equivalents. The whole issue is this : with the transformation of the independent variable x into x_1, the dependent variable turns into y_1, but $x_1 - x$ is of necessity equal to some difference, for example h. This is contained in the very concept of the variable. However from this it does not at all follow, that this difference, which is equal to dx, is a vanishing [magnitude], i.e., in reality it $= 0$. It may present itself also as a finite difference. If we assume before hand, that the increasing x turns into $x + \dot{x}$ (Newton's τ plays no role in his analysis of the basic functions and perhaps that is why it has been omitted[60]) or, as per Leibnitz, into $x + dx$, then the differential expressions at once turn into operational symbols, without further advance of their algebraic descent.

Ad [P] 15* (*Newton*).

Take the Newtonian initial equation for the product uz, which has to be differentiated. Then

$$y = uz, \; y + \tau\dot{y} = (u + \dot{u}\tau)(z + \dot{z}\tau)$$

Having cast away τ — as he himself gladly does, as he expands the first differential equation — we shall get :

$$y + \dot{y} = (u + \dot{u})(z + \dot{z}),$$
$$y + \dot{y} = uz + \dot{u}z + \dot{z}u + \dot{u}\dot{z},$$
$$y + \dot{y} - uz = \dot{u}z + \dot{z}u + \dot{u}\dot{z}.$$

Consequently, since $uz = y$,

$$\dot{y} = \dot{u}z + \dot{z}u + \dot{u}\dot{z}.$$

* See PV, 49-51.

And in order to obtain the correct result $\dot{u}\dot{z}$ has to be discarded. But where did the forcibly discarded term $\dot{u}\dot{z}$ come from ?

That is very simple : the main point is this, that the differentials of y in the form of $\dot{y}$, of u in the form of $\dot{u}$ and of z in the form of $\dot{z}$ are introduced from the very beginning, by definition, as existing side by side with the variables, from which they come into being, as independent of them, and are not introduced mathematically at all.

On the one hand, we see, of what benefit is this presupposed existence of dy, dx or $\dot{y}, \dot{x}$; as soon as the variables grow, I am only to substitute in the algebraic function, the binomials $y+\dot{y}$, $x+\dot{x}$ etc., and after that they are only to be manoeuvred as ordinary algebraic magnitudes.

Thus for example, having $y = ax$, I get

$$y + \dot{y} = ax + a\dot{x} \ ;$$

thus,

$$y - ax + \dot{y} = a\dot{x} \ ;$$

hence

$$\dot{y} = a\dot{x}.$$

Thus, I at once get the result : differential of the dependent variable is equal to the increment of [the function] ax, i.e., $a\dot{x}$ is equal to the *real value deduced from* ax, a (that here it is a constant, is accidental and does not change the general character of the result, since only this condition is obligatory, that the variable x is situated here in the first power) [multiplied by $\dot{x}$]. If I generalise this result [61], then I know [that] $y = f(x)$, for it signifies, that y is dependent upon the variable x. If the magnitude derived from $f(x)$, i.e., the real element of the increment, is called $f'(x)$, then the general result will be

$$\dot{y} = f'(x)\,\dot{x}.$$

Thus, it is known to me beforehand, that the equivalent of the differential of the dependent variable y is equal to the first derived function according to the independent variable multiplied by its differential, i.e., by dx or $\dot{x}$.

Thus, in general, if

$$y = f(x),$$

then

$$dy = f'(x)\,dx,$$

or, $\dot{y}$ = the real co-efficient in x (excluding the case, where a constant appears, owing to the fact that x enters into the first power), multiplied by $\dot{x}$.

But $\dot{y} = a\dot{x}$ at once gives $\frac{\dot{y}}{\dot{x}} = a$ and generally

$$\frac{\dot{y}}{\dot{x}} = f'(x).$$

Thus two further developed operational formulae are found for the differential, and for the differential co-efficient, forming the basis of the entire differential calculus.

Apart from this, generally speaking, thanks to the *a priori* supposition of the differentials dx, dy etc. or $\dot{x}$, $\dot{y}$ etc., as independent isolated, increments of x and y, a great advantage is obtained, permitting the differential calculus to express all functions of the variables in differential forms, from the very beginning.

Since I developed the basic functions of the variables, like ax, $ax \pm b$, xy, $\frac{x}{y}$, x^n, a^x, $\log x$ as well as the elementary circular functions along this path, I can, while searching dy, $\frac{dy}{dx}$, use them just like the multiplication tables of arithmetic. But now if we look at the opposite side of the affair, we shall at once observe, that the entire initial operation is mathematically wrong.

Take the simplest example : $y = x^2$. If x grows, then it obtains some indeterminate increment h ; owing to this y, the variable dependent on it, also obtains an indeterminate increment k, and we shall have —

$$y + k = (x + h)^2 = x^2 + 2hx + h^2$$

— a formula provided by the binomial [theorem]. Hence

$$y + k - x^2 \quad \text{or} \quad y + k - y = 2hx + h^2 \;;$$

consequently,

$$(y + k) - y \quad \text{or} \quad k = 2hx + h^2.$$

Having divided both the sides by h, we get

$$\frac{k}{h} = 2x + h.$$

Now assuming $h = 0$, we shall have

$$2x + h = 2x + 0 = 2x.$$

On the other hand, $\frac{k}{h}$ becomes $\frac{k}{0}$: but since y turned into $y + k$ only because x turned into $x + h$, so $y + k$ again turns into y when h turns into 0, on the strength of which $x + h$ again becomes $x + 0$, i.e., x. Hence, k also becomes 0 and $\frac{k}{0} = \frac{0}{0}$, which may be presented now in the form of $\frac{dy}{dx}$ or $\frac{\dot{y}}{\dot{x}}$. Thus we get $\frac{0}{0}$ or $\frac{\dot{y}}{\dot{x}} = 2x$.

If in

$$y + k - x^2 = 2hx + h^2 \text{ or } (y + k) - y = 2hx + h^2$$

[we assume that $h = 0$] (h turns into the symbol dx only because, in its initial form, it was supposed to be equal to 0), then we shall get $k = 0 + 0 = 0$, and the sole result obtained by us is but an assertion of our supposition, that y simply becomes $y + k$, when x becomes $x + h$, ... hence, when $x + h = x + 0 = x$, then $y + k = y$, or $k = 0$.

But by no means do we get, after Newton,

$$k = 2x\,dx + dx\,dx$$

or, in Newtonian notation

$$\dot{y} = 2x\dot{x} + \dot{x}\dot{x} \ ;$$

h turns into $\dot{x}$, and on the strength of that, k — into $\dot{y}$, only since h has gone down into hell through 0, i.e., since the differences $x_1 - x$ (or $(x+h) - x$), and that is why also $y_1 - y$ ($= (y+k) - y$), are reduced to their absolute minimum expressions : $x - x = 0$ and $y - y = 0$. In so far as Newton gets [the differentials] from the increments of the variables x, y etc., not with the help of mathematical deduction, but at once puts the stamps of differentials $\dot{x}, \dot{y}$ etc., over the increments, these increments cannot be $= 0$, for otherwise the result would be null, since, expressed algebraically, the supposition that these increments are equal to zero, from the very beginning, is tantamount to supposing at once $h = 0$, and that is why $k = 0$, as above, in the equation

$$(y+k) - y = 2hx + h^2$$

and hence, to obtaining in the last instance of $0 = 0$. The supposition that $h = 0$ is impermissible before the first derived function of x, in the given case $2x$, is freed from the multiplier h by division, and thus

$$\frac{y_1 - y}{h} = 2x + h$$

is obtained.

Only after this may the final difference be removed. That is why, in the same way the differential coefficient

$$\frac{dy}{dx} = 2x$$

must be initially expanded[62], before we can obtain the differential

$$dy = 2xdx.$$

Thus, there remains nothing else to do, but to present the increments of h as infinitely small [magnitudes] and to register them, as such, as *independent beings*, for example, in the symbols $\dot{x}$, $\dot{y}$ etc., or dx, dy [etc]. But infinitely small magnitudes are also magnitudes, as are the infinitely big (the word infinitely (small) signifies only the fact that it is indefinitely small); that is why, these dy, dx etc. or $\dot{y}$, $\dot{x}$ [etc] also figure in the computation as ordinary algebraic magnitudes, and in the equation reduced above

$$(y+k) - y \quad \text{or} \quad k = 2xdx + dxdx,$$

the term $dx\,dx$ has as much right to exist, as has $2xdx$. But most astonishing is that argument, by which this term is forcibly cast away, namely, on the strength of utilising the relativity of the concept of infinitely small; $dxdx$ is discarded because it is infinitely small as compared to dx, and consequently, also as compared to $2xdx$ or $2x\dot{x}$. Or, if in

$$\dot{y} = \dot{u}z + \dot{z}u + \dot{u}\dot{z},$$

[the addend] $\dot{u}\dot{z}$ is discarded in view of its infinite smallness in comparison with $\dot{u}z$ or $\dot{z}u$, then only the mention of the fact, that in our eyes $\dot{u}z + \dot{z}u$ has an approximate value, conceivably as proximate to the exact [value] as possible, may serve as a mathematical justification for this. We meet with the same type of manoeuvre, also in the ordinary algebra. But, then we face an even greater miracle : owing to this method, we get , for the derived function [in] x,

by no means an approximate, but an entirely exact value (though, as above, it is correct only symbolically), as in the example $\dot{y} = 2x\dot{x} + \dot{x}\dot{x}$. Here discarding $\dot{x}\dot{x}$,

$$\dot{y} = 2x\dot{x} \text{ is obtained}$$

and

$$\frac{\dot{y}}{\dot{x}} = 2x,$$

which is the correct first derived function of x^2, as it is proved, already in the binomial [theorem].

But this miracle is no miracle at all. Conversely, it would have been a miracle, had the forcible casting away of $\dot{x}\dot{x}$ *not given an exact result*. *For what is discarded is but some mistake in computation*, which, however, was the unavoidable consequence of the method, permitting the introduction of an indeterminate increment, for example h, of the variable at once as the differential dx or $\dot{x}$, as a ready-made operational symbol and thus of immediately obtaining the differential calculus as an independent means of computation, distinct from the ordinary algebra.

The course of the algebraic method used by us may be depicted in the general from as under. If $f(x)$ is given, then at first the "preliminary derivative" is expanded, which we shall call $f^1(x)$:

1) $f^1(x) = \dfrac{\Delta y}{\Delta x}$, or $\dfrac{\Delta y}{\Delta x} = f^1(x)$.

From this equation it follows that:

$$\Delta y = f^1(x)\Delta x.$$

Thus, also;

$$\Delta f(x) = f^1(x)\Delta x$$

(since $y = f(x)$), [so] $\Delta y = \Delta f(x)$.

Supposing $x_1 - x = 0$, and consequently, also

$y_1 - y = 0$, we get

$$[2)] \quad \frac{dy}{dx} = f'(x).$$

Then

$$dy = f'(x)\,dx,$$

hence, also

$$df(x) = f'(x)\,dx$$

(since $y = f(x)$, $dy = df(x)$).

Since we have already expanded

1) $\Delta f(x) = f^1(x)\Delta x$,

we see that

2) $df(x) = f'(x)\, dx$

is only the differential expression for 1)

[————————]

1) If x turns into x_1,then

A) $x_1 - x = \Delta x$;

hence the following conclusions :

Aa) $\Delta x = x_1 - x$; a) $x_1 - \Delta x = x$;

Δx, the *difference* between x_1 and x— expressed positively— is consequently, the *increment* of the variable x, because, if it is taken, conversely, away from x_1, then the latter returns to its initial condition, to x.

Hence the difference may be expressed in two ways : *immediately, as the difference* between the increased variable and its condition prior to increase — and this is its *negative expression* ; and positively, as the increment *, *as the result* : as the *increment* of x to that condition of it, when it does not increase further, and this is a positive expression.

We shall see later, what role this two-fold understanding plays in the history of calculus.

[2)] b) $x_1 = x + \Delta x$.

x_1 is increased x itself, its growth is not separated from it ; x_1 is the entirely indeterminate form of its increase; this form distinguishes the increased x, i.e., x_1 from its initial form prior to increase, from x, but it does not distinguish x from its increment, as such. Owing to this the relation of x_1 and x may be expressed only negatively, *as a difference*, as $x_1 - x$. As opposed to this, in $x_1 = x + \Delta x$:

1) The difference is expressed *positively*, as the increment of x.

2) That is why its increase is expressed not as a *difference*, but as the *sum* of it in its initial condition + its increment.

3) Technically speaking , from a monomial x turns into a binomial , and wherever in the initial function x is met with , in some power, there in place of the increased x appears the binomial, *consisting of x itself and its increment* ; generally [speaking] in place of x^m the binomial $(x + h)^m$. Thus , expansion of the increase of [the variable] x in reality becomes a case of simple application of the *binomial theorem*. Since x comes forth as the first , and Δx — as the second term of this binomial — which is indicated by their very interrelation, in so far as x must have existed prior to the emergence of its increment Δx, so in reality only a function of x is being deduced with the help of the binomial, meanwhile Δx figures side by side as a multiplier with increasing powers ; and herein Δx in its first power, i.e., Δx^1, must appear

* Here Marx has written is pencil " or decrease " . — Ed.

in the second term of the expansion as the multiplier of the first derived function of x_1, deduced with the help of the binomial theorem. This is observed at once, when x is given in its second power. x^2 turns into $(x + \Delta x)^2$, which is nothing else but the *multiplication* of $x + \Delta x$ by itself, [and which] gives $x^2 + 2x\Delta x + \Delta x^2$, i.e., the first term must be the initial function of x, and the first derived function of x^2, i.e., in the given case [2] x forms the second term with the multiplier Δx^1 ; this [multiplier] appears in the first term only as $\Delta x^0 = 1$. Thus, the derivative is found not through differentiation, but by applying the binomial theorem i.e., through *multiplication*, and besides, because the increment x_1 figures from the very beginning as a binomial, as $x + \Delta x$.

4) Though in $x + \Delta x$, as a magnitude Δx is as indeterminate, as is the indeterminate variable x itself, nevertheless, Δx is determinate, as distinct from the particular magnitude x, as the foetus beside its own mother before she became pregnant. $x + \Delta x$ does not simply indeterminately express the fact that, as a variable x has increased, it also expresses *how much* x has increased, namely, by Δx.

5) When the derivative is found by applying the binomial theorem, i.e., by substituting $x + \Delta x$ for x in the determinate power [of the variable] x, then x never appears as x_1 ; the entire expansion moves around the increment Δx. Only on the left hand side, when in $\frac{y_1 - y}{\Delta x}$ [the increment] Δx turns into zero, does Δx finally appear again as equal to $x_1 - x$, such that

$$\frac{y_1 - y}{\Delta x} = \frac{y_1 - y}{x_1 - x}\,^{*}.$$

Thus, the positive side of the equalisation of $x_1 - x$ with zero, namely, the turning of x_1 into x, can no where appear in the expansion in so far as x_1, as such, never figures on that side, where the expanded series is situated ; thus the real secret of differential calculus remains unrevealed.

6) If $y = f(x)$ and $y_1 = f(x + \Delta x)$, then we can say, that in this method, the *expansion of* y_1 *solves the problem of searching the derivative.*

c) $x + \Delta x = x_1$ (hence also, $y + \Delta y = y_1$). Here Δx may appear only in the form of $\Delta x = x_1 - x$, i.e., in the *negative* form of a *difference* between x_1 and x, and not in the *positive* form of an increment of x, as in $x_1 = x + \Delta x$.

1) Here the increased x, i.e., x_1 is distinct *from itself*, from what it was prior to the increase, i.e., from x , *but* x_1 *does not appear as* x increased by Δx ; that is why, in fact x_1 remains as indeterminate as x.

2) Further, just as x enters into an initial function, so does the increased x_1 — enter into the initial function transformed by the increase. Thus, for example, if x appears in the

---

* Marx wrote in pencil : " $(= \frac{\Delta y}{\Delta x})$ " . — Ed.

function x^3, then x_1 appears in the function x_1^3. While at first we substitute x by $(x + \Delta x)$ in the initial function, the derivative is furnished by the binomial in an entirely ready-made form, though endowed with the multiplier Δx and appearing as the leader of the other terms in x with the multipliers Δx^2 etc.; now from this immediate form of the monomial x_1^3, of the augmented x, we may immediately deduce as little, as [may be deduced] from x^3. However, hereby the *difference* $x_1^3 - x^3$ is given. We know from algebra, that all differences of the form $x^3 - a^3$ are divisible by $(x - a)$, i.e., in the given case by $x_1 - x$. That is why when we divide $x_1^3 - x^3$ by $x_1 - x$ (in place of multiplying, as before, $(x + \Delta x)$ by itself, as many times, as many are the units in the index of power), we obtain [preliminarily] an expression of the form $(x_1 - x)P$ irrespective of the fact, whether the initial function of x is a polynomial, (i.e., contains x in various powers), or, as in our example, a monomial. In the course of division this $(x_1 - x)$ becomes the denominator for $y_1 - y$ in the left hand side, and, thus, there emerges $\frac{y_1 - y}{x_1 - x}$ — the ratio of the difference of the function to the difference of the independent variable x in its abstract difference form. Expansion of the difference between the functions expressed in x_1, and those expressed in x, into such terms, each of which has the multiplier $x_1 - x$, may,in view of the characteristics of the initial function of x, demand greater or lesser algebraic manoeuvres; perhaps, it is a demand which is not always fulfilled as easily, as in the case of $x_1^3 - x^3$. But this changes nothing in the method.

Where, owing to its own nature, the initial function does not permit a direct expansion [of the difference] $f(x_1) - f(x)$ in $(x_1 - x)P$, as it happened for $f(x) = uz$ (two variables dependent upon x), there [the expression] $(x_1 - x)$ appears [in the] multiplier $\frac{1}{x_1 - x}$. Further, when after the transfer of $x_1 - x$ to the left hand side, through the division of both the sides by it, $x_1 - x$ still remains in P itself (as, for example, while deducing the derivative of $y = a^x$, then we find

$$\frac{y_1 - y}{x_1 - x} = a^x \left\{ (a-1) + \frac{(x_1 - x) - 1}{1 \cdot 2} \; (a-1)^2 + \text{etc.} \right\},$$

and there the supposition that $x_1 - x = 0$ gives

$$= a^x \left\{ (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \text{etc.} \right\}).$$

This, as in the example just cited, may happen always, only if the supposition of $x_1 - x$ equal to zero, leading to its disappearance, always left positive results in its place. In other words, these $x_1 - x$, still remaining in P, cannot be connected with the other elements of the expression P as multipliers (as multiplicators). In the opposite case, P could be represented as $P = p\,(x_1 - x)$ and, it means, that in so far as $x_1 - x$ was already assumed to be equal to zero, as $p \cdot 0$; this would signify that $P = 0$ [63] ....

When $y = x^3$, and $y_1 = x_1^3$, the first finite difference $x_1^3 - x^3$ is, consequently, expanded again into

$$y_1 - y = (x_1 - x)\,P,$$

hence

$$\frac{y_1 - y}{x_1 - x} = P\,.$$

The expression P, presenting itself as a combination of x_1 and x, is equal to f^1— the derivative of the first finite difference ; hence, $x_1 - x$ is excluded , as are the higher powers of it $(x_1 - x)^2$ etc. Owing to this, in the given case, x_1 and x can only combine in positive expressions, like $x_1 + x$; $x_1 x$, $\frac{x_1}{x}$, $\sqrt{x_1 x}$ etc. Hence, if now we assume that $x_1 = x$, then these expressions will correspondingly turn into $2x, x^2, \frac{x}{x}$ or 1, $\sqrt{x x}$ or x etc., and only on the left hand side, where $x_1 - x$ forms the denominator will 0 appear, and consequently, also the symbolic differential co-efficient etc.

II. THE HISTORICAL COURSE OF DEVELOPMENT

1) Mystical differential calculus. $x_1 = x + \Delta x$ at once turns into $x_1 = x + dx$ or $x + \dot{x}$, where dx is postulated by a metaphysical *explanation*. At first it exists, and is explained only subsequently.

But even then $y_1 = y + dy$ or $y_1 = y + \dot{y}$. From this arbitrary postulation it follows, that in the expansion of the binomial $x + \Delta x$ or $x + \dot{x}$ the terms in x and Δx, which, for example, were obtained side by side with the first derivative , must be *removed by jugglery*, so that a correct result may be obtained etc. etc. Since, upon actual substantiation, the differential calculus proceeds from this last result, namely, from the *differential particles*, which are anticipated, and not deduced, but *premised* with the help of an elucidation, so $\frac{dy}{dx}$ or $\frac{\dot{y}}{\dot{x}}$, the symbolic differential co-efficient, is also *anticipated* by this elucidation.

If the increment of x is equal to Δx, and the increment of the variable dependent upon it is equal to Δy, then it stands to reason all by itself, that $\frac{\Delta y}{\Delta x}$ expresses the ratio of the increments of x and y. But that Δx figures in the denominator, i.e., the increment of the independent variable stands in the denominator, and not, conversely, in the numerator, is a consequence of the fact, that the final result of the development of differential forms themselves, namely, the *differential*, is also given already before hand, by the premised differential particles.

Let us consider the simplest relation between the dependent variable y and the independent variable x, as in $y = x$. In this case it is known that $dy = dx$ or that $\dot{y} = \dot{x}$. But since we seek the derivative of the independent [variable] x, here which $= \dot{x}$, so both the sides must be divided by $\dot{x}$ *or* dx[64], and hence

$$\frac{dy}{dx} \text{ or } \frac{\dot{y}}{\dot{x}} = 1.$$

Thus, we know once and for all, that in the symbolic differential co-efficient, increment [of the independent variable] must be situated in the denominator, and not in the numerator.

But now for the functions of x in the second degree, the *derivative* is at once sought with the help of the binomial theorem, [providing the expansion, in which] the derivative appears in a ready-made form already in the second term with the multiplier dx or $\dot{x}$, i.e., with the increments in first power + the terms to be discarded. However, this *trick* is, though unconsciously, mathematically correct, since it only eliminates that mistake in computation, which emerged at the very beginning, from the initial tricks.

Only $x_1 = x + \Delta x$ is to be transformed into

$$x_1 = x + dx \quad \text{or into } x + \dot{x},$$

so that then we can lord over this differential binomial, just as [we do] over the ordinary binomials, which would be very comfortable from the technical point of view.

The only question which may still be raised is as follows : why the terms blocking the path are forcibly eliminated ? It has been assumed to be well known, that they stand in our path and, that, in reality, they do not belong to the derivative.

The *answer* is very simple : it has been found out experimentally. Not only for many more complex functions of x, including those in their analytical forms, like the equations for curves etc., were the actual derivatives long since well known, but this was also discovered immediately in the first possible experimental solution, namely, upon consideration of the simplest algebraic functions of the second degree, for example :

$$y = x^2,$$
$$y + dy = (x + dx)^2 = x^2 + 2xdx + dx^2,$$
$$y + \dot{y} = (x + \dot{x})^2 = x^2 + 2x\dot{x} + \dot{x}^2.$$

If from both the sides the initial function x^2 ($y = x^2$) is subtracted, then :

$$dy = 2xdx + dx^2,$$
$$\dot{y} = 2x\dot{x} + \dot{x}\dot{x} \;;$$

discarding the last terms from both [the right hand] sides, we shall get :

$$dy = 2xdx\,,\ \dot{y} = 2x\dot{x},$$

and, further,

$$\frac{dy}{dx} = 2x,$$

or

$$\frac{\dot{y}}{\dot{x}} = 2x.$$

But from $(x + a)^2$ it is known, that x^2 is the first term, and $2xa$ — the second ; if this expression is divided by a, as we have divided above $2xdx$ by dx or $2x\dot{x}$ by $\dot{x}$, then we would get $2x$, as the first derivative of x^2, as the accretion in x [65], which the binomial added to x^2. Thus, for seeking the derivative they had to discard dx^2 or $\dot{x}\dot{x}$, not to speak of the fact, that there was nothing to be done with dx^2 or $\dot{x}\dot{x}$ in itself.

Thus, already at the second step along the path of experimentation, they inevitably arrived at the conclusion, that, not only for obtaining the correct result, but also for obtaining any result at all, it is essential to discard dx^2 or $\dot{x}\dot{x}$.

But on the other hand, in $2xdx + dx^2$ or $2x\dot{x} + \dot{x}\dot{x}$, they had before them a correct mathematical expression (the second and the third term) of the binomial ($x + dx$)^2 or ($x + \dot{x}$)^2. That this *mathematically correct result is based upon an equally mathematically wrong presupposition at the very foundation*, that supposedly, from the very beginning $x_1 - x = \Delta x$ is nothing else but $x_1 - x = dx$ or $\dot{x}$— this they did not know [66]. In other words, that very result could have been obtained and proposed in the mathematical world, not with the help of jugglery, but with algebraic operations of the simplest type .

Thus, they themselves believed in the mysterious character of the newly discovered calculus, which provided correct (and more over in the geometrical applications, really astonishing) results by a positively incorrect mathematical procedure. They were thus self-mystified, valued the new discovery all the higher, enraged the crowd of old orthodox mathematicians all the more, and thus called forth the cry of opposition; it aroused an echo even in the lay world, and that is necessary for paving the path for something new.

2) Rational differential calculus. d'Alembert starts directly from *the starting point of Newton and Leibnitz* : $x_1 = x + dx$. But he at once makes a fundamental correction : $x_1 = x + \Delta x$, i.e., $x +$ an *indeterminate*, but first of all a *finite increment*. This he calls h. With him, the transformation of this h or Δx into dx (like all Frenchmen he uses the Leibnitzian notations) takes place only as the last result of the development or at least just at the eleventh hour, while with the mystics and the initiators of calculus it appears as the starting point (d'Alembert himself operates from the symbolic side, but before that side turns into a symbol). By this at once a two fold result is obtained [67].

a) Ratio of the differences

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x+h)-f(x)}{x_1 - x}$$

has as the starting point of its own formation 1) [the difference] $f(x+h) - f(x)$, which corresponds to the algebraic function given in x, obtained, when in the initial function of [the variable] x, for example, in x^3, x is substituted by that very x with its increment, i.e., by $x + h$. This form ($= y_1 - y$, if $y = f(x)$) is the form of the *difference of functions*, which is required by the development, so that the increment of the function in the ratio may be transformed into the increment of the independent variable ; hence, it plays a real role, and not purely a nominal one, as with the mystics. For, if I have with the latter

$$f(x) = x^3,$$
$$f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3,$$

then I know beforehand, that the opposite sides of the equality

$$f(x+h) - f(x) = x^3 + 3x^2h + 3xh^2 + h^3 - x^3,$$

are reduced to increments. This may not have been written, since in the second side [R.H.S] I see, that of the increment of [the function] $x^3 =$ the next three terms, just as in $f(x+h) - f(x)$, there remains only the increment of the [function] $f(x)$, i.e., dy. Thus, the first difference equation again, beforehand, plays only a vanishing role. The increments beforehand stand opposite each other on both the sides and if I have them, then from the definitions of dx and dy I can conclude, that $\frac{dy}{dx}$ or $\frac{\dot{y}}{\dot{x}}$ is the ratio etc. Hence , in order to form $\frac{dy}{dx}$ or $\frac{\dot{y}}{\dot{x}}$, I do not need the first difference, obtained through the subtraction of the initial function in x, from the changed (by the substitution of x by $x + h$) function (from the increased function).

It is essential for d'Alembert to hold fast to this difference, because the process of development must emerge from it. That is why in place of the positive expression of difference, i.e., in place of the increment, the negative expression of increment, i.e., namely, the difference $f(x+h) - f(x)$ comes to the fore, in the left hand side. And of this stressing of the difference in place of the increment (the fluxions of Newton), we at least have a presentiment in the Leibnitzian notation of dy, as opposed to the Newtonian $\dot{y}$.

2) $f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3$.

Dividing both the sides by h, we get

$$\frac{f(x+h)-f(x)}{h}=3x^2+3xh+h^2 .$$

Here in the left hand side

$$\frac{f(x+h)-f(x)}{h}=\frac{f(x+h)-f(x)}{x_1-x} \text{ is formed,}$$

appearing, thus, as the *derived ratio of finite differences*, whereas for the mystics it was a ready-made ratio of increments, provided by the definitions of dx or $\dot{x}$ and dy or $\dot{y}$.

3) Now assuming $h=0$ or $x_1=x$, i.e., $x_1-x=0$ in

$$\frac{f(x+h)-f(x)}{h}=\frac{f(x+h)-f(x)}{x_1-x},$$

we thereby transform this expression into $\frac{dy}{dx}$; simultaneously with this, owing to the transformation of h into 0, the terms $3xh+h^2$ also turn into zero, moreover, by a correct mathematical operation. Hence, now they are removed without a trick. We get :

4) $\frac{0}{0}$ or $\frac{dy}{dx}=3x^2=f'(x)$.

The latter came into being, as with the mystics, as already given, when x turns into $x+h$; since in place of x^3, $(x+h)^3$ gives x^3+3x^2h+ etc., where $3x^2$ already appears in the second term of the series as the *co-efficient* of h in first degree. Thus, the conclusion is the same as with Leibnitz and Newton, however, here the entirely ready-made derivative $3x^2$ is *disentangled* from its surroundings, strictly algebraically. This is no *development,* but rather the *disentanglement* of $f'(x)$, here of $3x^2$, freed from its multiplier h, and from the other terms marching in a row along with it. But what has really been developed, is the left symbolic side, namely, dx, dy and their ratio, the symbolic differential co-efficient $\frac{dy}{dx}=\frac{0}{0}$ (more correctly, conversely, $\frac{0}{0}=\frac{dy}{dx}$), which, in its turn, again called forth a couple of metaphysical horrors, though this time the symbol has been wormed out mathematically.

d'Alembert tore off the shroud of mystery from the differential calculus, and thereby took a great step forward. However, in spite of the appearance of his "*Treatise on the liquids*" already in 1744 (see : p. 15 *), the method of Leibnitz prevailed in France for many more years. There is hardly any need to point out, that Newton ruled in England till the first decades of the 19th century. But here too, as in France earlier, the d'Alembertian foundation — with certain modifications — prevailed till the present moment.

3) Purely algebraic differential calculus. Lagrange, "*Theory of analytical functions*" (1797 and 1813). As in 1) and 2), [here also] the first starting point was the increasing x, if y or $f(x)=$ etc., then y_1 or $f(x+dx)$ — as in the mystical method , and y_1 or $f(x+h)$ $(=f(x+\Delta x))$ — as in the rational. This binomial starting point at once gives us a binomial expansion on the other side, for example :

* See PV, 66. —Ed,

$$x^m + mx^{m-1}h + \text{etc.},$$

where the second term $mx^{m-1}h$ already gives the unknown real differential co-efficient mx^{m-1} in an entirely ready-made form.

a) $f(x+h)$ standing on the left hand side is related to the expanded series standing opposite it, as soon as $x+h$ is substituted in place of x in the given initial function, just as in algebra the *unexpanded general expression*, and first of all, the very binomial, is related to the corresponding *expanded series*, as in

$$(x+h)^3 = x^3 + 3x^2h + \text{etc.},$$

$(x+h)^3$ is related to the expanded series $x^3 + 3x^2h$ + etc., equivalent to it. Thereby itself $f(x+h)$ appears in the same algebraic correlation (only in application to variables), in which, in the whole of algebra, the general expression finds itself to its expansion, as, for example, in

$$\frac{a}{a-x} = 1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3} + \text{ etc.},$$

$\frac{a}{a-x}$ is related to the expanded series 1 + etc., or as in

$$\sin(x+h) = \sin x \cos h + \cos x \sin h,$$

$\sin(x+h)$ is related to the expansion standing opposite it.

d'Alembert simply algebraicised $(x+dx)$ or $(x+\dot{x})$ into $(x+h)$, hence also $f(x+h)$, from $y+dy$, $y+\dot{y}$. Lagrange imparted a purely algebraic character to the entire expression, having counterposed to it, as a *general unexpanded expression*, the expanded series, which must be deduced from it.

b) In the first method 1), as well as in the rational 2), the unknown real co-efficient is manufactured in a ready-made form by the [use of] the binomial theorem, and is met with already as the second term of the expanded series, thus, in the term necessarily containing h^1. Consequently, as in 1), so also in 2), the entire further course of differentiation is a luxury. That is why, let us cast aside this useless ballast. From the binomial expansion we know once and for all, that the first real co-efficient is the multiplier of h, the second co-efficient — that of h^2 etc. These real differential co-efficients are nothing but successive binomial developments of the *derived functions from the initial function* in x (and the introduction of this category of *derived functions* is one of the most important [achievements]). Concerning the separate differential forms, we know that, Δx turns into dx, Δy — into dy, that the first derivative finds symbolic expression in the form of $\frac{dy}{dx}$, the second derivative, the co-efficient of $\frac{1}{2}h^2$ — in the form of $\frac{d^2y}{dx^2}$ etc. Hence, thanks to the symmetries, we can present the results obtained by us purely algebraically and at the same time in the form of their symbolic differential equivalents. Of the differential calculus proper the nomenclature alone remains. Under such circumstances, the entire task is in essence reduced to : finding out the (algebraic) methods "of expanding all types of functions of $x+h$ according to increasing integral powers of h, which cannot be done in many cases without [recourse to] highly cumbersome operations"[68].

Thus far there is nothing in Lagrange, which could not have been obtained directly proceeding from the method of d'Alembert (for the latter method also contains, only in a corrected form, the entire course of deduction of the mystics).

c) But consequently, in so far as the expansion of y_1 or $(x+h)$ = etc., appears in place of the earlier differential calculus [[and thereby, in reality, clearly comes forth the secret of those methods, which proceed from $y+dy$ or $y+\dot{y}$, $x+dx$ or $x+\dot{x}$, namely, that their actual expansion is based upon the application of the binomial theorem, in so far as, from the very beginning they present the increased x_1 as $x+dx$, the increased y_1—as $y+dy$, transforming thereby the monomial into a binomial]], the [following] task emerges : since in $f(x+h)$ we have a function of x without power, only *its general unexpanded expression*, the general, i.e., suitable also for the functions of x of any power, *series of expansion* is to be algebraically deduced from this very unexpanded expression.

Here, for the purpose of algebricising the differential calculus, Lagrange takes that theorem of *Taylor,* as his immediate starting point, which *has outlived the Newtonians and Newton*[69]. In reality this most general and most comprehensive theorem, is also at the same time the operational formula of differential calculus, namely, expressed in symbolic differential co-efficients, the expanded series for :

$$y_1 \text{ or } f(x+h), \text{ i.e.,}$$

$$y_1 \text{ or } f(x+h) =$$

$$= y \text{ (or } f(x)) + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{[2]} + \frac{d^3y}{dx^3}\frac{h^3}{[2\cdot 3]} + \frac{d^4y}{dx^4}\frac{h^4}{[2\cdot 3\cdot 4]} + \cdots$$

d) Here the investigations on the theorems of MacLaurin and Taylor are to be inserted[70].

e) The Lagrangian algebraic expansion of $f(x+h)$ into its equivalent series, substitutes the Taylorian $\frac{dy}{dx}$ etc. and retains them only as the symbolic differential expressions of the algebraically derived functions of x. (This is to be subsequently developed further[71] .)

*III. CONTINUATION OF THE DRAFTS

c) Continuation of p.25*

Initially we have $x_1 - x = \Delta x$ as an expression of the difference $x_1 - x$; *here the difference exists only in its difference form* (analogously, when y is dependent upon x, we often write $y_1 - y$) . Assuming $x_1 - x = \Delta x$, we thereby impart upon the difference , an expression which is already different from itself . We express, though in an indeterminate form, *the value of this difference* as something different from the difference of magnitudes itself . Thus, for example , 4 –2 is the pure expression of the difference between 4 and 2 ; but 4 –2 = 2 is the difference expressed through 2 (on the right hand side) : a) in a positive form, hence, no more as a difference ; b) the subtraction executed, the difference computed, and 4 –2=2 gives us 4= 2+ 2 . Here the second 2 appears in the positive *form of an increment of the initial* 2, thus, in a form, which is *directly opposed to the form of difference* . (Exactly in the same way $a - b = c$, $a = b + c$, where c comes forth as an increment of b , same also for $x_1 - x = \Delta x$, $x_1 = x + \Delta x$, where Δx figures immediately as an increment of x .)

Thus the simple initial assumption of $x_1 - x = \Delta x =$ something , puts something else in place of the *form of difference,* namely, the form of sum $x_1 = x + \Delta x$; along with this $x_1 - x$, expressing only a difference, [becomes] the equivalent value of this difference, the magnitude Δx. Also in the same way , from $x_1 - x = \Delta x$ we obtain $x_1 - \Delta x = x$ Here, again we have the form of difference in the left hand side, but as the difference between increased x_1 and its increment proper, appearing beside it independently. The difference between it and the increment of x, equal to Δx, is now a difference, already expressing — though in an indeterminate form — a determinate value of x.

But if we proceed from the mystical differential calculus, where $x_1 - x$ at once appears as $x_1 - x = dx$, and if at first dx is corrected into Δx, then we proceed from $x_1 - x = \Delta x$; hence, from $x_1 = x + \Delta x$; but this can, in its turn, be again transformed into $x + \Delta x = x_1$, so that the increase of x again attains the indeterminate form x_1 , and, as such, immediately appears in the calculus.This is the starting point of our algebraic method.

d) From these simple differences in form, we at once, and immediately, obtain that fundamental difference in the treatment of calculus, which we characterised in particular (see the corresponding separate sheets)[72] , while analysing the method of d'Alembert. Here I shall limit myself to some remarks of a general character.

1) If the *difference* $x_1 - x$ (and hence also $y_1 - y$) appears at once as its opposite, *as the sum* $x_1 = x + \Delta x$, and that is why the magnitude of its value instantly assumes the *positive form of the increment* Δx, then if in the *initial function* in x , in place of x everywhere we substitute $x + \Delta x$, then a binomial series of a determinate degree is required to be expanded and, the expansion of x_1 is reduced to an *application of the binomial theorem.* The binomial theorem is nothing but a general expression for the binomial of first degree *multiplied* with itself

* See : PV, 71-72 .—Ed.

m number of times. That is why, if we at once present a *difference* as *its opposite*, as a *sum*, then *multiplication* becomes the method of expanding x_1 [or] $(x + \Delta x)$.

2) Since in the general expression $x_1 = x + \Delta x$, the difference $x_1 - x$ given in the positive form of Δx, i.e., in the form of an *increment*, is *the latter or the second* term of the expression, so x becomes the first, and Δx — the second term of the initial function in x, when the latter appears as a function in $x + \Delta x$. But we know from the binomial theorem, that the second term figures only as a multiplier in increasing degrees beside the first term, and besides as such that the multiplier of the first expression containing x (of a determinate degree of the binomial) is $(\Delta x)^0 = 1$, the multiplier of the second term is $(\Delta x)^1$, that of the third $(\Delta x)^2$ etc. Thus, in the positive form of an increment the difference appears only as a multiplier, and besides at first really as a multiplier in the second term (since $(\Delta x)^0 = 1$) of the expanded binomial $(x + \Delta x)^m$.

3) On the other hand, if we consider expansion of the functions according to x itself, then the binomial theorem gives us for this first term, here for x, the derived functions according to this x— one after the other. For example, if we have $(x + h)^4$, where, in the algebraic binomial h is considered to be a known, and x — an unknown magnitude, then we get

$$x^4 + 4x^3h + \text{etc.}$$

$4x^3$ which stands in the second term, and has as multiplier h in the first degree, is hence the first derived function of x or, algebraically speaking : if we have the *unexpanded binomial expression* $(x + h)^4$, then the expanded series gives us as the first addition to x^4 (as its accretion), $4x^3$, which appears as the co-efficient of h. If x is a variable and we have $f(x) = x^4$, then the very increase of the latter turns this [expression] into $f(x + h)$ or, in the first form, into

$$f(x + \Delta x) = (x + \Delta x)^4 = x^4 + 4x^3\ \Delta x + \text{etc.}$$

x^4, which was obtained by us, in the ordinary algebraic binomial $(x + h)^4$, as the first term of the binomial[expansion], now appears in the binomial expression of the variable x, in $(x + \Delta x)^4$, as a reproduction of the initial function in x, before it increased and became $(x + \Delta x)$. From the very nature of the binomial theorem it is clear beforehand, that if $f(x) = x^4$ turns into $f(x + h) = (x + h)^4$, then the first term in [the expansion of] $(x + h)^4$ is equal to x^4, i.e., it must be equal to the initial function in x; $(x + h)^4$ must contain both the initial function in x (here x^4) + all the terms acquired by x^4, while it was turned into $(x + h)^4$, hence, the first term [in the expansion] of the binomial $(x + h)^4$ [is the initial function].

4) Further, the second term of the binomial expansion $4x^3h$ instantly gives us the first derived function of x^4, namely, $4x^3$, in an *entirely ready-made form*. Thus this derivative was obtained through the expansion of

$$f(x + \Delta x) = (x + \Delta x)^4;$$

it was obtained owing to the fact that, from the very beginning the difference $x_1 - x$ was presented as its *opposite*, as the *sum* $x + \Delta x$.

Thus, the binomial expansion of $f(x + \Delta x)$ or y_1, obtained from $f(x)$ through the increase of x, provides us the first derivative, the co-efficient of h (in the binomial series), and besides already at the beginning of the binomial expansion, in its second term. Hence, the derivative is not at all obtained through differentiation, but with the help of the expansion of $f(x + h)$ or y_1 into some determinate expression, obtained by simple multiplication.

Thus, the corner-stone of this method is the expansion of the indeterminate expression $f(x+\Delta x)$ or y_1, into a determinate binomial form, and by no means the expansion of $x_1 - x$, and hence, also of $y_1 - y$ or $f(x+h) - f(x)$ as differences.

5) The sole difference equation, which is met with in this method, consists of the fact that since we instantly get :

$f(x+\Delta x) = (x+\Delta x)^4 = x^4 + 4x^3\Delta x + 6x^2\Delta x^2 + 4x\Delta x^3 + \Delta x^4$,

so, when we write :

$x^4 + 4x^3\Delta x + 6x^2\Delta x^2 + 4x\Delta x^3 + \Delta x^4 - x^4$,

i.e., in the end, we again substract the initial function x^4, which is the beginning of the series ; we have before us an *increment,* which the initial function in x obtained through binomial expansion. That is why, Newton also writes like that. Hence, we have the increment

$$4x^3\Delta x + 6x^2\Delta x^2 + 4x\Delta x^3 + \Delta x^4,$$

i.e. the increment of the initial function in x. That is why, in the opposite side we don't need a *difference expression of any kind.* The increment of x corresponds to the increment of y, as y or $f(x) = x^4$. Not for nothing Newton at once writes :

$$dy\text{, for him } \dot{y} = 4x^3\dot{x} \text{ etc.}$$

6) Now the entire further development consists of freeing the completely ready-made derivative $4x^3$ from its multiplier Δx and from the neighbouring terms, of disentangling it from its surroundings. Thus, this is not a *method* of development, but of *disengagement.*

e) Differentiation of $f(x)$ (as a general expression).

We note at first, that the concept of "derived function" for the successive real equivalents of the symbolic differential co-efficients, which was quite unknown to those who first discovered the differenctial calculus and to their first successors, was in fact, for the first time, introduced by Lagrange. In [the writings of] the earlier[authors] only the dependent variable, for example y, figures as the *function of x,* which fully corresponds to the initial algebraic meaning of [the word] *function,* applied at first to the so called indeterminate equations, where the number of the unknowns is greater than the number of equations, and thus, where, for example, y takes different values depending upon the different values put in place of x. Whereas in Lagrange the initial function is a determinate algebraic expression of x, which is to be differentiated ; hence, if y or $f(x) = x^4$, then x^4 is the initial function, $4x^3$ — the first derivative etc. That is why, to avoid confusion, we shall call, y the dependent [variable], or $f(x)$ — *the function of x,* the initial function, in the Lagrangian sense — the *initial function in x,* correspondingly the "derivatives" are functions *in x.*

In the algebraic method, where we at first expand f^1 — the preliminary derivative or [the ratio] of finite differences and, only from that, the final derivative f', we know beforehand, that $f(x) = y$, hence :

a) $\Delta f(x) = \Delta y$, and that is why conversely also, $\Delta y = \Delta f(x)$. What is now to be developed first of all, is $\Delta f(x)$, the value of the finite difference of $f(x)$.

We find that :

$$f^1(x) = \frac{\Delta y}{\Delta x}\text{, i.e., } \frac{\Delta y}{\Delta x} = f^1(x).$$

That is, also

$$\Delta y = f^{1}(x)\,\Delta x,$$

and since $\Delta y = \Delta f(x)$, so

$$\Delta f(x) = f^{1}(x)\,\Delta x.$$

Further, the unfolding of the differential expression, which in the final count gives us

$$df(x) = f'(x)dx,$$

is simply the differential expression of the finite difference unfolded earlier.
In the usual method

$$dy \quad \text{or} \quad df(x) = f'(x)dx$$

is not at all expanded, but , see above , the fully ready-made $f'(x)$ furnished by the binomial $(x + \Delta x)$ or $(x + dx)$ or, is only *disengaged* from its multiplier and from the accompanying terms

*THEOREMS
OF TAYLOR AND MACLAURIN
LAGRANGE'S THEORY OF
ANALYTICAL FUNCTIONS

1. FROM THE MANUSCRIPT "TAYLOR'S THEOREM, MACLAURIN'S THEOREM AND LAGRANGIAN THEORY OF ANALYTICAL FUNCTIONS"[73]

I

Discovery of the binomial theorem by Newton (in its application to the polynomial), also gave rise to a revolutionary transformation in the whole of algebra — first of all, because it made the *general theory of equations* possible.

But the binomial theorem is also an important foundation of the differential calculus — and this is definitely recognised by the mathematicians, especially from the time of Lagrange. Even a cursory glance shows that, with the exception of the circular functions emanating from trigonometry, all differentials of the monomials like x^m, a^x, log x etc., are deduced only with the help of the binomial theorem[74].

Now it has even become a fashion to show in the textbooks that, as the binomial theorem can be deduced from Taylor's and MacLaurin's theorems, so also conversely[75]. However, nowhere, not even in Lagrange — whose theory of derived functions provided a new basis for the differential calculus, is this connection between the binomial theorem and the two others, laid bare in all its virgin simplicity, and here, as everywhere, it is important to strip the veil of secrecy from science.

Taylors theorem, which is historically prior to MacLaurin's theorem, gives us — under definte presuppositions — a sequence of symbolic expressions for all the functions of x, when x is incresed by a positive or negative increment h[76], i.e., generally for $f(x \pm h)$, indicating that series of differential operations,by means of which $f(x \pm h)$ may be expanded. Thus, what is at issue here, is the expansion of *any function of x , as soon as x changes.*

In contrast to this MacLaurin gives — also under determinate assumptions — *for every function of x*, the general expansion of this very function of x, also in a series of symbolic expressions, indicating how it is easy to expand , with the help of the differential calculus, those functions, the algebraic expansions of which are often very cumbersome and difficult.But the expansion of any function of x signifies nothing other than *obtaining constant functions combining with [the powers of] the independent variable x*[77], since the expansion of the very variable was, as it were, identical with its variation, i.e., with the object of Taylor's theorem.

Both of these theorems are colossal generalisations in which the differential symbols themselves become the content of the equation.Instead of actually successively deduced functions of x, the derivatives are presented only in the form of their symbolic equivalents, each of which, independently of the type of the functions $f(x)$ or $f(x+h)$, prescribes some operational stategy to be carried out. Thus, two formulae are obtained ; they are, with certain restrictions, applicable to all particular functions of x or $x + h$.

Taylor's formula :

$$f(x+h) \text{ or } y_1 = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1\cdot 2} + \frac{d^3y}{dx^3}\frac{h^3}{1\cdot 2\cdot 3} + \frac{d^3y}{dx^4}\frac{h^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

MacLaurin's formula :

$$f(x) \text{ or } y = (y) + \left(\frac{dy}{dx}\right)\frac{x}{1} + \left(\frac{d^2y}{dx^2}\right)\frac{x^2}{1\cdot 2} + \left(\frac{d^3y}{dx^3}\right)\frac{x^3}{1\cdot 2\cdot 3} + \left(\frac{d^4y}{dx^4}\right)\frac{x^4}{1\cdot 2\cdot 3\cdot 4} \text{ etc.}$$

Even at a first glance it is visible that, historically, as well as theoretically, here it is already assumed, that the *arithmetic(of* what may be called) *the differential calculus, i.e.* ,the development of its basic operations, is both available and well known. This should not be forgotton latter on , when I shall assume this acquaintance.

II

. . . .

Maclaurin's theorem may be viewed as a *particular instance* of Taylor's theorem. In Taylor's [theorem] we have :

$y = f(x)$,

$$y_1 = f(x+h) = f(x) \text{ or } y + \frac{dy}{dx}h + \frac{1}{2}\frac{d^2y}{dx^2}h^2 + \text{etc.} + \left[\frac{1}{1\cdot2\cdot3\cdots\cdots n}\right]\frac{d^ny}{dx^n}h^n + \text{ etc.}$$

If in $f(x+h)$ and , also on the right hand side, in y or $f(x)$ and , in its derived functions, symbolically derived in the form of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc., we assume that $x = 0$, such that, all these functions will no more contain anything other than a reversal of the constant element of x [78] , then

$$f(h) = y + \left(\frac{dy}{dx}\right)h + \left(\frac{d^2y}{dx^2}\right)\frac{h^2}{1\cdot2} + \left(\frac{d^3y}{dx^3}\right)\frac{h^3}{1\cdot2\cdot3} + \text{ etc.}$$

Then $y_1 = f(x+h) = f(0+h)$ will be that very function of h , which $y = f(x)$ is , in respect of x, for h enters into $f(h)$, just as x does into $f(x)$, and (y) into $\left(\frac{dy}{dx}\right)$ etc.— [herein] all trace of the variable x has vanished. That is why, on both the sides we can put x instead of h, and then we get :

$$f(x) = (y) \text{ or } f(0) + \left(\frac{dy}{dx}\right)x + \left(\frac{d^2y}{dx^2}\right)\frac{x^2}{1\cdot2} + \text{ etc.} + \left(\frac{d^ny}{dx^n}\right)\frac{x^n}{1\cdot2\cdot3\cdots n} + \text{ etc.}$$

Or, as is usually written :

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{1\cdot2} + f'''(0)\frac{x^3}{1\cdot2\cdot3} + \text{ etc.},$$

as, for example, in the expansion of $f(x)$ or $(c+x)^m$:

$$(c+0)^m = f(0) = c^m,$$
$$m(c+0)^{m-1}x = mc^{m-1}x = f'(0)x \text{ etc.}$$

Afterwards, in the transition to Lagrange, I shall no more especially dwell upon the theorem of MacLaurin, which is only a particular instance of Taylor's theorem.

Here, let us only note further that like Taylor's theorem , it also has its own so called "exceptions".In the former, the exceptions always arise out of the irrational nature of the constant function, in the latter — from the similar nature of the variable function [79].

Now one may ask oneself : isn't the case like this, that Newton having disclosed to the world only his results — as he does for example, in the most difficult instances of "*Arithmetica universals*" — quietly extracted both Taylor's and MacLaurin's theorems, for his personal use, from the binomial theorem already discovered by him? To this it may be said with full confidence : that no, he was not one of those, who would give his pupils the opportunity to appropriate such a discovery *. In reality he was still too absorbed with the elaboration of those very differential operations, which were already assumed to be well known and available to Taylor and MacLaurin. This is testified by the first elementary formulae of his calculus. Clearly, Newton initially approached them from the points of departure of mechanics, and not from those belonging to pure analysis.

On the other hand,Taylor and MacLaurin operated upon the basis of differential calculus itself, from the very beginning of their work ; and that is why, nothing prompted them to seek the simplest possible algebraic points of departure of this calculus; the more so, as the controversy among the followers of Newton and Leibnitz revolved around the definitions of the ready-made forms of calculus, just discovered by an absolutely special kind of mathematical discipline , [forms], which are as far away as the stars in heaven, by way of ordinary algebra.

The connection of their respective *initial equations* with the binomial theorem, was, for them, in some way self-evident. But by far this connection was not understood by them, in a manner in which, for example, it is understood while differentiating xy or x/y , that these expressions are given by ordinary algebra.

The real, and accordingly the simplest, interconnection between the new and the old, is always discovered only when this new itself already attains its final form , and it may be said, that in the differential calculus this return(taking) backwards was carried out by the theorems of Taylor and MacLaurin. That is why, the idea of leading the differential calculus on to a strictly algebraic foundation, was conceived only by Lagrange. Perhaps in this respect he was preceded by *John Landen* , the mid-18th century English mathematician, in his "*Residual Analysis*" . But before forming a final opinion on this, I must first of all , go through this book in the Museum.

III. THE LAGRANGIAN THEORY OF FUNCTIONS

Lagrange proceeds from the algebraic foundations of Taylor's theorem, i.e., from the most general formula of differential calculus.

Regarding the initial equation of Taylor

$$y_1 \text{ or } f(x+h) = y \text{ or } f(x) + Ah + Bh^2 + Ch^2 + \text{ etc.}$$

* The publication of the eight volumes of the mathematical manuscripts of Newton edited by D.T. Whiteside *et al* (1967-1981) has changed our ideas on this issue. Newton did discover the expansions of Taylor and MacLaurin. —Tr.

only the following is to be noted :

1) This series is not at all proved ; $f(x+h)$ is not a binomial of any *definite* degree ; $f(x+h)$ is rather an indefinite general expression for every function [of the variable] x, this [x] increases by the positive or negative increment h ; thus $f(x+h)$ includes in itself functions of x of every degree, but at the same time it excludes every definite degree of that very series of expansion. That is why, Taylor puts "+ etc." at the end of the series. But it should be *proved* further, that the rule of expansion into a series, true for the definte functions of x, subject to increment — independently of the fact, whether they are now presented as a finite equation [80] or as an infinte series — can be unconditionally extended to the indefinite general $f(x)$, and that is why, also to the equally indefinite and general $f(x_1)$ or $f(x+h)$.

2) This equation is translated into the differential language with the help of a double differentiation of y_1 — first in respect of h as variable and x as constant, and then in respect of x as variable and h as constant.Thus two equations are obtained, of which the first sides [L.H.S.] are identical, whereas the second sides [R.H.S.] are different in form. But in order to equalise the indeterminate co-efficients of these second sides, which are, all, in point of fact, functions of x, it is assumed further — and this is indispensable — that the separate co-efficients A, B etc. are — *though indeterminate*, [*still*] *finite magnitudes*, and [it is to be] also [assumed] that the multipliers h accompanying them increase in *integral and positive powers*[81]. Even if it is assumed — which was in reality not the case — that Taylor proved all these for $f(x+h)$, in so far as x in $f(x)$ remains *general*, then from this it still does not at all follow, that it holds good even then, when the functions of x assume definte particular values. Conversely, the latter can be incompatible with the transformations [effected] with the help of his series

$$y_1 = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}h^2 + \text{etc.}$$

In brief : the conditions or assumptions, included in the unproved initial equation of Taylor, are contained, it goes without saying, also in the theorem

$$y_1 = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}h^2 + \text{etc.},$$

deduced from it. That is why it is not applicable to certain functions of x, which contradict these assumptions. Hence the so-called exceptions to the theorem.

Lagrange bases the initial equation upon algebraic foundations, and shows at the same time, by its very development, to which particuluar instances, contradicting its *general* character, i.e., the *general*, indeterminate character of the functions of x, it is inapplicable, owing to this character of theirs.

N) 1) The great merit of Lagrange lies not only in laying the foundations of Taylor's theorem and differential calculus in general through a purely algebraic analysis , but in particular also in the introduction of the very concept of derived functions, which all the successors of him have in fact used in some way or the other, though without ever

mentioning it. But he does not confine himself to this alone. He gives a purely algebraic expansion of all possible functions of $(x+h)$, in ascending integral positive powers of h, and then christens all the co-efficients thus obtained, with the names of differential calculus. All simplicities and short cuts, which the differential calculus itself allows (Taylor's theorem and others), thereby suffer a damage and are very often replaced by algebraic operations of a much more cumbersome and complicated character.

2) So far as it is a question of pure analysis, Lagrange is in fact free of all that appears to him as metaphysical transcendence in Newton's fluxions, Leibnitz's infinitesimals of various order, the theory of limits of vanishing magnitudes, the substitution of the symbol $\frac{0}{0}\left(=\frac{dy}{dx}\right)$ in place of the differential co-efficients etc. However, thereby he himself is not deterred from constantly using one or the other of these "metaphysical " notions, while he applies his theory to curves etc.

*2.FROM THE INCOMPLETE MANUSCRIPT "TAYLOR'S THEOREM"

Thus, if in Taylor's theorem [82], 1) for certain *specific forms* of the binomial theorem,where for $(x+h)^m$ — under the supposition that m is an *integral and positive power*, and that is why the multipliers in h are equal to h^0, h^1, h^2, h^3, etc. — it is accepted , that h [enters into] an ascending, positive and integral power, then it is [also] accepted, 2) that as in the algebraic binomial theorem of a *general* form, the *derived fuctions of* x are determinate , in as much as [they] are *finite functions in* x. But a third condition is added to this. The derived functions of x may turn into $0, +\infty, -\infty$, and $h^{[k]}$ may also become $= h^{-1}$ or $h^{m/n}$ (for example, $h^{1/2}$), only when the variable x takes a *particular value* , for example $x = a$ [83]. Let us sum up what has been stated : *Taylor's theorem* is generally applicable to the expansion in series of [those] functions in x, in which x becomes equal to $x+h$ or, increases, turning into x_1 from x, only if : 1) the independent varibale x retains the general *indeterminate* from of x, 2) the initial function in x is itself decomposable, through differentiation, into a series of determinate and, in so far as it is so, of finite derived functions in x with the corresponding multipliers h in ascending, positive and integral powers, i.e., h^1, h^2, h^3 etc.

But in other words, all these conditions are but expressions of the fact that, this theorem is merely the binomial theorem with *integral and positive indexes of power*,translated into the language of differential calculus.

Where these conditions are not fulfilled and , hence,*Taylor's theorem is not applicable*, there appears that [situation], which comes forth in the differential calculus as"*exceptions*" to this theorem.

But the biggest *mistake* of Taylor's theorem is not these particular exceptions to its applicability , but that *general mistake,* which consists of the fact that

$$y = f(x) \quad [\text{and}] \quad y_1 = f(x+h),$$

which are only symbolic expressions of binomials of some power[84], turn into such expressions , in which $f(x)$ is a function of x, which includes all powers in itself and that is why, it itself *has no power*, such that $y_1 = f(x+h)$ also includes in itself all powers and itself has no power, appearing as it were, as an *unexpandable general expression* for any function of the variable x , when the latter increases. That is why, that series of expansion, which serves as an expansion of this $f(x+h)$ without power, namely, $y = Ah + Bh^2 + Ch^3 +$ etc ., includes in itself all the powers, whereas it itself has no power.

This leap from the *ordinary algebra*, and besides *with the help of ordinary algebra*, into the *algebra of variables*, is accepted as an *accomplished fact ;* it is not proved and , first of all, it *contradicts all the rules* of ordinary algebra, where $y = f(x)$ and $y_1 = f(x+h)$ can never have this meaning.

In other words: not only is the initial equation

$$y_1 \text{ or } f(x+h) = y \text{ or } f(x) + Ah + Bh^2 + Ch^3 + Dh^4 + Eh^5 + \text{etc.}$$

not proved, but — consciously or unconsciously — the substitution of *variables* in place of the *constants,* is also assumed. This contradicts all the rules of algebra, since algebra , and hence also the algebraic binomial, admits only constants, and besides constants of merely

two kinds — *known and unknown*. That is why, the deduction of this equation from algebra rests upon a fraud.

However, if in fact *Taylor's theorem* — the exceptions to which have hardly any significance for applications, since actually they are confined to such functions in x, which are undifferentiable [85],i.e., do not at all yield to a treatment with the means of differential calculus — in parctice showed itself to be the most comprehensive, the most general and fruitful *operational formula* of the whole of calculus, then it is only due to the accomplishment of that entire task, which arose, from the school of Newton to which he [Taylor] belonged, and generally, from the entire Newtono-Leibnitzian period of the developmet of differential calculus, which from its very first step elicits correct result from mistaken premises.

Lagrange gave us the algebraic proof of Taylor's theorem , and genrally speaking based it upon *his* algebraic method of differential calculus . If I write the historical part of this manuscript [86], then I shall dwell upon this in detail.

Here — as a *freak of history* — I only mention that, Lagrange never returned, to that which unconsciously served as the bais of Taylor, i.e., to the binomial theorem, and besides in its simplest form, where it [i.e., the binomimal] consists of only two magnitudes $(x + a)$ or, here $(x + h)$, and has positive integral index of power.

Further, in a much lesser measure does he return farther backward and ask himself the question as to why the Newonian binomial theorem translated into the differential form, and at the same time forcibly freed from its algebraic conditions, appears as the all comprehensive general operational formula of the differential calculus, based upon it .The answer is simple: because from the very beginning Newton assumes that $x_1 - x = dx$ and, hence $x_1 = x + dx$. Thus the expansion of the *difference* instantly turns into the expansion of a *sum*, into an expansion of the binomial $(x + dx)$ (wherein we entirely digress from the fact that it should have been written as $x_1 - x = \Delta x$ *or* h) (hence, $x_1 = x + \Delta x$ *or* $x + h$). Taylor merely developed this basis of the system into its most general and comprehensive form, which , generally speaking, became possible, for the first time, when all the fundamental operations of differential calculus were already discovered, because what meaning would his $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc. have, if for all the important fuctions in x their corresponding $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc. were not already obtained ?

Conversely, Lagrange immediately sides with Taylor's theorem — naturally under such circumstances, where , on the one hand, the successors of the epoch of Newton-Leibnitz furnished him with the already corrected version [of the formula] $x_1 - x = dx$, that is, also : $y_1 - y = f(x + h) - f(x)$ and, on the other hand, having just algebricised Taylor's formula, he constructed his own theory of *"derived" functions*. [[Thus did Fichte side with Kant, Schelling — with Fichte and, Hegel with — Schelling, wherein neither Fichte , nor Schelling, nor Hegel did investigate the general basis of Kant, of idealism in general ; or else they could not have developed in further.]]

*APPENDIX TO THE MANUSCRIPT "ON THE HISTORY OF DIFFERENTIAL CALCULUS" ANALYSIS OF D'ALEMBERT'S METHOD

*ON THE NON-UNIVOCALITY OF THE TERMS "LIMIT" AND "LIMITING VALUE " 87

I) x^3;

a) $(x+h)^3 = x^3 + 3hx^2 + 3h^2x + h^3$;

b) $(x+h)^3 - x^3 = 3hx^2 + 3h^2x + h^3$;

c) $\dfrac{(x+h)^3 - x^3}{h} = 3x^2 + 3xh + h^2.$

When h becomes $= 0$, then

$\dfrac{(x+0)^3 - x^3}{0}$ or $\dfrac{x^3 - x^3}{0} = \dfrac{0}{0}$ or $\dfrac{dy}{dx}$ and the right hand side $= 3x^2$,

hence,

$$\frac{dy}{dx} = 3x^2 .$$

$y = x^3$; $y_1 = x_1{}^3$

$y_1 - y = x_1{}^3 - x^3 = (x_1 - x)(x_1{}^2 + x_1 x + x^2)$;

$\dfrac{y_1 - y}{x_1 - x}$ or $\dfrac{dy}{dx} = x^2 + xx + x^2$;

$$\frac{dy}{dx} = 3x^2.$$

II) If we assume that $x_1 - x = h$, then,

1) $(x_1 - x)(x_1{}^2 + x_1x + x^2) = h(x_1{}^2 + x_1x + x^2)$;

2) hence,

$$\frac{y_1 - y}{h} = x_1^2 + x_1x + x^2.$$

In 1) the co-efficient of h *is not a ready-made derivative*, like f' above, but f^1; that is why the division of both the sides by h does not give $\dfrac{dy}{dx}$, but gives

$\dfrac{\Delta y}{h}$ *or* $\dfrac{\Delta y}{\Delta x} = x_1^2 + x_1x + x^2$ etc. etc.

If, on the other hand, in I c), i.e., in

$\dfrac{f(x+h) - f(x)}{h}$ or $\dfrac{y_1 - y}{h} = 3x^2 + 3xh + h^2,$

we start from the supposition that, on the right hand side, the value of the terms $3xh + h^2$ diminishes further and further, commensurately with the dimunition of the value of h [88], and hence, the value of the entire right hand side, $3x^2 + 3xh + h^2$, becomes more and more proximate to the value of $3x^2$, then , however, it should be added that : it [this proximation] does happen, without ever conciding with it [$3x^2$].

Thus, $3x^2$ becomes the value to which the series $3x^2 + 3xh + h^2$ constantly approximates, never attaining it , and what is more, consequently, never going beyond it. In this sense $3x^2$ becomes the *limiting value* [89] of the series $3x^2 + 3xh + h^2$.

On the other hand, the magnitude $\frac{y_1 - y}{h}\left(\text{or } \frac{y_1 - y}{x_1 - x}\right)$ also gets diminished all the more, when the denominator h gets diminished [90].

But since $\frac{y_1 - y}{h}$ is the equivalent of $3x^2 + 3xh + h^2$, so the limiting value of this series is also its proper *limiting value* — in that very sense , in which it serves as the limiting value of the series equivalent to it.

However, as soon as we assume that $h = 0$, in the right hand side, the terms , which made $3x^2$ the limit of its value, vanish; now $3x^2$ is the first derivative of x^3 and, hence $= f'(x)$. As $f'(x)$ it shows that, from it, in its turn, $f''(x)$ may be derived (in the given case, which $= 6x$) etc., that, consequently, the increment $f'(x)$ or $3x^2$ is not equal to the sum of the possible increments of the expanded $f(x) = x^3$. Had $f(x)$ itself been an infinte series, then, of course, so would have been the series of all possible increments obtained from it.But in this sense the expanded series of increments, as soon as I stop it abruptly, would be the *limiting value* of its expansion, consequently, here the *limiting value*, is the *limit* in that usual algebraic or arithemtic sense, according to which the expanded part of an infinite decimal fraction is the *limit* of its possible expansion.This limit is sufficient for practical or theoretical considerations.This has nothing in common with the limiting value in the first sense.

Here, in the second sense, the limiting value *can be increased* at will, whereas there [the value of the expression] can only *diminish*. Further, so long as h only diminishes

$$\frac{y_1 - y}{h} = \frac{y_1 - y}{x_1 - x}$$

can only approximate the expression $\frac{0}{0}$; the latter is the limit, which this ratio can never attain and what is more [can never] cross, in so far as $\frac{0}{0}$ may be considered as its *limiting value* [91] .

But as soon as $\frac{y_1 - y}{h}$ turns into $\frac{0}{0} = \frac{dy}{dx}$, the latter ceases to be the limiting value of $\frac{y_1 - y}{h}$, for this latter [expression] has itself vanished into its limit [92]. In respect of its earlier

form $\frac{y_1 - y}{h}$ or $\frac{y_1 - y}{x_1 - x}$, it can only be said, that $\frac{0}{0}$ is its absolutely minimal expression, which, isolatedly considered, is no expression at all, it has no value; but now $3x^2$, i.e., $f'(x)$ stands opposite the expression $\frac{0}{0}\left(\text{or } \frac{dy}{dx}\right)$, as its real equvalent.Thus, in the equation

$$\frac{0}{0}\left(\text{ or }\frac{dy}{dx}\right) = f'(x),$$

neither of the two sides is the limiting value of the other.They are situated, not in a *limiting relation* to each other, but in an *equivalence relation.*

If I have $\frac{6}{3} = 2$, then neither is 2 the limit of $\frac{6}{3}$, nor $\frac{6}{3}$ — the limit of 2. It would be a banal tautology to assert, that the value of any magnitude is equal to the limit of its value.

Thus , perhaps the concept of limiting value has been incorrectly interpreted, and is constantly so interpreted. In application to the differential equations[93], as a means preparatory to the supposition of $x_1 - x$ or h equal to zero and, to make the latter more graphic, it is infantile; its emergence should be sought in the first mystical and mystificatory method of calculus. In application, however, of the differential equations to curves etc., it actually serves the purpose of gemoetrically graphic representation.

*COMPARISON OF D'ALEMBERT'S METHOD WITH THE ALGEBRAIC METHOD

Let us compare the method of d'Alembert with the algebraic one [94].

I) $f(x)$ or $y = x^3$;

a) $f(x+h)$ or $y_1 = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$;

b) $f(x+h) - f(x)$ or $y_1 - y = 3x^2h + 3xh^2 + h^3$;

c) $\frac{f(x+y) - f(x)}{h}$ or $\frac{y_1 - y}{h} = 3x^2 + 3xh + h^2$;

if $h = 0$, then :

d) $\frac{0}{0}$ or $\frac{dy}{dx} = 3x^2 = f'(x)$.

II) $f(x)$ or $y = x^3$;

a) $f(x_1)$ or $y_1 = x_1{}^3$;

b) $f(x_1) - f(x)$ or $y_1 - y = x_1{}^3 - x^3 = (x_1 - x)(x_1{}^2 + x_1x + x^2)$;

c) $\frac{f(x_1) - f(x)}{x_1 - x}$ or $\frac{y_1 - y}{x_1 - x} = x_1{}^2 + x_1x + x^2$.

When x_1 becomes $= x$, then $x_1 - x = 0$, hence :

d) $\frac{0}{0}$ or $\frac{dy}{dx} = (x^2 + xx + x^2) = 3x^2$.

In both [the methods] one and the same [thing happens] : if the independent variable x increases, then the dependent variable y also increases. Everything is reduced to the manner of expressing the increase of x. When x become x_1 , then $x_1 - x = \Delta x = h$ (an indeterminate, infinitely diminishable difference, which however, always remains finite)[95]. Δx or h is the increment , by which x has increased, for :

a) $x_1 = x + \Delta x$, but also, conversely,

b) $x + \Delta x$ or $x + h = x_1$.

Differential calculus historically starts from a), i.e., from [the position] that, the difference Δx or the increment h (both express one and the same thing, the first — negatively, as the difference Δx, the second — positively, as the increment h) *exists independently* along with the magnitude x , of which it is the increment, and hence, which it expresses as *increasd ,* and increased by h. With this, from the very beginning an advantage is gained, which — in consonance with the expression of the initial function of the variable, as soon as the latter increases — is expressed in binomials of determinate power, and that is why, from the very beginning, the binomial theorem becomes applicable to it. Actually, already in the general left hand side, we have the binomial, namely $x + \Delta x$, [such that $f(x + \Delta x)$] or $y_1 =$ etc.

Mystical differential calculus at once turns: $x + \Delta x$ into $(x + dx)$ or, according to Newton, into $x + \dot{x}$ [96]. Owing to this, also on the right algebraic side, we at once get the binomials $x + dx$ or $x + \dot{x}$, which are then treated as ordinary binomials. Instead of being derived mathematically, the transformation of Δx into dx or $\dot{x}$ is assumed *a priori ;* that is why, afterwards, the mystical rejection, of certain terms of the expanded binomials, becomes possible.

d'Alembert starts from $(x + dx)$, but corrects this expression, changing it into $(x + \Delta x)$, and correspondingly into $(x + h)$; now a development becomes indispensable, with the help of which Δx or h turns into dx, but the entire development, which actually takes place, is reduced to this.

Whether one starts incorrectly from $(x + dx)$ or correctly from $(x + h)$, the substitution of this indeterminate binomial in the given algebraic [powered] function of x, turns it into a binomial of some determinate power, just as in I a), in place of x^3 there appears $(x + h)^3$, besides in the binomial — where, in one case dx, and in the other h, figures as its last term, and hence, also in the expansion of this binomial — only in the form of a multiplier, externally attached to the functions to be derived, with the help of the binomial.

That is why *already* in I a) we find the *first derivative* of x^3 in a ready-made form, namely $3x^2$ as a co-efficient in the second term of the series, endowed with the multiplier h. From this moment $3x^2 = f'(x)$ remains invariable. This derivative is in no way obtained as a result of some process of differentiation, but from the very beginning, it is given by the binomial theorem, and besides, because, from the very beginning we presented the increased x in the form of the binomial $x + \Delta x = x + h$, i.e., as x increased by h. Now the whole task consists of freeing not some merely embryonically existing $f'(x)$, but an entirely ready-made one, from its multiplier h and from the other neighbouring terms.

Conversey in II a), the increased x_1 enters into the algebraic function exactly in that form, in which initialy x entered into it; x^3 turns into $x_1{}^3$. Thus the derivative $f'(x)$ can be obtained only as a result of two successive differential operations, and besides, [operations] of entirely different character.

In the equation I b), though the difference $f(x + h) - f(x)$ or $y_1 - y$, also paves the way for the appearence of symbolic differential co-efficient, this difference does not introduce any change in the real co-efficient, it is only shifted from the second place in the series to the first, and that is why it becomes possible to free it from h.

In II b) we get the expression of differences in both the sides; in the algebraic side, the difference is so expanded, that $(x_1 - x)$ appears in the form of a multiplier of some derived function in x and x_1, obtainable by dividing $x_1^3 - x^3$ *by* $x_1 - x$. Only the presence of the difference $x_1^3 - x^3$, made its expansion into two factors possible.Since $x_1 - x = h$, so the multipliers, into which $x_1^3 - x^3$ is expanded, could also be writtern in the form of $h\,(x_1^2 + x_1 x + x^2)$. Herein emerges the novelty, which distinguishes it from I b). As a *multiplier in the preliminary derivative*, h itself is deduced only with the help of an expansion of the difference $x_1^3 - x^3$ into a product of two multipliers, whereas h as a multiplier in the "derivative", as it itself is in I a), already exists in the ready-made form, even before any difference whatever was formed. That the indeterminate increment of x into x_1 atttains beside x the solitary form of the multiplier h, is implied in I) from the very beginning; in II) however (since $x_1 - x = h$), it is proved deductively. Though in I), on the one hand h is indeterminate, however, on the other hand, it is determinate all the same, in so far as the indeterminate increment of x already appears as an *independent* magnitude, *by which* x has increased, and which, that is why, as such appears beside it.

Further in I c) $f'(x)$ is freed from its multiplier h ; in the left hand side we get $\frac{y_1 - y}{h}$ or, $\frac{f(x+h) - f(x)}{h}$, i.e., some expression of the differential coefficient, still finite. But we get it on the other side, when in $\frac{f(x+h) - f(x)}{h}$ we assume $h = 0$, consequently we turn it into $\frac{0}{0} = \frac{dy}{dx}$. In I d), on the one hand we get the symbolic differential co-efficients, and on the other, $f'(x)$, already existing in a ready-made form in I a), now freed from its neighbouring terms, and it figures alone in the right hand side.

Positive development takes place only in the left hand side, for it is here, that the symbolic differential co-efficient is obtained. In the right hand side the development is confined only to freeing $f'(x) = 3x^2$ — which was found already in I a) with the help of the binomial — from its initial accompaniments. In the right hand side, the turning of h into 0 or $x_1 - x = 0$ has only this negative purport.

Conversely, in II c) at first, some *preliminary derivative* is obtained by dividing both the sides by $x_1 - x (= h)$.

Finally, in II d) the positive assumption of $x_1 = x$ gives us the *final derivative*. But this assumption of $x_1 = x$, at the same time stands for the assumption of $x_1 - x = 0$, and owing to this, in the left hand side the finite ratio $\frac{y_1 - y}{x_1 - x}$ is turned into $\frac{0}{0}$ or $\frac{dy}{dx}$.

In I) the search for the "derivative" is as little felicitated by the assumption $x_1 - x = 0$ or $h = 0$, as it was in the mystical differential method. In both the cases, the accompanying terms are removed from the path of $f'(x)$ — emergent at once in a ready-made form. Now this removal is mathmatically correct, there it was done by a *coup d'état*.

*ANALYSIS OF D'ALEMBERT'S METHOD IN THE LIGHT OF YET ANOTHER EXAMPLE [97]

Let us operate according to the method of d'Alembert :

a) $f(u)$[98] or $y = 3u^2$;
b) $f(x)$ or $u = x^3 + ax^2$.

$$y = 3u^2, \tag{1}$$

$$f(u) = 3u^2. \tag{1a}$$

$$f(u + h) = 3(u + h)^2,$$

$$f(u + h) - f(u) = 3(u + h)^2 - 3u^2 = 3u^2 + 6uh + 3h^2 - 3u^2 = 6uh + 3h^2 \tag{2}$$

(here the derived function is given by the binomial theorem in a ready-made form — in the form of the co-efficient of h),

$$\frac{f(u + h) - f(u)}{h} = 6u + 3h\ .$$

with the help of this division $f'(u) = 6u$, given already in a ready-made form in (2), is freed from its multiplier h :

$$\frac{f(u + 0) - f(u)}{h} = 6u,$$

$$\frac{y_1 - y}{u_1 - u}\ ,\ \text{then}\ \frac{0}{0} = \frac{dy}{du} = 6u.$$

Here, if we put the value of u from the equation b), we shall have

$$\frac{dy}{du} = 6(x^3 + ax^2).$$

Since in a) y is differentiated in respect of u ,so

$$(u_1 - u) = h\ ,\ \text{or}\ \ h = (u_1 - u),$$

because, u is the independent variable.

Thus,

$$\frac{dy}{du} = 6(x^3 + ax^2).$$

(This is obtained from $f(u)$ or $y = 3u^2$).

[Now we shall operate with b) ; according to the same method , namely :]

b) $f(x)$ or $u = x^3 + ax^2$,

$$f(x + h) = (x + h)^3 + a(x + h)^2,$$

$$\begin{aligned} f(x + h) - f(x) &= (x + h)^3 + a(x + h)^2 - x^3 - ax^2 = \\ &= \left.\begin{aligned} &x^3 + 3x^2h + 3xh^2 + h^3 \\ &+ ax^2 + 2axh + ah^2 \end{aligned}\right\}\begin{aligned} &-x^3 \ + \\ &-\ ax^2 = \end{aligned} \\ &= (3x^2 + 2ax)\,h + (3x + a)\,h^2 + h^3, \end{aligned}$$

$$\frac{f(x + h) - f(x)}{h} = 3x^2 + 2ax + (3x + a)\,h + h^2.$$

If now we assume that $h = 0$, then in the second side [R.H.S.]

$\frac{0}{0}$ or $\frac{du}{dx} = 3x^2 + 2ax.$

But the derived function $3x^2 + 2ax$ is already contained in a ready-made form in

$f(x+h) = (x+h)^3 + a(x+h)^2,$

since the latter gives,

$x^2 + 3x^2h + 3xh^2 + h^3 + ax^2 + 2axh + ah^2,$

hence,

$x^3 + ax^2 + (3x^2 + 2ax)\,h + (3x + a)\,h^2 + h^3.$

It already appears as the ready-made coefficient of h. Hence, this derivative is not obtained through differentiation; but owing to the increment of $f(x)$ into $f(x+h)$, i.e., of $x^3 + ax^2$ into $(x+h)^3 + a\,(x+h)^2$.

It is obtained simply owing to the fact, that with the transformation of x into $x + h$, on the other side we obtain the binomials $x + h$ in determinate powers , besides the second term, with the multiplier h, contains the derived function of u or $f'(u)$ in a ready-made form.

All further procedures lead only to the freeing of $f'(x)$ given from the very beginning , from its coefficient proper h, and from all the other remaining terms.

The equation

$$\frac{f(x+h) - f(x)}{h} = \text{etc.}$$

has a two-fold significance : firstly, it allows us to obtain the numerator of the first side [L.H.S.] in the form of a difference [of the values] of $f(x)$, [i.e.,] at first as equal to $\Delta f(x)$; in the second side [R.H.S.] it gives only an algebraic advantage , permitting the removal of the initial function $x^3 + ax^2$ etc., given in x, from the result of fulfilling the operations in $(x+h)^3 + a(x+h)^2$.

But let us proceed further. For a) we got

$\frac{dy}{du} = 6(x^3 + ax^2),$

and for b)

$\frac{du}{dx} = 3x^2 + 2ax.$

Multiplying $\frac{dy}{du}$ by $\frac{du}{dx}$, we got

$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx},$

i.e., the unknown[sought]. Here, in place of $\frac{dy}{du}$ and $\frac{du}{dx}$,let us put the values obtained for them and we shall get $\frac{dy}{dx} = 6(x^3 + ax^2)\,(3x^2 + 2ax),$

and, generally speaking , if we have ;

$$y = f(u), \frac{dy}{du} = \frac{df(u)}{du}, u = f(x), \frac{du}{dx} = \frac{df(x)}{dx},$$

then,

$$\frac{dy}{du} \cdot \frac{du}{dx} \text{ or } \frac{dy}{dx} = \frac{df(u)}{du} \cdot \frac{df(x)}{dx}.$$

If in the equation a) we assume that $h = u_1 - u$, and in the equation b) $h = x_1 - x$, then the affair takes the following form :

y or $f(u) = 3u^2$,

$f(u + (u_1 - u)) = 3(u + (u_1 - u))^2 = 3u^2 + 6u(u_1 - u) + 3(u_1 - u)^2$,

$f(u + (u_1 - u)) - f(u) = 3u^2 + 6u(u_1 - u) + 3(u_1 - u)(u_1 - u) - 3u^2$,

hence ,

$f(u + (u_1 - u)) - f(u) = 6u(u_1 - u) + 3(u_1 - u)^2$,

$$\frac{f(u + (u_1 - u)) - f(u)}{u_1 - u} = 6u + 3(u_1 - u).$$

Hence, [if] in the first term $u_1 - u = 0$, then

$$\frac{dy}{du} = 6u + 0 = 6u.$$

This shows that, if from the very beginning $f(u)$ turns into $f(u + (u_1 - u))$, such that in the second side [R.H.S.], its increment appears in the form of the positive second term of the determinate binomial, then the second term, which has as its multiplier $(u_1 - u)$ or h according to the binomial theorem, instantly shows the unknown coefficient. If the second term is a polynomial, as we see in $x^3 + ax^2$, turning into $(x + h)^3 + a(x + h)^2$ or in

$$(x + (x_1 - x))^3 + a(x + (x_1 - x))^2,$$

then for obtaining the co-efficient of h, or of $x_1 - x$, only the terms with $x_1 - x$ (or h) in the first power are required to be written down — and the co-efficient is ready . This result shows :

1) that if, in d'Alembert's expansion, in place of $x_1 - x = h$, conversely we put $h = x_1 - x$, then, by this absolutely nothing is changed in the method itself, only the peculiarity of this method is revealed more clearly. It consists of the fact that, with the help of $f(x + h)$ or $f(x + (x_1 - x))$ in place of the initial function in the algebraic expression on the second side, in the given instance in place of $3u^2$, the binomials are obtained instantly.

The second term with the multiplier h or $x_1 - x$, thus obtained, is the ready-made first derivative of the function. Now the task consists of freeing it from h or $x_1 - x$, which has already been made easy. The derivative is already present in a ready-made form; hence , it

is not sought by assuming $x_1 - x = 0$, but is freed from its multuplier $(x_1 - x)$ and from others. Just as it is obtained, by simple multiplication (binomial expansion) as the second term [with the multiplier] $x_1 - x$, [so] is it freed, in the final count, from the latter, through a division of both the sides by $x_1 - x$. The intervening operations consist of an expansion of the equation $f(x+h) - f(x)$ or $f(x + (x_1 - x)) - f(x) = [\cdots]$.

This equation is necesary only to force the disappearance of the initial function in the second side [R.H.S.], since the expansion of $f(x + h)$ inevitably contains $f(x)$ along with its binomially expanded increment. Thus, these [terms, corresponding to the initial function] are removed from the second side [R.H.S.].

Consequently, what happens, for example, in

$$(x+h)^3 + a(x+h)^2 - x^3 - ax^2,$$

consists of the removal of the first terms x^3 and ax^2 from the binomials $(x+h)^3 + a(x+h)^2$; thus, we obtain, the ready-made derived function endowed with the multiplier h or $x_1 - x$ as the first term of the equation.

In the second side [R.H.S.], the first differentiation is nothing but a simple subtraction of the initial function from its increased expression ; hence, it gives us the increment, by which it has increased ; besides, its first term, endowed with the multiplier h, is already the ready-made derived function. The other terms cannot contain anything else, apart from the co-efficients of h^2 or $(x_1 - x)^2$ etc. ; the first division by $x_1 - x$ in this and in the other side lowers the indices of power of the latter by one unit; besides the first term will appear without h.

2) The difference from the method of $f(x_1) - f(x) =$ etc., consists of the fact that, if, for example, we get $f(x)$ or $u = x^3 + ax^2$, when $f(x_1)$ or $u_1 = x_1{}^3 + ax_1{}^2$ — [then] the first increment of the variable x, by no means gives us a ready-made $f'(x)$, from the very beginning

[by forming the difference $f(x_1) - f(x)$ we get]

$$f(x_1) - f(x) \quad \text{or} \quad u_1 - u = x_1{}^3 + ax_1{}^2 - (x^3 + ax^2).$$

Here the issue is not one of again removing the initial function,since $x_1{}^3 + ax_1{}^2$ does not contain x^3 and ax^2 in any form. Conversely, the first difference equation gives us a certain moment of development, namely, the transformation of each of the two initial terms in the difference [of the powers] of x_1 and x. Namely :

$$[u_1 - u] = (x_1{}^3 - x^3) + a(x_1{}^2 - x^2).$$

Now it is already clear, that if in each of these two terms we again isolate the multiplier $x_1 - x$, then as coefficients of $x_1 - x$ we shall get functions in x_1 and x, namely:

$$f(x_1) - f(x) \quad \text{or} \quad u_1 - u = (x_1 - x)(x_1{}^2 + x_1x + x^2) + a(x_1 - x)(x_1 + x).$$

Having divided this, and hence also the left hand side by $x_1 - x$, we shall get

$$\frac{f(x_1) - f(x)}{x_1 - x} \quad \text{or} \quad \frac{u_1 - u}{x_1 - x} = (x_1{}^2 + x_1x + x^2) + a(x_1 + x).$$

With the help of this division we obtained the preliminary derivative.Each of its parts contains terms with x_1.

Hence , we may finally obtain the first function in x, subject to deduction, only having put $x_1 = x$, hence $x_1 - x = 0$. Then

$x_1{}^2 = x^2$, $x_1 x = x^2$

and, hence,

$(x_1{}^2 + x_1 x + x^2) = 3x^2$ and $x_1 + x = x + x = 2x$,

whence,

$a(2x) = 2ax$.

In the other [side] the result is

$\frac{df(x)}{dx} = \frac{du}{dx} = \frac{0}{0}$.

Hence, here the derived function is obtained only by assuming $x_1 = x$, i.e., $x_1 - x = 0$. [The equality] $x_1 = x$ gives the final determinate result in the form of a proper function of x.

But $x_1 = x$ also gives $x_1 - x = 0$, and that is why at the same time, along with this determinate result, in the other side [it] gives us the symbolic [expression]

$$\frac{0}{0} \text{ or } \frac{dy}{dx}.$$

It could have been said earlier that : in the end we must get the *derivative* in x_1 and x. It can only be transformed into a derivative in x, as soon as we put $x_1 = x$, but to assume $x_1 = x$ is the same thing as assuming $x_1 - x = 0$. This turning into zero, positively expressd in the formula $x_1 = x$, is essential for turning the derivative into a function of x , whereas the negative form of $x_1 - x = 0$ must give us the symbol.

3) Even if this treatment of x, where its increment, for example, $x_1 - x = \Delta x$ or h, is not introduced along with it independently, was already known, even then it is highly likely — and after seeing [the works of] J. Landen in the Museum, I shall be able to convince myself about it — that all the same, its essential distinction from the other treatments could not be understood.

The difference of this method from [the method of] Lagrange consists of this : in the given method a proper differentiation is carried out, on the strength of which the differential expressions appear also in the symbolic side, whereas with him the deduction does not present the differentiation algebraically, but it algebraically deduces the functions immediately from the binomial, and their differential form is introduced only for the purpose of "symmetry", since from the differential calculus it is well known that, the first derivative $= \frac{dy}{dx}$, and the second $= \frac{d^2y}{dx^2}$ etc.

PART II

DESCRIPTION OF THE MATHEMATICAL MANUSCRIPTS

MANUSCRIPTS OF THE PERIOD PRIOR TO 1870

ARITHMETICAL AND ALGEBRAIC CALCULATIONS AND GEOMETRICAL DRAWINGS IN THE NOTE BOOKS ON POLITICAL ECONOMY

It seems that for the period prior to the 60s of the last century there are no well connected mathematical manuscripts of Marx. In some note books of excerpts on political economy, there are separate pages containing mathematical calculations. But even there, where the note book has the dating in Marx's hand, it is difficult to ascertain the time of these calculations. It is quite possible that certain blank pages were left in a note book and, Marx later on used this blank space of an old note book, for mathematical calculations. The following manuscripts consist of such pages containing calculations. They do not contain any text.

S.U.N. 147

It is a note book with excerpts on political economy (from Shutz, List Ociander and Ricardo), dated 1846. On the last pages(64-71) of this note book there are some algebraic calculations related to the generalisation of the concept of power in the cases of fractional and negative indices, to the exponential function and logarithms, to combinatorics and, the binomial of Newton. In these sheets there are no texts in words.

S.U.N. 210

It is a note book on political economy, containing excerpts from Kenny's book. Since the excerpts are from a book, which was published in 1846, so, in any case, this note book can not be dated earlier than that year.

On the sheets 12-17 there are mathematical calculations. Their content is not very clear. At first we find an equality

$10 : 2 = (10 + 3\frac{1}{3}) : (2 + \frac{2}{3})$, and a system of two equations

$10 : 2 = (10 + \frac{4}{x}) : (2 + \frac{4}{y}),\ \frac{4}{x} + \frac{4}{y} = 4,$

connected with it. Solution of the system turns the first equation exactly into the equality cited above. Further, $(x + a)^6$ and $(x + a)^5$ are expanded according to Newton's binomial. There are other systems of equations with two or three unknowns, of some sort (it is difficult to say of what sort).

On sheet 16 there are two diagrams : of a circle and, apparently, of a parabola. Here also we have equations of the straight line and the circle, and some arithmetical calculations, which continue in sheet 17 (in particular, division of the number 15911729 by 2 ; of the obtained result, i.e., of 7955864 by 4 ; of the number 1988966, obtained as a result of that division, by 751195; it is difficult to follow the calculation further).

Thus, it is clear, that Marx by chance retained these calculations. Here they are being mentioned only to make the picture complete.

S.U.N. 1052

We have in view sheet No. 36 of the note book entitled "M", containing the "Introduction" and index of seven note books of preparatory work, for the book "Critique of Political Economy". This note book has the dates : 23/VIII and IX/1857, first half of June 1858.

This sheet contains certain calculations, having in view some expansions in series, and solution of the problem of putting n arithmetic means between the numbers a and b (at first generally, and then, when $a = 1$, $b = 23$ and $n = 10$).

S.U.N. 1153

Sheets 15 and 17 of the above mentioned note book containing the thematic indexes of the note books I-VII related to the preparatory work leading to the book "Critique of Political Economy". Apparently half of it initially remained unused. Here we have :

on sheet 15 — geometrical drawings : of a rectangle and a triangle ;

on sheet 17 — fractional indices of power and logarithms, two triangles and a square, division of triangles with a common vertex at some internal point.

In manuscript 497, we find, for the first time, an well ordered text with historico-mathematical contents.

NOTES AND EXTRACTS FROM POPPE'S BOOK ON THE HISTORY OF MATHEMATICS AND MECHANICS

S.U.N. 497

Among the photocopies of a note book containing extracts on the history of technology, taken down by Marx in September-October 1851, there are three sheets : 19-21(in Marx's numeration pp. 10-11), containing notes taken from some parts of the book : J.H.M. Poppe, "Geschichte der Mathematik seit der ältesten bis auf die neueste Zeit", Tübingen, 1828 (J.H.M. Poppe, "History of Mathematics from the most ancient to the most modern time", Tubingen, 1828). This note is a very short description of the introduction to this book and of some informations about the history of pure and applied mathematics. To give an idea of what, namely, drew Marx's attention in Poppe's book, here we reproduce Marx's text in full.

Introduction. The method by which the ancient Egyptians determined the height of pyramids in terms of the length of their shadows, itself shows how incomplete was the mathematics of the Chaldeans and the Egyptians. The Greeks are our teachers in mathematics. Plato invented geometrical analysis. Euclid, 284 B.C., studied in Athens, under the Platonists. After him little has changed in elementary geometry. Roman mathematicians were mere translators and commentators of the celebrated Greek authors. Towards the 7th century the mathematical sciences flourished in the countries under Arab rule, and later on, in those under the Persians. The Moors brought them to Spain, and from there, they spread into the rest of Europe. In the 10th, 11th, 12th and 13th centuries mathematics found its refuge only with the Arabs.Astronomy was especially cultivated by them. They translated Euclid, Apollonios, Archimedes and others. *Roger Bacon* in the latter half of 13th century (1-14).

The numbers (1-14) indicate the corresponding pages of Poppe's book. The extracts that follow have been subdivided, as in the book into two parts : History of Pure and of Applied Mathematics. However, Marx took a more or less detailed note, of only a part related to the history of arithmetic or "the art of counting". On this we read in the manuscript :

First Part. History of Pure Mathematics.

1) History of Arithmetic, or the art of counting.

The Phoenicians. Even the most ancient people, to the exclusion of the Chinese and the Tatars, counted in tens. To all appearance, it was suggested to them by the fingers on both the

hands. The letters of their alphabet served them as *numerical signs*. Different powers of tens were distinguished by strokes, as with the Greeks, or by suitable combinations of letters, as with the Romans. The so-called Arabic numerals are 1, 2, 3, 4, 5, 6, 7, 8, 9. It is one of the most beautiful discoveries. By using them, the biggest number can be written down with the help of zero and definite place indications. It came to Europe in the 10th or 11th century, through the *Arabs*. Even Archimedes had to deal with very big numbers, for this he applied orders of ten thousand, or myriads. But with this he could not carry out the computation of the circumference of a circle further than the limits of $3\frac{1}{7}$ to $3\frac{10}{70}$, taking the diameter of a circle as the unit. Initially the Arabic numerals and their place values were used only by the mathematicians, in no way were they used in ordinary life. In the 15th century even in the source materials, these numerals were still very rare : till then most often the Roman numerical symbols were used. Arabic numerals became more common place, only from the middle of the 16th century. In the 15th century these numerals were more to be seen on stones, than on parchment. They still remained nearly unused in printed publications. In the older printed books even the year is almost always indicated in words or by Roman letters. Thus, in Roman times and later on even the small computations, for example, agricultural or commercial calculations, were carried out not with the help of numerals but, with pebbles and other analogous symbols on the *computation-board.* On it, a few parallel lines were drawn ; and there one and the same pebble or some other symbol on the first line signified units, on the second — tens etc.

Ancient number games. Superstitions. As, also, in more recent times, especially in the 16th century. *Discovery of the Pythagoreans* : a multiplication table (true, very inconvenient and cumbersome), polygonal-pyramidal etc., trivial and corporeal numbers in general ; and also calculation of musical ratios. The Greeks knew the four operations of arithmetic, as well as the properties of geometrical ratios and proportions, arithmetic and geometric progressions and, the doctrine of those magnitudes, whose ratio can not be exactly expressed in numbers. And also the means of extracting the quadratic and the cubic roots. Towards the end of the 16th century , extraction of roots — also approximate when they are irrational — was carried forward further than the way it was done earlier, when simply the fractions attached to the whole numbers indicating the root , were considered enough. For this Simon Stewin used decimal fractions. *Naming and designating powers* created a lot of trouble in the ancient period... Partnership rules etc., were not rare in the 16th century. At that time began the computation, of compound interests on capital... It appears that in 1731 Graumann first discovered the chain rule....*The Rule of False Position* was used, when algebra was still not well known or was hardly used.

Logarithms. In 1614 the Scotlander *Johann Napier* gave the world its first logarithmic tables. These were improved upon by *Briggs.* His logarithmic tables were published for the first time in London in 1624. Counting machines. Already from the beginning of the 16th century text books of arithmetic appeared in very large numbers. Spaniard Juan de Ortega...... Adam Riese (15-19).

Here it is not clear, as to what these numbers 15-19 signify. In Poppe's book, these pages contain two introductory paragraphs of the first part of the book. Marx did not take any note from these

pages. The aforementioned part of the notes is related to §§ 19-51, pp. 19-51 of this book. The latter part of the notes proceeds in the following order :

2) History of Geometry. For its emergence Geometry is indebted to the art of measuring fields. Thales. Pythagoras. Oenopides of Chios. 500 B.C. Inventor of some simple geometrical problems. Hippocrates of chios, 450 B.C., was the first to discover the equivalence of certain spaces enclosed by curves and others enclosed by straight lines. Plato, 400 B.C. Upto Plato's time the circle was the only curve investigated in geometry. He introduced the conic sections (ellipses, parabolas and hyperbolas), their inventor proper was Menachmus. Later on Aristaus wrote 4 books and Apollonios 8 books on the same. *Eudoxus* of *Cindus. Euclid*, 300 B.C. *Archimedes*, 250 B.C..... Towards the end of the 17th century a new epoch began in geometry. It occurred in connection with the discovery of the analysis of the infinite, by Newton and Leibnitz.

From the next, 3rd, section of the book (pp. 99-118) Marx took down only the heading.

3) *History of practical geometry in particular.*

From the 4th section (pp. 118-128) apart from the heading, Marx took down only a sentence.

4) *History of trigonometry in particular.* In the Orient tables of tangents existed before the Europeans had them.

Notes from the 5th section (pp. 128-162) of Poppe's book, taken by Marx reads :

5) *History of Algebra and Analysis.* Greek *Diophantus* is considered to be the inventor of algebra, because of his studies on equations. The Arabs knew it at the beginning of the 10th century. In the 16th century the Italians were ahead of others. Towards the end of the 16th century the Frenchman François Vieta [introduced] the general art of calculations with letters. End of the 17th and beginning of the 18th century is the brightest period of mathematics, thanks to Newton, Leibnitz, Bernoulli etc.

From the second part of the book (pp. 165-568), Marx took note only of the beginning (pp. 165-166,170).

Second Part. History of applied mathematics.

1) *History of the science of mechanics.* Statics or the study of equilibrium of solid bodies; mechanics, or the study of movement of solid bodies ; hydrostatics, or the study of equilibrium of liquids, flowing substances ; hydraulics, or the study of movement of liquids, flowing substances ; aerostatics, or the study of equilibrium of air like substances; pneumatics, or the study of movement of air like substances; atomometry, or the study of equilibrium and movement of vaporous substances. During the last 150 years these sciences produced more results, than in the previous 1000 years. From the very beginning people [must have possessed] some *natural mechanics. Archimedes* conducted the following investigations with the balance : if both the arms of a balance are of the same length, then for constituting an equilibrium, the weights lying on both the pans of the balance must also be equal ; if one arm is longer than the other, as in the case of the so-called steel yard, then the weight attached to the longer arm, must be less than that attached to the other — in that ratio, by which the longer [arm] is lengtheir than the shorter one. Thus, he arrived at the conclusion, about the balances with unequal arms : that for equilibrium to take place, the two weights,

suspended from the unequal arms of such balances, must be *inversely proportional*. The *entire theory of lever* and of all machines based on it, is included in this rule.

There are no original comments of Marx in this manuscript. But he did a lot of work to collect those informations on the history of mathematics and mechanics from Poppe's book, which were of interest to him.

S.U.N. 2055

From Marx's letter to his uncle, Leon Philips dated the 14th of April 1864 (vide, Works, V. 30, pp. 538-539) (Eng. ed. V. 41, pp. 514-516) we learn, that Marx was specially interested in the history of arithmetic and especially in the computing instruments. In this letter we find that in the British Museum Marx was studying the old classic of Boetius (480-524 C.E.) "De institutione arithmeticae", even before it was republished anew by Friedlein in Leipzig, in 1867. In this letter he writes, that he also used some other works and compared them with Boetius' book. A comparison of this letter of Marx, with an excerpt taken form Poppe's book "History of Mathematics..." on sheet 3 of the note book of extracts, carrying the heading "Diversa (1867-69)", shows that Poppe's book was, in any case, one of those "other writings" which he had read already by 1864.

This extract from Poppe's book is being reproduced below in full. Numerals within brackets indicate Marx's page number, as well as the paragraph number of Poppe's book.

At the time of the Romans and even later, never were the ordinary calculations, for example in domestic affairs and trade, carried out with the numerals. These were conducted with stones and analogous symbols on the *counting board*. On this board certain parallel lines were drawn ; and there one and the same stone or other material symbol indicated the units on the first line, tens — on the second, hundreds — in the third, thousands — on the fourth etc. (even now the Chinese use such counting boards) (22).

The Pythagorean multiplication table was still very inconvenient and cumbersome because in part it consisted of special signs, and in part — the letters of the Greek alphabet (23, 24).

infinitely small, it coincides with the corresponding part of the curve itself. Consequently, I can consider mnR to be a Δ (triangle), and the $\Delta\, mnR$ and $\Delta\, mTP$ are similar triangles. That is why

$dy\ (= nR) : dx\ (= mR) = y\ (= mP) : PT$ (which) is the sub-tangent (for the tangent Tn). Hence, the subtangent $PT = y\dfrac{dx}{dy}$. And this is the *general differential equation* for any point of tangency of *any* curve. If I am now required to operate further with this equation and to determine with its help, the magnitude of the sub-tangent PT (on having the latter, it remains for me only to connect the points T and m with a straight line, and the tangent is obtained), then I must know the *specific character of* the curve. In keeping with its character (as a parabola, ellipse, cissoid [98a] etc.) it will have a *determinate general equation* for its ordinate and abscissa at any point, which is well known from the algebraic geometry. If, for example, the curve mAo is a parabola, then I know that y^2 (y is the ordinate of any arbitrary point)$= ax$, where a is the parameter of the parabola and x is the abscissa corresponding to the ordinate y.

If I put this value of y in the equation $PT = y\dfrac{dx}{dy}$, then, consequently, I must at first seek dy, i.e., find out the differential of y (an expression, which represents its infinitely small increase). If $y^2 = ax$, then I known from the differential calculus, that $d\,(y^2) = d\,(ax)$ (it stands to reason, that I must differentiate both the parts of the equation) gives $2ydy = adx$ (d everywhere designates the differential).

Hence, $dx = \dfrac{2ydy}{a}$. If I put this value of dx in the formula $PT = \dfrac{ydx}{dy}$, then I shall get

$$PT = \frac{2y^2\,dy}{ady} = \frac{2y^2}{a}\ \text{(since } y^2 = ax\text{)} = \frac{2ax}{a} = 2x.$$

Or : the subtangent at any point m of the parabola = twice the abscissa of the same point. The differential values are cancelled in the operation.

THE PROBLEM OF TANGENT TO THE PARABOLA

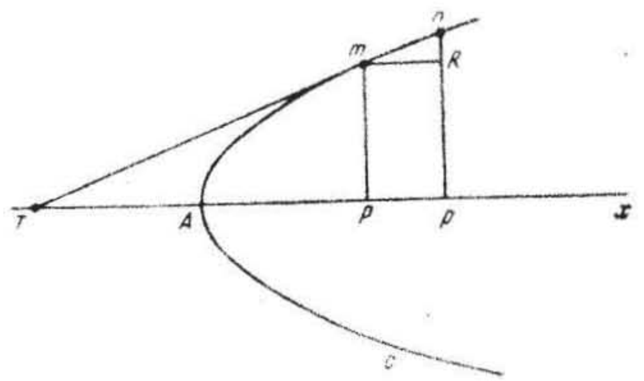
(Appendix to a letter to Engels)

S.U.N. 1922

This "Appendix" to a letter to Engels belonging to the end of 1865-beginning of 1866 (vide, MECW(E), 42, 208-210), is the first text with a proper mathematical content within the manuscripts of Marx. The letter itself has not reached us. In this appendix Marx explains the essence of the differential calculus to Engels, in the light of the problem of tangent to the parabola. Marx's source is the third volume of a book by Abbe Sauri (pp. 13-14) : *Sauri, "Cours complet de mathématiques"*, Paris, 1778. Here differentiation is understood exactly along the lines of Leibnitz.

APPENDIX

During my last stay in Manchester [from October 20th to early November 1865] one day you asked me to explain the differential calculus. You will be able to size up the question in full, in the light of the following example. The whole of differential calculus sprang up, initially from the problem of constructing *tangents* to an arbitrary curve, at any of is points. I wish to explain the essence of the matter to you in the light of this example.



Let the line *nAO* be an arbitrary curve. We *do not know* its nature (whether it is a parabola, ellipse etc.), and at a point *m* on it a tangent is required to be drawn.

Ax is the axis. We drop the perpendicular *mP* (the ordinate) on the abscissa *Ax*. Now assume that the point *n* on the curve is infinitesimally *proximate* to the nearby *m*. If I drop a perpendicular *np*, on the axis, then *p* must be infinitesimally proximate to the point *P*, and *np* must be infinitesimally proximate to the parallel line *mP*. Now iet us drop an infinitely small perpendicular *mR* upon *np*. If now you take the abscissa *AP* as *x* and the ordinate *mP* as *y*, then *np* = *mP* (or *Rp*), increased by the infinitely small increment [*nR*] , *or* [*nR*] = *dy* (the differential of *y*), and *mR* *(Pp)* = *dx*. Since the *mn* part of the tangent is

infinitely small, it coincides with the corresponding part of the curve itself. Consequently, I can consider *mnR* to be a Δ (triangle), and the $\Delta\, mnR$ and $\Delta\, mTP$ are similar triangles. That is why

$dy\ (= nR) : dx\ (= mR) = y\ (= mP) : PT$ (which) is the sub-tangent (for the tangent Tn). Hence, the subtangent $PT = y\dfrac{dx}{dy}$. And this is the *general differential equation* for any point of tangency of *any* curve. If I am now required to operate further with this equation and to determine with its help, the magnitude of the sub-tangent PT (on having the latter, it remains for me only to connect the points T and m with a straight line, and the tangent is obtained), then I must know the *specific character of* the curve. In keeping with its character (as a parabola, ellipse, cissoid [98a] etc.) it will have a *determinate general equation* for its ordinate and abscissa at any point, which is well known from the algebraic geometry. If, for example, the curve *mAo* is a parabola, then I know that y^2 (y is the ordinate of any arbitrary point)= ax, where a is the parameter of the parabola and x is the abscissa corresponding to the ordinate y.

If I put this value of y in the equation $PT = y\dfrac{dx}{dy}$, then, consequently, I must at first seek dy, i.e., find out the differential of y (an expression, which represents its infinitely small increase). If $y^2 = ax$, then I known from the differential calculus, that $d\,(y^2) = d\,(ax)$ (it stands to reason, that I must differentiate both the parts of the equation) gives $2ydy = adx$ (d everywhere designates the differential).

Hence, $dx = \dfrac{2ydy}{a}$. If I put this value of dx in the formula $PT = \dfrac{ydx}{dy}$, then I shall get

$$PT = \frac{2y^2\,dy}{ady} = \frac{2y^2}{a} \quad (\text{since } y^2 = ax) \; = \frac{2ax}{a} = 2x.$$

Or : the subtangent at any point m of the parabola = twice the abscissa of the same point. The differential values are cancelled in the operation.

THE FIRST NOTES ON TRIGONOMETRY

S.U.N. 2759

Advising Engels to study the differential calculus, Marx wrote to him on the 6th of July 1863, that " *Save for a knowledge of the more ordinary kind of algebra and trigonometry, no preliminary study is required except a general familiarity with conic sections.*" (vide, MECW(E), 41, 484).

The pages of his notes on trigonometry and theory of conic sections, which have been preserved, are a testimony to the fact, that Marx considered such a preliminary labour essential, for himself too.

The first of these notes, related, to all appearance, to the beginning of the 1860s, is devoted to a summary of trigonometric formulae. It has been put together according to the first volume of Sauri's book (pp. 433-482) and is related to the sections: "On Trigonometry" (pp. 433-447),"On the solution of triangles" (pp. 448-452), "On the solution of oblique-angled triangles" (pp. 452-482).

Photocopies of the manuscripts contain 24 sheets :

S.1. — Summary of trigonometric formulae, under Marx's heading : "*Resumé. I*".

S.2.— Calculation of certain values of trigonometric functions under the heading :"*Calculation of trigonometrical functions*".

Sheets 1 and 2 have not been numbered by Marx. Further on the sheets are numbered by the numerals, from 1 to 18, wherein some numerals have been repeated.

S.3.— (p. 1 in Marx's numeration) contains a very brief description of the section. "On Trigonometry" of Sauri 's book

S.4.— Carries Marx's heading : "*II. Résolution des triangles*". This section continues upto the lower part of s.5 entitled " *Résolution des triangles obliqueangles*".

This section continues upto s.15 (p. 10 in Marx's numeration), entitled : "*Recherches ultérieures trigonométriques*". At issue here are : trigonometric functions of multiple angles, construction of some formulae with a view to convenient logarithmisation, "impossible" problems in view of the emergence of imaginary numbers etc.

The manuscript comes to an end in ss. 23-24, with four tables of formulae for trigonometric functions of multiple angles.

This note is very concise in form and contains only the formulations of theorems or summaries of formulae (without proofs). In spite of such brevity, Marx included in his note all the questions of an applied character from the sections on trigonometry of Sauri's book : measurement of the width of a river, of the height of a tower or mountain, of the distance between two inaccessible places, methods of survey for maps and plans etc.

The manuscript contains many drawings ; however Marx executed them only by hand (without the help of any drawing instrument).

THE FIRST NOTES ON COMMERCIAL ARITHMETIC

The first notes on commercial arithmetic, related to 1869, are distinguished by a characteristic trait of Marx, consisting of the fact that having met with some special question, with which he was insufficiently acquainted, Marx considered a special study of that question essential for himself.

In the manuscripts 2388 and 2400 we see how the study of political economy leads Marx to the necessity of mastering the technique of settling the bills of exchange, in connection with which, in its turn, arises the need to solve some general "types" of arithmetical problems.

From the days of yore people devised special rules for solving these problems — special rules for each type of problem : (simple and complex) rule of three, chain rule, rule of partnership, rule of mixture etc. And though Marx was not very fond of arithmetical calculations (in a letter dated the 30th of may 1864, Engels even wrote to him regarding his "arithmetic" : "*you would seem pretty well to have ignored [it], if the failure to correct the scandalous printing errors in the figures is anything to go by") (MECW(E), 41, 532)*, he studied all these rules with unusual patience and integrity, from Feller and Odermann's course on commercial arithmetic, and with great care took detailed notes from it.

S.U.N. 2388

30 pages, pp. 109-139 (ss. 109-139), of the note book containing extracts on Political Economy entitled "*1869, Note book I*", contain Marx's notes on commercial arithmetic.

In connection with the study of circulation of capital, Marx took notes from G. J. Goschen's "The Theory of the Foreign Exchanges", on pp. 87-109 of this note book. While studying this book Marx was in need of some special informations regarding the settlement of international bills of exchange, that is why he turned to the section on the settlement of bills of exchange in the book : F.E. Feller and C.G.Odermann, "Das ganze der kaufmännischen Arithmetic", ("A Complete Course of Commercial Arithmetic"), Leipzig, 1859, and took detailed notes from it. Having taken notes from paragraphs 382-407 (pp. 318-365 of this book), related to the technique of direct settlement of bills of exchanges between cities and countries, which have an unmediated exchange relation, Marx interrupts the notes with the words (s. 118): *Before passing over to arbitration in an indirect way*, an insertion(chain rule and calculation of percentages).*

After this he referred to the continuation of the text on p. 135, where the author has gone back to the calculation of bills (Ch. XIV). The insertion is a note taken from the chapters V-X, of that very book (§§ 129-316, pp.98-245) by Feller and Odermann.

On sheet 127 we find the following comment of Marx, related to the section devoted to the calculation of rebates and, expressing his critical attitude to the book :

Customary rebates are pure charlatanism. Be they calculated *from* 100 or *upon* 100, they are already, before hand, added to the *selling* price [99].

At the end of s. 138 Marx's postscript : *"Continuation, Note Book 2, 1869"* and, on s. 139 : "*Contents*" of the note book from p. 87. Here we have the detailed headings of those parts of the books by Goschen, and by Feller and Odermann, from which notes have been taken, as well as indications to the corresponding pages of Marx's note book.

In order to give an idea of the contents of this manuscript, here we reproduce this table of contents in full (s. 139).

---

* That is when there is no direct relation or, when bills issued from two points are exchanged in a third point. —Ed.

CONTENTS

From this list we find that at first Marx dealt only with certain "types" of arithmetical problems, for the solution of which, special rules have been proposed for each "type".

S.U.N. 2400

It is a big note book, consisting of 125 sheets with Engels' superscription :

1869

1) Commercial Calculations, Note book II, End, pp. 1-36.

2) Foster, Commercial Exchanges, 37-51.

3) Hausner, Comp. Statistics, 1865.

4) Sadler, Ireland, 1829.

The first 36 pages (ss. 3-38) are continuation of manuscript 2388. Here, first of all the notes from chapter XIV come to an end and, notes are taken from chapter XV of the book by Feller and Odermann §§ 413-426, pp. 382-400 — about the calculation of Bills of Exchange, calculations of values of shares and of other government papers. Further on, Marx returns to the chapters XI -XIII, §§ 317-380, pp. 246-318 of this very book — which he skipped earlier — on the gold and silver contents of the currencies of different countries. This note comes to an end with the notes from chapters XVI - XVIII, §§ 428-471, pp. 402-481. These are about calculations of weights and measures, estimate of commodities and calculation of losses in cases of shipwreck.

MANUSCRIPTS OF THE 1870s

THE MANUSCRIPTS ON THE THEORY OF CONIC SECTIONS

Here the following manuscripts are in view : S.U.N. 2760, 2761 and 2762.

S.U.N. 2760

It consists of 9 sheets (ss.1-9) of notes, taken from : J. Hymers, "A treatise on conic sections and the application of algebra to geometry", 3rd ed., Cambridge, 1845. This book was found in Marx's personal library. Marx took notes from the first 12 pages of it. These pages are related to the introduction of coordinates; the problem of finding the distance between two points, given their coordinates; the equation of the straight line and, the problems of : determining the equation of a straight line, in terms of the segments cut off from it by the axes of coordinates and, the equation of a straight line passing through one and two points, their coordinates being given.

S.U.N.2761

5.5 double page sheets of rough notes, on the theory of conic sections, from Sauri's book cited above, volume 2, pp. 2-27, in French and English.

S.U.N. 2762

4 double page sheets in French . Fair notes on the theory of conic sections from the same book by Sauri, volume 2, pp. 2-27.

THE FIRST NOTES ON THE DIFFERENTIAL CALCULUS

S.U.N. 3704

4 sheets of photocopies ; the beginning is not there, we have only pp. 3-6 in Marx's pagination. From the content it is clear, that to all appearance, this manuscript is the very first note taken from the initial paragraphs of Boucharlat's text-book ("An Elementary Treatise on the Differential and Integral Calculus " by J.-L. Boucharlat. Translated from French By R. Blakelock, B.A., Catharina Hall, Cambridge-London, 1828), i.e., [this note] is related to that time, when Marx, having got acquainted with the basics of differential calculus according to Sauri's course, turned to a newer treatise on this calculus by Boucharlat.

The preserved sheets of the manuscript contain notes from §§ 5-18 this text book. The following beginning of page 3 of Marx's manuscript indicates that, in the missing pages 1-2 notes were taken from the paragraphs preceeding the fifth one. It reads :

He would say

$$(x + dx)^3 = x^3 + 3x^2dx + 3x(dx)^2 + (dx)^3.$$

Now, if we subtract the given quantity x^3, there remains $3x^2dx + 3x(dx)^2 + (dx)^3$; the two latter terms disappear as infinities of the second [and third] order [s] and, we get $d(x^3) = 3x^2dx$, which is the differential of x^3, e.g. $= d(x^3)$, and there is less to be said against this ; as in the other equation $y =$ etc., x changes independently of y and the changes of y are only correlative to those of x.

Here Marx considers an example, which Boucharlat investigated in § 3 (for the full text of this paragraph see PV, 326-327, but he differentiates it according to Sauri, i.e., using the infinitesimal method of Newton-Leibnitz. Marx's objections to it are still to be met with. It appears, that the words "he would say" [above] refer to what Boucharlat would say, if he would act according to Sauri.

The following circumstance too speaks in favour of this conjecture : in his critical comments following the examples of differentiation of the functions $y = a + 3x^2$, $y = ax^3 - b^3$ and $y = \frac{1-x^3}{1-x}$ (§§ 5-7 according to Boucharlat), Marx writes (in p.3), taking notes from § 9 of this book :

dx " is itself the differential of x". If we had said from the beginning [that] we call the infinitely small increment of x — dx, the thing was simple, as with Sauri. But having introduced h, and spoken till now of the differential of the algebraic expression of it as the differential of the function y (y being the function of x, put on the other side of equation), he wants some hocus pocus[100].

For [instance] let us have :

$y = x,\ y_1 = x + h,\ y_1 - y = h,\ \frac{y_1 - y}{h} = 1$; as h does not enter into the second side [R.H.S.] of the equation, we *pass to the limit* in making $y_1 = y$ or $y_1 - y = dy$ and h into dx. And as all expression of x has disappeared in $\frac{y_1 - y}{h} = 1$, we have not even a pretext to say, that h becomes dx, and therefore $y_1 - y = dy$ or that $\frac{0}{0} = 1$.

It is only true in that sense that $\frac{0}{0} =$ every quantity q whatever, because $\frac{0}{0} = q$, gives $0 = q \cdot 0$ or $0 = 0$[100].

The remaining part of this manuscript (pp. 4-6 according to Marx) is a note taken from §§ 10-18 of that very book of Boucharlat. (For the contents of these paragraphs see PV, 327-330.) Here it is also evident that Marx is still in favour of Sauri's method. Thus, in the case of the theorem about the differential of the product of functions, which is proved in Boucharlat (see § 14 in pp. 329-330) as per Lagrange, i.e., formally multiplying the expansions

$$y_1 = y + Ah + Bh^2 + Ch^3 + \text{etc.},$$
$$z_1 = z + A'h + B'h^2 + C'h^3 + \text{ etc.},$$

Marx writes (p. 5 in his pagination) :

He develops out of this $d\,(xyzu$ etc.), which might have been done much more simply and directly (see *Sauri*).

He says : ...

Further on, the proof according to Boucharlat is set forth in full (see PV, 330). After this Marx adduces a simpler proof (p.5) from Sauri and notes that factually speaking Boucharlat also omits the terms with the higher powers of h. This comment of Marx reads :

Instead of which he might simply have said :

$$(z + dz)\,(y + dy) = zy + dz \cdot y + dy \cdot z + dz \cdot dy\,,$$

$$d\,(zy) = z\,dy + y\,dz$$

by subtracting the given quantity *zy and suppressing* $dz{\cdot}dy$; he does the same in suppressing $(Bz + A\,A' + B'\,y)\,h$ + etc.

"ON THE METHOD OF FINITE DIFFERENCES "

S.U.N. 4039

Two small pages (two sheets of photocopies). It is a very brief note(almost without words, only calculations, very little explained) taken from §§ 171, 172 of Sauri's book "Cours complet de mathématiques", vol. III, pp. 303-304. In Sauri's course, with these paragraphs begins the section : "Calculus of finite differences."

Here Sauri compares the values of the variable x :

$$x,\ x+p,\ x+2p,\ x+3p,\ x+4p \text{ etc.},$$

with the values of the "variable magnitude" y :

$$y,\ y^{I},\ y^{II},\ y^{III},\ y^{IV} \text{ etc.},$$

noting herein, that if $y = x^0$, then all values of y remain constant (being equal to one). And with this comment, Marx begins his notes. After that Marx adduces the following comment of Sauri : if $y = ax + b$, then the series for the y-s will be arithmetic ; if $y = a^x$, then it will be a geometric series ; if $y = \frac{a}{bx + c}$, then it will be harmonic.

Later on the differences between the successive values of y-s and the differences of the second and higher orders (differences between the differences of the first, then of the second etc. orders) have been considered. The corresponding notations have been introduced :

$Dy = y^{I} - y,\ Dy^{I} = y^{II} - y^{I}, \ldots\ldots,\ D^2 y = Dy^{I} - Dy = y^{II} - 2y^{I} + y,\ D^3 y = y^{III} - 3y^{II} + 3y^{I} - y$.

With this the note abruptly comes to an end.

NOTE BOOKS CONTAINING EXTRACTS ON COMMERCIAL ARITHMETIC

S.U.N. 3881

The note book with Marx's superscription : "II. Begun on March 1878", contains in its sheets 144-146 (142-144 in Marx's numeration) extracts from the book "Das Ganze der kaufmännischen Arithmetic" by Feller and Odermann(§§ 385 and 382), on mathematical evaluations of the impact of discount upon the rate of exchange.

S.U.N. 3931

It is a note book with extracts on commercial arithmetic, in German ; 69 sheets.

Sheets 1-20. Extracts from the book by Feller and Odermann : Chapter IX— on the calculation of discount, §§ 294-309, pp. 226-238 ; Chapter XIV— on the calculation of bills of exchange, §§ 382-418, pp. 320-390.

Sheets 21-22. Extracts from Sauri's book, vol. I, pp. 109-121, on arithmetic and geometric progressions.

Sheets 23-54. Extracts from the book by Feller and Odermann : chapter VII — on the calculation of percentages , §§ 210-260, pp. 162-196 ; chapter VIII — on the calculation of interests, §§ 261-293, pp.197-227.

Sheets 54-69. Extracts from Sauri, vol. I, pp. 109-132, under the general heading given by Marx : "*Insertion (Progression etc.)*" — on progressions and logarithms.

A NOTE BOOK
CONTAINING NOTES ON MATHEMATICAL ANALYSIS ACCORDING TO THE BOOKS OF SAURI, NEWTON, BOUCHARLAT AND HIND

S. U. N. 2763

81 sheets in German and French.This note book, containing extracts and notes on mathematics, is related to a long period of Marx's study of mathematics, starting with the work on Sauri's course, drawn up according to Leibnitz and Newton, right through the works of Newton himself (ss. 24-28), and upto a detailed acquaintance (according to the books of Boucharlat and Hind) with the ideas of the "algebraic" differential calculus of Lagrange (ss. 37-81).

Later on Marx no more considered these ideas of Lagrange to be suitable as a "basis", upon which the differential calculus may be constructed (see the first part of the present volume). But in the note book herein described, he still did not arrive at this conclusion. [Here] the task remains, first of all to look into the "method of Lagrange", according to the sources available to him and, to ascertain its value and shortcomings. In this connection the question of dating this manuscript is of considerable interest. The answer to this question may throw light also upon the question, as to when, namely, did Marx's own dialectical understanding of the operational nature of differential calculus finally mature. Unfortunately, a number of difficulties are connected with this [problem of] dating.

It is clear from certain bibliographical indications and dates on sheets 33-36, not related to mathematics, that those pages of this manuscript which are devoted to Lagrange, were, in any case, written after 1872. It may be said with confidence that they were written before 1881, [the year] with which is associated Marx's first work "On the Concept of the Derived Function", wherein Marx has already come out with his own way of treating the basic concepts of differential calculus. However, these boundaries could not be significantly narrowed down. The problem is this : in the last two pages of the note book containing manuscript 2763 (sheets 79-80), under the heading — "*Continuation of another note book(III , next to Kaufmann II) (last page)* (see description of the manuscripts 3881 and 3888), Marx once more begins to take notes from those paragraphs of Boucharlat's book, which are devoted to the method of Lagrange. Evidently these notes were taken not with an intention to criticise Lagrange, but to look (and besides, sufficiently sympathetically) into his "method".It is true, that Marx soon stopped this note abruptly and crossed it out with a pencil. But the note book "Kaufmann II"(see manuscript 3881) begins with the following superscription in Marx's hand :"*Begun on March 1878*". We do not know when it was finished. Note book III (manuscript 3888) follows the note book"Kaufmann II". Thus, in any case, it belongs to a period after March 1878. Further, the continuation of this note book — its last pages, have been placed in manuscript 2763. It is natural to think that this continuation could hardly have been writen earlier than 1879.

On the contrary, the first pages of manuscipt number 2763 contain notes from Sauri's book. This provides a basis to assume that these pages could have been written before 1872. From all this it is clear that this note book is actually related to a sufficiently long span of time, in the course of which Marx's mathematical studies also found expression in certain other note books.

In manuscript 2763 Marx's own point of view on the nature of differential calculus has still not been formulated; he still, in the main, highly evaluates the "algebraic method" of Lagrange; however, here we already have a number of Marx's observations regarding the choice and role of the symbols of differential calculus, the dialectic of quality and quantity, of form and content, of unity and opposition connected with them, and about the interrelationship of algebra and mathematical analysis(the "algebraic" roots of symbolic differential calculus). These observations

are already, in part, a preparation towards Marx's own conception, which also begins to take shape at this stage of his mathematical studies.Many of these [ideas] were later on elaborated by Marx. In the detailed description of manuscript 2763, which follows herein under, Marx's own observations have been brought out in full.Many such places[of the manuscript], as are not simply extracts, but [his] own account of the noted material, have also been brought forth. Titles of all the four parts of the manuscript belong to Marx himself. The sub-titles belong to us.

"CONIC SECTIONS"

Sheets 1-24 (in Marx's numeration : ad p.1,1-23). Extracts from Sauri's book "Cours complet de mathématiques", 1778, vol. II, pp. 2-49, entitled *"Conic Sections"*. This part of the book contains the chapters on parabola (pp. 2-12), circle, ellipse and hyperbola (pp. 12-34) and, part (pp. 34-49) of the chapter on the asymptotes to hyperbola and the diameters of ellipse and hyperbola.The account still has a very archaic character. Though already from the geometrical properties of the curves, equations are deduced, connecting the abscissae and ordinates of their points, the general method of coordinates is still absent. Abscissae and ordinates of a given curve exist in it, just like its axes of symmetry, tangents and diameters, and are defined through the later. Thus, the "ordinate of the parabola" is defined as "the line PM, perpendicular to the axis and ending at the parabola" (p.2) ; the "abscissa" — as "the part AP of the axis, enclosed between the ordinate and the point A , where the axis meets the parabola" (ibid).The "tangent " is understood as a straight line having one, and only one point in common with a given curve. The area of the "semi-parabola" acb (where a — is the vertex of the parabola, c — a point on it, ab —the abscissa of the point c) is determined in the mode of Archimedes, simplified according to the methods of Newton and Leibnitz (i.e., through the "characteristic triangles" of Pascal and Leibnitz — through the identification of an infinitely small arc of the curve with a segment of its tangent, of the infinitely small curvilinear trapeziums with rectangles, differing from them by infinitesimals of the second order). Area of the ellipse is sought differently — by means of the badly formulated principle of Cavalieri : "Sum of all y is equal to the area of the ellipse" (p. 22). A large number of theorems about the different properties of each of the conic sections, their diameters, axes, foci, asymptotes and the other elements, about the ways of constructing tangents to them, construction of any number of points on these curves, computation of the areas connected with them, and others are proved by the usual arguments of elementary geometry (i.e., synthetically, and not with the help of calculus).

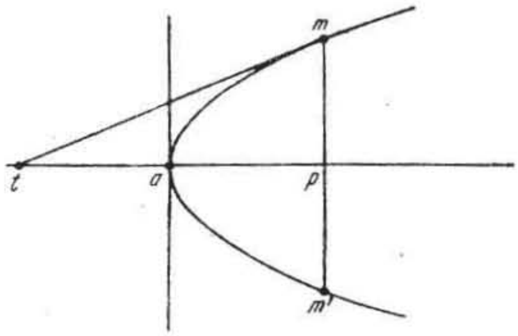
It is not possible to extract any general method or principles for systematising the materials from this part of the book. One may think, that Marx read it, having in view the introduction to the English translation of Boucharlat's text book(1828), where the reader has been specially forewarned, that the main body of the work assumes an acquaintance "with the elementary principles, relating to curves", and under the latter, first of all, he had in view the conic sections. In fact, already on the 6th of July 1863 Marx wrote to Engels, that he had " a superfluity of works" on the differential and integral calculus and that : "Save for a knowledge of the more ordinary kind of algebra and trigonometry, no preliminary study is required except *a general familiarity with conic sections*" (editor's stress) [MECW(E), 41,484]. The choice of Sauri's book is naturally explained by the fact that, at first Marx very much appreciated the simplicity of its methods (see the description of manuscript 3704). However, a closer acquaintance with the content of the section on conic sections, from which Marx took notes in great detail, must have disappointed him, and in fact, Marx did abruptly stop his notes, from the chapter on the asymptotes to hyperbola and the diameters of ellipse and hyperbola — which occupies pp. 35-68 of Sauri's book — at p. 49 of the book. And later on he took notes from only one of the remaining chapters : from the chapter "On Conic

Sections of Higher Orders" (see below — description of sheets 29-33). Even he did not have the patience to study all of those large number of particular theorems, not connected by any kind of general idea.

The content of the latter parts of manuscript 2763 (beginning with sheet 33) shows that, these must have been written after manuscript number 3704, wherein Marx is still in favour of Sauri's (i.e., Leibnitz's and Newton's) method. However, there are direct indications in the manuscript to the effect that sheets 37-81, were, in any case, written after 1872, because on sheets 33-36 (pp. 32-35 in Marx's numeration) Marx provides a list of the (then!) latest books on the history of Germany. In this list there are publications of the years 1866,1868,1871 and one of them: "Von 1806-1866. Zur Vorgeschichte des neuen Deutschen Reich" von H . Langwerth von Simmern, was published from Leipzig in 1872. It is true, that the sheets 33-36 are related to that part of the manuscript, which is situated (immediately) after the last extracts from Sauri's book. But it is clear, that the pages that follow could not have been written before 1872.

Sheet 1 (ad p.1 according to Marx), attached to the beginning of the manuscript, contains definitions, which were at first omitted by Marx : of the "parabola" (as it was usually done, i.e., through "focus" and "directrix"), of the "axis", "diameter", "tangent", "ordinate" and "abscissa", of the "subtangent", "parameters" of the axis or diameter, "normal", "subnormal", "applicates to the diameter"*, "radius-vector", and "vertex" — all these specially for the parabola. The list comes to an end with the following observation, which is not there in Sauri. It appears, that it belongs to Marx himself :

From these data, given by the very construction of the curve, follows the explanation :



If a point m is given on a curve and I am required to draw a tangent through it, then it is clear, that, if I have, corresponding to this point m, for example, to the point m in the figure inserted here, the *subtangent* pt, i.e., the corresponding extension of the abscissa ap, then I am only to connect the terminal point of the subtangent pt, i.e. the point t, with m by a straight line — and then, mt is the *tangent*.

Further, since ap is the *abscissa*, by extending which the subtangent is obtained, so mp is the ordinate.

* What we have in view here under "ordinate " or "applicate (i.e., attached) to the diameter" — if we speak about it in a more modern language — is the ordinate of a point of a parabola in a system of coordinates, the ordinate axis of which is a tangent at a fixed point m of the parabola,and the axis of abscissae — is the diameter passing through that very point m.—Ed.

Since a point *m* on a curve is obtained, corresponding to a given abscissa, when the curve is intersected by the perpendicular arising from the end point of the abscissa, hence, also, conversely, the straight line joining *m* with *p* is perpendicular to the axis.

From this it follows further, that the point *m*, where the ordinate, corresponding to the given abscissa intersects the curve, is the point in which the tangent, corresponding to this ordinate, touches the curve. It is the point of tangency. Hence it follows that : if a given *point m* of a parabola is its *vertex a*, then at the same time it is the point, where the *ordinate*, i.e., the perpendicular to the axis, is to be raised; and herein it is the only point, where (corresponding to this ordinate) the tangent touches the curve. Hence, *this tangent and this ordinate coincide*: instead to two — here we have the *one and only one line*, perpendicular to the axis at *a*. Thus, while the tangent became infinitesimal*, abscissa *pa* turned into 0, and along with this also its extension, the subtangent, or, to be more precise, the difference between the abscissa and the subtangent has vanished, both of them have coincided with the starting point of the ordinate and of the tangent at *a*, the difference between them has also vanished **.

The remaining part of this section is written in French, i.e., in the language of the text book studied, though a large part of this is no mere extract, but summarized statement of the material noted. However, Marx carries out all the calculations in full, the diagrams (including the highly intricate ones) have been copied (with the designations exactly retained). This part of the manuscript does not contain any comment that is Marx's own. It comes to an end with a diagram related to §55 of Sauri's book, left unnoted, where the concept of "adjoining circles" has been introduced. Under the diagram, at the beginning of a line the number "12" has been written (it is the number of the next point of the note) and a blank space has been left. Perhaps, later on something was written there with a pencil, traces of which are now unrecognizable. Thereby it appears, that for some time Marx interrupted his mathematical studies.

QUADRATURES OF CURVILINEAR AREAS (ACCORDING TO NEWTON)

Sheets 25-28 in German.

Extracts from the works of Newton, along with Marx's critical observations. These are being reproduced here in full. Marx indicated his own large comments by a vertical line on the left. In other cases it is clear from the context, as to where Marx is enunciating Newton, and where it is a comment on him.For a commentary on this section of the manuscript see :[editorial] notes [101-107].

QUADRATURES OF CURVILINEAR AREAS

(From Newton's communication on "*Analysis with the help of equations with infinite number of terms*", addressed to the President of the Royal Society in London [101], in 1669).

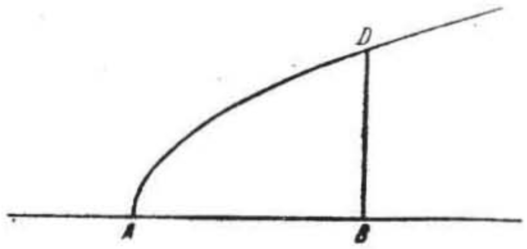
* Evidently, in this observation, what is had in view under "tangent", is a segment of the tangent between the point of tangency and that point where the tangent intersects the axis of abscissa, i.e., (see the figure) the segment *mt*, if the points *m* and *t* are different. If *m* and *t* coincide, then they do not univocally determine a straight line — the tangent — and , then not the segment *mt* is to be taken for the "tangent", but the *entire* tangent. It appears that, here this is what Marx wanted to say. — Ed.

** That is to say, the difference between the abscissa of the "starting points" *m* and *t* of the ordinate and the tangent has vanished : both the points have coincided with *a*.—Ed.

At issue here are the simple curves of the type

$$y = ax^{m/n}$$

(where , for example, for the parabola $a = p^{\frac{1}{2}}$ and $x^{m/n} = x^{\frac{1}{2}}$).



$AB = x$, $BD = y$, a, b, c are given magnitudes, $[m, n]$ — whole numbers.

If $y = ax^{m/n}$, then

the *area* $ABD = \dfrac{an}{m+n} x^{\frac{m+n}{n}}$.

(This has been put forward [102] without proof.)

As examples he gives :

1) $x^2 = 1 \cdot x^{\frac{2}{1}} = y$; $a = 1$, $m = 2$, $n = 1$; then

$$\text{the area } ABD = \frac{1}{3}x^3.$$

2) If $4\sqrt{x} = 4x^{\frac{1}{2}} = y$ $(a = 4,\ m = 1,\ n = 2)$, then

$$\text{the area } ABD = \frac{8}{3}x^{\frac{3}{2}}$$ [103] .

This enumeration is not accompanied by any kind of proof: neither a proof of the general theorem, nor any explanation for the examples.

Take the *first example* : $y = x^2$; then an element of the area $= ydx = x^2\,dx$.

Hence,

$$\text{the area } = \int x^2 dx = \frac{x^{2+1}}{2+1} = \frac{1}{3}x^3 .$$

Second example : $y = 4x^{\frac{1}{2}}$; an element of the area $= ydx = 4x^{\frac{1}{2}}\,dx$; hence,

$$\text{the area } = \int 4x^{\frac{1}{2}}\,dx = \frac{4x^{\frac{1}{2}+1}}{\frac{3}{2}} = \frac{8}{3}x^{\frac{3}{2}} .$$

And generally : if $y = ax^{\frac{m}{n}}$, then

an element of the area $= y\,dx = ax^{m/n}\,dx$;
hence,
the area $= \int ax^{m/n}\,dx$:

$$\int ax^{m/n}\,dx = \frac{ax^{m/n+1}}{\frac{m}{n}+1} = \frac{ax^{\frac{m+n}{n}}}{\frac{m+n}{n}} = \frac{an}{m+n}x^{\frac{m+n}{n}}.$$

Newton knew from analytical geometry [104], that an element [of the area] of the parabola etc. $= ydx$, i.e., equal to the differential of the unknown curvilinear area ; and since according to the equation of the curve $y = ax^{m/n}$, so this differential $= ax^{m/n}\,dx$, i.e., equal to y expressed in the abscissa[multiplied by dx]. He knew , further, from that very source, that the area is considered to be the infinite sum of these elements, i.e., to be

$$\int ax^{m/n}\,dx.$$

He knew further, that the result for $x^m dx$ (where m may be any $\frac{m}{n}$, only m and n are whole numbers ; for example when $n = 1$, $x^{m/n} = x^m$) [is] $\frac{x^{m+1}}{m+1}$, which gives

$$\int ax^m\,dx = a\frac{x^{m+1}}{m+1} = \frac{1}{m+1}ax^{m+1}.$$

However, the differentiation, for example of x^m, already showed him, that its differential is $mx^{m-1}\,dx$, i.e., the second term of the binomial expansion, again becomes the first [and], thus, gets integrated according to the formula :

$$\frac{mx^{m-1+1}\,dx}{(m-1+1)\,dx} = x^m.$$

Having known this formula, he knew from analytical geometry, that it is the integral of $y\,dx$, i.e. of the differentiated function of x in the equation of the curve [105]. However, that he could not at all cope with the application of integral and differential calculus to analytical geometry, shows his following proof of the general theorem, where :

1) the auxiliary ▭ or, more correctly, the auxiliary trapezium is not constructed from dx and $y + dy$, but from dx and some height, which is not the ordinate, that is why [dy] does not vanish when dx vanishes ; 2) he does not construct the curve from the equation $y =$ etc., but does it geometrically, assuming the area to be *given* ; 3) the height y is found with the help of differentiation of the given function of x, and then it is conversely concluded that : if now the height $y =$ such and such, then, conversely, if y is given by such an expression, then the area must be such and such. However, he actually avoids integrating or showing, how this inverse process may be accomplished with the help of calculus; 4) in other words he also observed, that the formula is not enough for all simple curves, and [one should also write] $+ C$; in many cases this constant is not 0, but must be further defined. In the next page, we shall now give his so-called proof, word for word.

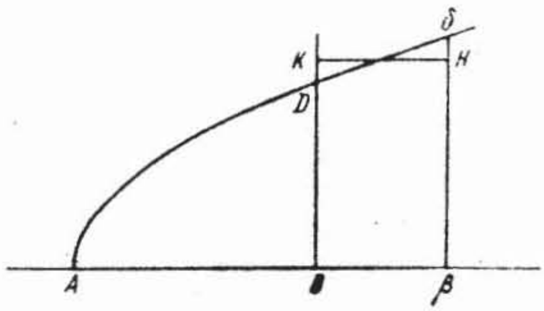
Notabene : Whenever he writes 0, 0^2 etc., we write dx, dx^2 etc.

1) *Preparation for the proof.* The area to be calculated $ABD = z$; $AB = x$; $BD = y$; $B\beta = dx$; $Bk = v$; $\square B\beta KH\,(vdx) =$ the area $B\beta\delta D$;

$A\beta = x + dx$; $A\delta\beta = z + vdx$.

From the arbitrarily chosen interrelationship of x and z, *I seek y as follows* :

A) Suppose $z = \frac{2}{3}x^{\frac{3}{2}}$ or $z^2 = \frac{4}{9}x^3$ ·



Putting $x + dx$ for x and $z + vdx$ for z, we get :

B) $\frac{4}{9}(x + dx)^3 = (z + v\,dx)^2$; hence :

$\frac{4}{9}(x^3 + 3x^2\,dx + 3xdx^2 + dx^3) = z^2 + 2zv\,dx + v^2\,dx^2.$

$\frac{4}{9}x^3 = z^2$; mutually curtailing these two terms and dividing the rest by dx, we get

$\frac{4}{9}(3x^2 + 3xdx + dx^2) = 2zv + v^2\,dx.$

Now assume that $B\beta = dx$ diminishes infinitely and finally vanishes, i.e., actually becomes 0 (earlier he already called it 0) ; then the terms multiplied by 0 vanish , and we get:

C) $\frac{4}{9}\cdot 3x^2 = 2zv$, or $\frac{4}{3}x^2 = 2zv$, whence

$\frac{2}{3}x^2 = zv.$

But now, when dx becomes = 0, v is equal to y (why ? Isn't v the ordinate , and dx the function of v ?[106]) and that is why

$\frac{2}{3}x^2 = zv = zy = \frac{2}{3}x^{\frac{3}{2}}y.$

But if $\frac{2}{3}x^2 = \frac{2}{3}x^{\frac{3}{2}}y$, then

$$y = \frac{\frac{2}{3}x^2}{\frac{2}{3}x^{\frac{3}{2}}} = \frac{x^2}{x^{\frac{3}{2}}} = x^{2-\frac{3}{2}} = x^{\frac{1}{2}} \cdot$$

[[Thus, we have found y, by 1) assuming that the differential rectangle $= ydx$, which by no means follows from the construction, and then the differentiation of $\frac{4}{9}x^3$ is obtained from the accepted expression for the area $vz = \frac{2}{3}x^2$.]]

Now *Newton* adds — and this must be proved — that he can *integrate*:

"If that is why (why?), conversely, $x^{\frac{1}{2}} = y$ (i.e., if we have the equation of the curve), then the area $z = \frac{2}{3}x^{\frac{3}{2}}$".

Hence, the course of the argument is like this : if the equation for the area is given , in which it is expressed through the abscissa, then I shall find out with the help of differentiation, the *equation* of *the curve*, for example $y = x^{\frac{1}{2}}$ (perhaps $y = 1 \cdot x^{\frac{1}{2}}$, and if I change 1 by a or by some given number, and $\frac{1}{2}$ by m/n [then] I shall get $y = ax^{\frac{m}{n}}$), i.e., a determination of the ordinate y in terms of the abscissa x. Then I conclude that, conversely , it must also be possible to find out the area from the *equation of the curve*, i.e. by integrating, and besides this area must be that only, from which I found

$$y = ax^{\frac{m}{n}}.$$

However, thereby I have still by no means *integrated*, any more than initially obtaining the differential of the area ABD directly by *differentiating* the equation of the curve, which, you see, I, conversely, obtain as the result [107]. However, had Newton thought, that this method, which he calls "preparation for the proof", gave the proof itself, then he would not have given any kind of *proof* after this. But this imaginary proof consists only of the fact that, one and the same thing is repeated *in a general algebraic form* instead of *determinate* numbers, i.e., instead of $z = \frac{2}{3}x^{\frac{3}{2}}$ what is assumed is $z = \frac{n}{m+n}\ ax^{\frac{m+n}{n}}$.

Now let us get acquainted with the following *proof.*

Proof. Now *in the general form*, when

$$z = \frac{n}{m+n}\ ax^{\frac{m+n}{n}}$$

(instead of $= \frac{2}{3}x^{\frac{3}{2}}$; a general algebraic expression, substituted for the numbers), or (it is only an algebraic simplification, substitution of simpler expressions, in place of which their values may be again substituted), or [if] $\frac{an}{m+n} = c$ and $m + n = p$, [then] when (with the help of this substitution which does not give any kind of new demonstrative argument, but

only *repeats the assumption* in other simpler designations) $z = cx^{\frac{p}{n}}$ or $z^n = c^n x^p$, then, if we substitute (just as we did in the preparation) $x + dx$ [[he writes as above, $x + 0$]] [for x] and $z + vdx$ (for him, as above, $x + v\,0$) or (what is then reduced to the same) $z + ydx$ for z [[the progress consists of the fact that the element of the $\square$ is defined directly as ydx without further ballyhoo]], [we shall get]

$$c^n (x^p + px^{p-1}\,dx + \ldots\ldots\ldots\ldots) = z^n + nz^{n-1}\,ydx + \ldots\ldots\ldots$$

Removing $c^n x^p$ along with z^n and dividing what remained by dx [then assuming it to be $= 0$], we shall get

$$c^n px^{p-1} = nz^{n-1}y = \frac{yn\,z^n}{z} = \frac{ync^n x^p}{cx^{\frac{p}{n}}},$$

or, dividing by $c^n x^p$,

$$\frac{c^n px^{p-1}}{c^n x^p} = \frac{yn\,c^n x^p}{cx^{\frac{p}{n}} c^n x^p},$$

$px^{-1} = \dfrac{yn}{cx^{p/n}}$, whence

$$ny = px^{-1}\,cx^{\frac{p}{n}} = pc\,x^{\frac{p}{n}-1} = pc\,x^{\frac{p-n}{n}}.$$

Substituting again for p and c, their values, we shall finally get : $y = ax^{\frac{m}{n}}$.(In fact, herein $ny = (m+n)\dfrac{an}{m+n}x^{\frac{m+n-n}{n}}$, $ny = an\,x^{\frac{m}{n}}$, $y = ax^{\frac{m}{n}}$.)

That is why, if *conversely*, it is assumed that $y = ax^{\frac{m}{n}}$ (i.e., the equation of the curve is given), then for z is obtained the value $\dfrac{m}{m+n}ax^{\frac{m+n}{n}}$, Q.E.D. That is, nothing is proved but only the "preparation for the proof" *has been repeated* in a general algebraic from.

Sheet 28 (p. 27 according to Marx) of the manuscript contains only one line : "....*but only the "preparation for the proof " has been repeated in a general algebraic form*". Rest of the sheet is blank. Evidently, between this section of the manuscript and its latter parts, there was a second break in Marx's work.

"CONIC SECTIONS OF HIGHER ORDERS"

Sheets 29-33. Here Marx returns to volume II of Sauri's book, but as has been noted earlier, this is no continuation of the notes from the chapter , from which he was taking notes earlier. He also omits the next two chapters, and begins taking notes in full , from the chapter "On Conic Sections of Higher Orders" (§§ 91-99, pp. 95-105). In this chapter Sauri considers those curves, whose equation has one of the following forms, under a system of coordinates chosen and specially indicated by him.

1) $y^{m+n} = x^m (2a - x)^n$ or, what is considered by Sauri to be different from it and is hence obtained through a transformation of the system of coordinates,

$y^{m+n} = x^m (a - x)^n$ —"circles of higher orders"(the order is determined by the number m, sequence in the order — by the number n), § 91;

2) $y^{m+n} = a^m x^n$—"parabolas of higher orders", § 92 ;

3) $\frac{p}{a} y^{m+n} = x^m (a-x)^n$—"ellipses of higher orders", § 93 ;

4) $\frac{p}{a} y^{m+n} = x^m (a+x)^n$ or, when taken to the asymptotes, $x^m y^n = a^{m+n}$—"hyperbolas of higher orders",§§ 94-95 ;

5) $a^{m-1} y = x^m + bx^{m-1} + acx^{m-1} + \ldots + a^{m-1} k$—"paraboloids",§96.

Here, "if we assume x to be infinite, positive or negative, then, disregarding all the terms, which may be considered as 0 in comparison to x^m, we get the equation $y = \frac{x^m}{a^{m-1}}$ " (p. 102). In Sauri the latter signifies, that when x changes from $-\infty$ to $+\infty$, then in the case of an odd m the curve, owing to its continuity , must intersect the x– axis (as the ordinate y from being equal to $-\infty$ must become equal to $+\infty$, i.e., from being negative must become positive), and besides must intersect it odd number of times. And analogously when m is an even number, then the curve must either not intersect the x-axis at all, or must intersect it even number of times, as distinct from zero. Marx's note comes to an end with the consequences that follow from here, and are related to the number of real and imaginary roots of the equations of higher degrees(with one unknown x) .

This note is very laconic and does not contain any observation, which is Marx's own. However, all definitions, diagrams and general assertions of Sauri have been reproduced by Marx in full. That he did not intend to study Sauri's book any longer, is clear from the fact that, he did not leave any blank space in the last sheet 33 (p. 32 of Marx) of this note, but began writing here itself, informations about the latest books on the history of Germany [referred to above], which he found interesting

Sheets 34-36 , were not numbered by Marx. The following sheet 37 has Marx's number "33" .That is why to obtain Marx's numeration for the sheets 37-52 , one must subtract four from the archival number of these sheets. Later on Marx commits two mistakes in numeration : he wrote "46" instead of "45" and "44" instead of "47" , and upto sheet 62 the archival number is already 10 more than Marx's number

"A SOMEWHAT MODIFIED VERSION OF THE LAGRANGIAN ACCOUNT OF TAYLOR'S THEOREM, BASING IT ON A PURELY ALGEBRAIC FOUNDATION"

Sheets 37-81, in German ; there are English and French phrases.

Marx united this entire part — more than half of the manuscript — under the general title : "*A somewhat modified version of the Lagrangian account of Taylor's theorem, basing it on a purely algebraic foundation*".It starts (sheets 37-44, Marx's pp. 33-40) with notes from the last chapter of the section on "Differential Calculus" of Boucharlat's book entitled :"On Lagrange's method of substantiating the differential calculus without recourse to limits, infinitesimals or any kind of vanishing quantities" (§§244-250,pp. 168-172, in the 5th French edition in our possession). In places this note alternates with extracts from chapter 5 of J. Hind's " The Principles of the Differential Calculus", Cambridge, 1831, §§95, 96, 99, pp. 120-121,124-129. Herein it acquires a systematic character and consists of points, successively numbered 1-9 by Marx. All these pages of the manuscript have been crossed out in pencil, with the exception of one place on sheet 39 (p. 35 according to Marx), put inside a frame by him, where, returning once more to the equations $y = f(x) + Ph$, $P = p + Qh$, $Q = q + Rh$, ..., found in Boucharlat (p. 168), Marx explains (having in view the expansion of their right hand sides in series):

So in each of these equations, only the second term, containing h in its first power, is to be found out.

ON THE EVALUATION OF LAGRANGE'S METHOD

In this conspectus Marx pays special attention to such places of the text-books under consideration, which are related to the modes of introducing the specific, symbolism of differential calculus as per Lagrange and to the evaluation of his method. Here, in places Marx mentions his sources and quotes from them (within quotation marks). Thus, on sheet 41(p .37 according to Marx) we read:

Thus Lagrange himself says, that he writes dx instead of h, for the sake of uniformity of notation*.

Boucharlat says, that the expression $\frac{dy}{dx}$ is the symbol of the operation by which we obtain the coefficient of h in the development of $f(x+h)$, and $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ etc. indicate that if repeated, the same process will make known to us the coefficients of the other powers of h.

In reality a comparison with Taylor's formula and the differential calculus generally shows that if the development of a proposed function of $x+h$ is by any means expressed in a series ascending by integral and positive powers of h, then the coefficients of h, $\frac{h^2}{2}$, $\frac{h^3}{2\cdot 3}$, ..., $\frac{h^n}{1\cdot 2\cdot 3\cdots n}$, will be so many *Derived Functions* equivalent to the corresponding *Differential Coefficients* denoted by

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ldots\ldots\ldots\ldots$$

This place of Marx's note is related to § 251(pp. 172-173 of the 5th French edition) of Boucharlat's book.

Following Boucharlat, Marx sets forth further, as an example, the way of seeking successive derivatives of ax^m, by expanding $a(x+h)^m$ according to the binomial theorem and listing the coefficients of h, $\frac{h^2}{1\cdot 2}$, $\frac{h^3}{1\cdot 2\cdot 3}$ etc., after which he writes (sheets 42, Marx's p. 38):

Consequently, this method does not permit expansion of algebraic and transcendental functions with the help of differential calculus, but, conversely, it is a means, for obtaining from the algebraic expansion of functions, the expressions for their differentials in a ready-made form.

Immediately after this place, there follows an evaluation of Lagrange's method. It is a summing up of a part (§§ 244-251) of the chapter on Lagrange's method in Boucharlat's book, from which Marx took notes [earlier]. However, in the 5th French edition of this book in our possession, there is no such formulation. It is not there in Hind's book too. (Not excluding the possibility, however, that it may be there in the notes of G. Peacock to the English translation of S.F.Lacroix's book: An elementary treatise on the Differential and Integral Calculus, Cambridge, 1816, [we are of the position that] this requires to be verified.) After this summing up and the evaluation of Lagrange's method contained therein, Marx cited the evaluations of Lagrange's method contained in the books of Boucharlat (§ 252, p. 173, 5th French edition) and Hind (§99, pp. 128-129),

---

* The words "for the sake of uniformity of notation" are there in Hind's book, §95,p.120(in the section on Lagrange's "derived functions").—Ed.

mentioning the surnames of the authors, within square brackets and quotation marks. The whole of this part of the note (sheets 42-43, pp. 38-39) has been crossed out in pencil, in the manuscript.

It is being presented below in full, since it throws some light upon a stage in Marx's path, treading which he arrived at his own point of view on the nature of differential calculus. Here (and later on) Marx's square brackets have been changed into double square brackets.

Thus, if in his theory of functions Lagrange did not do anything more than finding the expansions for functions, till then algebraically undecomposed, in order to give thereby also their differential expression*, then his contribution to differential calculus is limited to this, that :

1) he algebraically proved the *theorem of Taylor*, which, under certain constraints, may be viewed, in a determinate sense, as the basis of differential calculus, [having algebraically proved] that expansion of $f(x + h)$, from which Taylor proceeds [Since here Marx had to repeat, several times, a few words stating that Lagrange proved something algebraically, one such repetition appears omitted in the manuscript. It had to be restored, though thereby the clause became awkward. We note that in all the sources used by Marx, Taylor's theorem was proved with the assumption that $f(x+h)$ is expanded in a series of ascending integral positive powers of h. That is why, while speaking about this theorem, it was natural to connect the name of Taylor not only with its formulation, but also with its proof.(For this in detail, see PV,333-335.) However, from this it is already clear, that these observations are *not*, wittingly, mere extracts from the sources of Marx's notes.—Ed.];

2) algebraically proved once and for all, that $f(x)$ or y is the first term of the expansion of $f(x + h)$ and that the second term, contains only the first power of h as well as the coefficient of h, which is the value of the first differential coefficient $\frac{dy}{dx}$,

$$\text{or} \quad \frac{dy}{dx} = \text{the coefficient of } h,$$

and that is why the first differential is

the second term $\times$ dx (*instead of* h).

But proceeding along his own course, Lagrange not only found a new theorem for the differential calculus, to which his name is attached [Now, in the differential calculus, usually the theorem on proper value is called "Lagrange's theorem". In the text books used by Marx(see, for example, T.G.Hall, "A treatise on the differential and integral calculus", London, 1852,pp. 227-231), the formula (expansion in series) for handling functions, carrying his name, and published by him for the first time in the article "A new method for solving lettered equations with the help of series "(Mém. Ac. Berlin, 1768) (1770), was called "Lagrange's theorem ".—Ed.], but, as we shall see later on he also provided the differential calculus with its own rational basis, having presented the successive coefficients of h, h^2 etc. as the derived functions of x : $f'(x)$, $f''(x)$, $f'''(x)$[etc.].

It should be mentioned further, that Lagrange designates the derived functions differently : namely, instead of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc. he writes y', y''etc.

---

* Under "differential expression " of functions what we have in view here is the expression for their differentials or "differential coefficients" , i.e., for the derivatives.—Ed.

[[On the other hand, though in the method of Lagrange "the principles of differential calculus appear demonstrated in a manner independent of every considerations of *limits, infinitesimals or evanescent quantities*, he is himself bound constantly to have recourse to limits or infinitesimals, as soon as he comes to its applications, f.i. the determination of volumes, surfaces, the length of curves, or to obtain the expressions for the subtangents etc. "(*Boucharlat*) *.

At the same time "his method implies a previous knowledge of the methods of developing all kinds of functions of $x + h$ in integral ascending positive powers of h, which is often utterly difficult. Meanwhile, MacLaurin's and Taylor's theorems, when once established, enable us to determine, with great ease, the developments of many functions, whose expansion by common algebra, would be exceedingly tedious"(*Hind*).]]

The above quotation is from point 99,chapter V, pp. 128-129, of Hind's book.

After this (under number "9" , sheets 43-44, pp.,39-40 in the manuscript) Marx took notes from Hind (end of § 96, pp. 126-127), and cited it as an example of the proof of the theorem on the first and second differential of the product of two functions, by Lagrange's method. It simply consists of a formal multiplication of the Taylor's serieses for these functions and then of writing out the coefficients of h and $\frac{h^2}{2}$ as the first and second derivatives of the product (with their subsequent translation into the Leibnitzian language of differential symbols).

Marx separated the rest of point "9" from the preceeding text by a horizontal line (sheet 44,p. 40 according to Marx) and gave it the title :"*Towards equation III*, p.34". "Equation III"is obtained by substituting $h + i$ for h in the expansion of $f(x+h)$ into a series of powers of h, which gives one of the expansions of $f(x + h + i)$ into a series of powers of the two variables h and i. Then, substituting $x + i$ for x in the expansion of $f(x + h)$ according to the powers of h and, expanding therein the coefficients of the powers of h according to the powers of i (in the expansion for $f(x + h)$ these coefficients are functions of x),Boucharlat obtains another expansion for

$$f(x + h + i)$$

according to the powers of the variables h and i. Here Marx first of all writes out the second expansion, and abruptly stops the conspectus, without proceeding upto equation III. What, namely, did not satisfy him in the previous notes (or in Boucharlat), remains unclear. However, it is important, that this is part of the Lagrangian explanation of the connection between the coefficients of the expansion $f(x + h)$ into a series of the powers of i and the successive derived functions of $f(x)$, and here Marx goes over to a consideration of the corollaries which follow from it.

Here the crossed out part of the manuscript comes to an end. Marx did not cross out the following sheets 45-63.

ON THE DIFFERENT MEANS OF SEEKING (AND DETERMINING) THE SUCCESSIVE DERIVATIVES OF THE FUNCTION $f(x)$

The non-crossed-out part of the manuscript begins as under (s.45,Marx's p.41) (here $y_1 = f(x + h)$ and, h is assumed to be different from zero):

10) If at first we take the Lagrangian expansion as tne pure development of the *derived functions of x,*

* In the 5th French edition of Boucharlat, this is point 252, p. 173.—Ed.

$$y_1 = f(x)\ (\text{or } y) + f'(x)\,h + f''(x)\frac{h^2}{1\cdot 2} + f'''(x)\frac{h^3}{1\cdot 2\cdot 3} + \cdots,$$

$$y_1 - f(x) \text{ or } y_1 - y = f'(x)\,h + f''(x)\frac{h^2}{1\cdot 2} + f'''(x)\frac{h^3}{1\cdot 2\cdot 3} + \cdots,$$

$$(\text{ the first difference} = \Delta y\,) \qquad \text{(Ia)}$$

and

$$\frac{y_1 - y}{h} = f'(x) + f''(x)\frac{h}{2} + f'''(x)\frac{h^2}{2\cdot 3} + f^{IV}(x)\frac{h^3}{2\cdot 3\cdot 4} + \cdots$$

and designate

$$\frac{y_1 - y}{h} = y^{①} \qquad \text{(Ib)}$$

then

$$y^{①} = f'(x) + f''(x)\frac{h}{2} + f'''(x)\frac{h^2}{2\cdot 3} + f^{IV}(x)\frac{h^3}{2\cdot 3\cdot 4} + \cdots \qquad \text{(II)}$$

After this Marx forms the difference $y^{①} - f'(x)$ and calls it the second difference of $\Delta^2 y$; after that he introduces the function $y^{②} = \frac{y^{①} - f'(x)}{h}$. Subtracting $\frac{1}{2}f''(x)$ from the latter he gets the third difference and the function $y^{③} = \frac{2y^{②} - f''(x)}{h}$, with which he operates the way he did with $y^{①}$ and $y^{②}$. Later on Marx writes out the following equalities (sheet 46,p. 42 according to Marx):

1) $\frac{y_1 - y}{h} = f'(x) + f''(x)\frac{h}{2} + f'''(x)\frac{h^2}{2\cdot 3} + f^{IV}(x)\frac{h^3}{2\cdot 3\cdot 4} + \cdots,$

2) $\frac{y_1 - f(x)}{h^2} - \frac{f'(x)}{h} = \frac{1}{2}f''(x) + f'''(x)\frac{h^2}{2\cdot 3} + f^{IV}(x)\frac{h^3}{2\cdot 3\cdot 4} + \cdots,$

3) $\frac{y_1 - f(x)}{h^3} - \frac{f'(x)}{h^2} - \frac{\frac{1}{2}f''(x)}{h} = \frac{1}{2\cdot 3}f'''(x) + f^{IV}(x)\frac{h}{2\cdot 3\cdot 4} + \cdots,$

4) $\frac{y_1 - y}{h^4} - \frac{f'(x)}{h^3} - \frac{\frac{1}{2}f''(x)}{h^2} - \frac{\frac{1}{2\cdot 3}f'''(x)}{h} = f^{IV}(x)\frac{1}{2\cdot 3\cdot 4} + \ldots,$

Transforming (correspondingly) the left hand parts of these equalities into

$$\frac{f(x+h) - f(x)}{h},\quad \frac{f(x+h) - f(x) - f'(x)\,h}{h^2},$$

$$\frac{f(x+h) - f(x) - f'(x)\,h - \frac{1}{2}f''(x)\,h^2}{h^3},$$

$$\frac{f(x+h) - f(x) - f'(x)\,h - \frac{1}{2}f''(x)\,h^2 - \frac{1}{2\cdot 3}f''(x)\,h^3}{h^4}$$

[y_1 was changed into its value $f(x+h)$], and in them formally assuming $h = 0$ [it is possible to do this, since it has been assumed that $f(x+h)$ is decomposable into Taylor's series, and that is

why here all the necessary limits are meaningful] Marx further concludes (sheet 46, p. 42 according to Marx) :

Hence, we get :

1) $\frac{0}{0}=f'(x)$, 2) $\frac{0}{0}=\frac{1}{2}f''(x)$,

3) $\frac{0}{0}=\frac{1}{2\cdot 3}f'''(x)$, 4) $\frac{0}{0}=\frac{1}{2\cdot 3\cdot 4}f^{IV}(x)$,

etc. etc.

We get these different values of $\frac{0}{0}$ purely algebraically. They all emerge from [successive] deduction of derivatives from the initial function of x , since $f'(x)$ is the derivative of $f(x)$, $f''(x)$ — that of $f'(x)$ etc. Hence, what remains of the process itself, is attaching to the symbol $\frac{0}{0}$ its various values.

However, it is natural that, if a symbol can have *different* values depending upon the *means* of its generation, then it can not take the form of a constant, but must contain variable (parameters), indicating whatever supplementary information is, wittingly, still necessary, so that its value may be exactly determined (in any case, what is necessary, is to know for it, that concrete process, in which this value is formed). (For example, the symbol of derived function must contain an indication, that the prototype function and argument, according to which the derivative is sought, should be known). If such supplementary information is related to the *means* of generating the object, then it is clear, that it must determine this means *itself*, its difference from the other means , i.e., its *quality*. The symbol $\left(\frac{0}{0}\right)$ — even if it is provided with the indexes (1), (2), (3) ...., playing the role of the values of the numerical parameter(n) — does not satisfy this demand. The state of affair is different with the symbols $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ etc. That is why the question of substituting the symbol $\frac{0}{0}$ by these symbols — of the dialectic of quantity and quality connected with them — draws the attention of Marx, and he specially dwells upon it in sheets 47-49(pp. 43-45 in Marx's numeration). Part of this note, related to the substitution of the symbol $\frac{0}{0}$ by the symbol $\frac{dy}{dx}$, was published earlier(in Russian , see "Pod Znamenem Marxizma", 1933, No. 1, pp. 21-23) . This note is being presented below, in full.

ON SUBSTITUTING THE SYMBOL $\frac{0}{0}$ BY THE SYMBOLS $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ ETC.

The ratio

$\frac{f(x+h)-f(x)}{h}$ or $\frac{f(x+h)-f(x)}{x_1-x}$ or $\frac{y_1-y}{x_1-x}$ or $\frac{\Delta y}{\Delta x}$ expresses [the ratio] of the difference between the initial magnitude of the function $f(x)$ and its augmented magnitude $f(x+h)$ [to h], or the ratio between the part by which the function of x, i.e., $f(x)$ grew and the magnitude of increment of the variable x, of which it [i.e. f] is the function.

This is the ratio of the difference of the function of x to the difference of the variable x itself.

In the *numerator* we have the *difference* between the *functions of x*, in the *denominator* — the difference between the initial and augmented magnitudes of the variable x itself; in the denominator — a measure of the change in x, in the numerator — a measure of the change of its function.

Δy is the first difference of y, and Δx is the first difference of x.

If $\Delta x = 0$, then $\Delta y = 0$, since in so far as x became $= x + \Delta x$, y became $= y_1$.

We see in the first and the second expressions :

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x+h)-f(x)}{x_1 - x} = \frac{y_1 - y}{x_1 - x} = \frac{\Delta y}{\Delta x},$$

that if $h = 0$, then

$$\frac{f(x+0)-f(x)}{0} = \frac{f(x)-f(x)}{0} = \frac{0}{0}.$$

It is clear that, as soon as Δx or h becomes equal to zero, $y_1 - y$ or Δy becomes $= 0$.

Since $x_1 - x = (x + \Delta x) - x = \Delta x$, as soon as Δx becomes equal to 0, so becomes Δy.

Thus, it is clear, that here $y_1 - y$ or Δy does not only become 0, but it is also [true] that this happens only as a consequence of the transformation of Δx into zero or of the equalisation $x_1 = x$; since $x_1 - x = \Delta x$, i.e., $(x + \Delta x) - x = \Delta x$, so the first side [L.H.S.] can become zero or $x + \Delta x$ can become $= x$, only if Δx becomes 0.

Thus, even while the vanishing Δy displays dependence of the function y upon the variable x, of which it is the function, its final turning into 0, its final disappearance, itself remains a consequence of the disappearance of Δx, which is the increment of the variable x; dependence of the function y upon the variable x is retained right upto the nullification [108].

But in the expression $\frac{0}{0}$ this *qualitative relation* between the function y and the variable x, of which it is the function, has also *vanished.* In the expression $\frac{0}{0}$ all trace of the *qualitative* difference between the numerator and the denominator, between the function of a variable and the variable itself, has been erased.

That is why, in order to express the emergence and meaning of $\frac{0}{0}$, we put dx in place of the vanishing Δx, and with it the vanishing Δy is now naturally substituted by dy.

Hence, $\frac{dy}{dx}$ is not only the symbol for $\frac{0}{0}$, but it is at the same time also a symbol of the *process*, through which, under given determinate conditions, $\frac{0}{0}$ was obtained from the original equation, and it $[\frac{dy}{dx}]$ expresses that which $\frac{0}{0}$ can not express, namely the fact, that the transformation of Δy into 0 flows from the qualitative relation of the function y with the variable x, and that is why, the transformation of Δy into dy is a consequence of the transformation of Δx into dx. Thus, in the negation is retained that *qualitative relation*, of which this transformation is the negation [109].

On the other hand, $\frac{0}{0}$ does not indicate *what vanishes*; here only the *quantitative* side is expressed, namely, that the numerator has vanished , as also the denominator and thereby the ratio itself vanishes ;the *existing qualitative relation*, owing to which 0 of the numerator is only a *consequence* of the 0 of the denominator, i.e., the very expression of dependence of the function upon the variable, of which it is the function, is not expressed [in it]. It is quite true, that $\frac{0}{0}$ can express any magnitude, but to the same extent can x express any magnitude; the particular value of $\frac{0}{0}$, as well as of x, every time depends upon those determinate conditions or functions, in which this $\frac{0}{0}$ or x figures, and upon those determinate conditions, which lead to the emergence of a $\frac{0}{0}$ or to a change in x.

But not only did the investigation into the process of emergence of $\frac{0}{0}$ lead us to the symbol $\frac{dy}{dx}$ for the transformation of $\frac{\Delta y}{\Delta x}$ into $\frac{0}{0}$, but [also]to the result obtaining from the original equation. Namely, this result is :

$\frac{0}{0} = f'(x)$, and not $\frac{0}{0} = 0$, or any other arbitrary real value .

We have (see p. 42*):

1) $\frac{0}{0}\ (1) = f'(x)$, 2) $\frac{0}{0}\ (2) = \frac{1}{2} f''(x)$,

3) $\frac{0}{0}\ (3) = \frac{1}{2 \cdot 3} f'''(x)$, 4) $\frac{0}{0}\ (4) = \frac{1}{2 \cdot 3 \cdot 4} f^{IV}(x)$

etc. etc.

Thus, we see that the first real content of the symbol $\frac{0}{0}$ is equal to the *first derived function of x or the function $f'(x)$*, and that the further contents, or real values of $\frac{0}{0}$ all consist of determinate functions of the variable x, deduced from the original function of x, successively emerging one from the other, according to a determinate rule.

[[Here we can also say that when $\frac{0}{0}$ turns into 0, thereby the differential coefficient as well as the limit becomes equal to zero. This takes place, when the process leads to [a situation] when the variable itself vanishes or becomes equal to some constant.

For example, if we had $y = f.x$

[here the abbreviated notation signifies : y is a function of x, graphically coinciding with the expression "x"] ,

$$y_1 = f.(x + h)$$

---

* Sheet 46;PV, 137.

[i.e., y, graphically coincides with the function "$x + h$"], then

$$y_1 = f . x + h \, [= y + h] ;$$

and when h turns into 0, here

$$y_1 = f . x = y.$$

Hence

$$y_1 - y \text{ or } \Delta y = h; \frac{y_1 - y}{x_1 - x} = \frac{f(x+h) - f . x}{x_1 - x} = \frac{\Delta y}{\Delta x} = 1 ,$$

i.e., $f . \frac{dy}{dx}$ [here it means : i.e., equal to (the derived) function $\frac{dy}{dx}$].

So that $\frac{0}{0} = 1$. Here the variable x has vanished.

If we substitute $\frac{0}{0}$ by $\frac{dy}{dx}$, then $\frac{dy}{dx} = 1$, hence $dy = dx$ and $1 = 1$ (since $\frac{dy}{dx} = \frac{a}{a} = 1$).

Here $\frac{0}{0}$ became equal to some constant....

Let us differentiate $\frac{dy}{dx}$ for obtaining the second differential; since the increment of the constant is equal to zero, we shall get $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = 0$ or $\frac{0}{0} = 0$.]]

Thus, only by fixating (in symbols) the qualitative relation within $\frac{0}{0}$ by $\frac{dy}{dx}$, we get the opportunity to fixate the $\frac{0}{0}$ (2), $\frac{0}{0}$ (3) and $\frac{0}{0}$ (4) differing form the first $\frac{0}{0}$ or $\frac{0}{0}$ (1) but connected with it and emerging from it successively, to make them symbols of the processes connected amongst themselves in a law governed way, of the processes which give birth to them. [We note that] they themselves express this connection of theirs, through the second side [R.H.S.], where their different real values appear, values which have determinate relations among them, and they all emerge at a step either closer or more distant from the original function of x and the first equation A), in which this initial function still figures.

Marx now goes over to a discussion of the first of the four equations, cited above (see p.136), having noted (at the bottom of sheet 49, p. 45 according to Marx):

Now we should first of all consider the result of the first differentiation, taking into account the second side [R.H.S.] of our equation and the processes which lead to it.

The equation at issue here is the same expansion of $f(x+h)$ into Taylor's series.

Immediately after these words we find the note cited below (sheet 50, p. 46). What is at issue here is the differential as the principal part of the increment of a function [110].

ON THE DIFFERENTIAL AS THE PRINCIPAL PART OF THE INCREMENT OF A FUNCTION

A) *The original equation :*

$$f(x+h) \text{ or } y_1 = f(x) \text{ (or } y) + f'(x)\, h + \frac{1}{2} f''(x)\, h^2 + \frac{1}{2\cdot 3} f'''(x)\, h^3 + \frac{1}{2\cdot 3\cdot 4} f^{IV}(x)\, h^4 + \cdots$$

As the first task we take away $f(x)$ form $f(x+h)$ or y from y_1. Then we get, so long as $h > 0$,

$$\Delta y = f'(x)\, h + \frac{1}{2} f''(x)\, h^2 + \frac{1}{2\cdot 3} f'''(x)\, h^3 + \frac{1}{2\cdot 3\cdot 4} f^{IV}(x)\, h^4 + \cdots$$

Further, for obtaining the ratio of the difference between $f(x+h)$ and $f(x)$ to the increment of the variable x (since $x_1 - x = h$), we divide both the sides by h and get

$$\frac{\Delta y}{\Delta x} = f'(x) + \frac{1}{2} f''(x) h + \frac{1}{2\cdot 3} f'''(x) h^2 + \frac{1}{2\cdot 3\cdot 4} f^{IV}(x) h^3 + \cdots$$

The division by h, which is essential for obtaining the ratio $\frac{\Delta y}{\Delta x}$, freed *the second term of the original equation* and the first term of this new equation from h, in fact it is that term, which has h as its multiplier, only in the first power; but along with this all the remaining terms of the original equation also got modified, so that the *multiplier h, in each of them, lost its power by 1*, such that, for example, the third term has h^{3-1} or h^2 instead of h^3 etc. However, it is important to remember, once and for all, that the process of freeing the second term of the original equation from the multiplier h, also *modifies* the remaining terms.

Secondly, we see from the equation where Δy appears (i.e., *the original function of x still participates*, so long as

$$\Delta y = f(x+h) - f(x)),$$

that the more the magnitude h diminishes, the lesser becomes each of the successive terms, in comparison to the preceeding ones, such that the first term $f'(x)h$ where $f'(x)$ is accompanied only by the first power of h, expresses the *biggest* [part] *of the difference* between y_1 and y, and the smaller h becomes, the more does this term become *proximate to the sum of the partial differences*, the sum total of which is A or the series for Δy, and the less does Δy exceed * $f'(x)h$.

Marx does not call the product $f'(x)\,h$ (or, in another notation $f'(x)\,\Delta x$), the differential. However, we see, that he specially stresses the role of this product as the principal part of the increment of the function $f'(x)$, when x is increased by h (or Δx). Thus we have a basis for thinking, that Marx had a concept , equivalent to the concept of the *differential* as the *principal part of the*

---

* Here in the sense of "differs from". In a number of places Marx himself stipulates (see pp.88,91,225-226) the use of the term "increment" in the sense in which we now use the term "absolute value of an increment". — Ed.

increment of a function (for what it was like in Euler, see : Appendix, On Leonhard Euler's Calculus of Zeros, p.316).

ON TWO DIFFERENT WAYS OF DETERMINING THE DERIVATIVE

Discussion of the first of the four equations cited above in p.136 concludes with the words (sheet 50,p. 46 according Marx):

If h becomes = 0 and that is why $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$, then in the second side [R.H.S.] of the equation all the terms containing h vanish and there remains only $f'(x)$ as the real value of $\frac{dy}{dx}$. It should be remembered, that $\frac{0}{0} = f'(x)$ was obtained purely algebraically, without recourse to the differential calculus, and, that, conversely, rather, the symbolic differential expression $\frac{dy}{dx}$ for $\frac{0}{0}$ was developed from the algebraic result $\frac{0}{0} = f'(x)$.

There is not doubt, that it is not a note. These are the words of Marx himself, and besides they are such that just as above, they contain his characteristic ideas, which he developed later on : ideas like the counterposing of the "symbolic" differential expression against its "real", "algebraic " value.Having concluded the discussion on the first equation, Marx goes over to the next, observing therein (sheet 50, p.46 according to Marx) :

There are two difficulties here. They emerge, when we do not have, as it happened here, these four algebraic equations side by side, but get them though differentiation.

What, namely, are these two difficulties ? It is not fully clear. It is true that after this statement the text starts with point a), but there is no point b) in the manuscript.

However, from the context it is clear, that what is at issue here are the difficulties, which emerge in the transition from the equations taken down by Marx, to the equations used in the successive search for the derivative of the $(n+1)$-th order ($n \geq 0$) of $f(x)$, as the *first* derivative of the derivative of the n-th order. Judging by the beginning of point a), by the *first difficulty* connected with such transition — though it is still to come — Marx implies the question, as to how could h —in connection with the search for the first derivative, which was already subjected to (as we would say now) " a limiting transition to zero through h" * — again turn out to be different from zero, and be again subjected to a limiting transition to zero through h, while seeking the second derivative etc. In so far as, having explained this, Marx goes over to the general question of reconciling the usual definition of the derivative of $(n+1)$-th order as the first derived function of the derivative of the n-th order, with the definition of the expressions $\left(\frac{0}{0}\right)^1$, $\left(\frac{0}{0}\right)^2$, $\left(\frac{0}{0}\right)^3$ etc., through the differences obtainable from the ready-made expansion of $f(x+h)$ into Taylor's series (see : PV,136), it is natural to think that under the *"second difficulty"* Marx had in view : the question of the connection between the two different definitions of the derivative of $(n+1)$-th order :1) as the co-efficient of $\frac{1}{(n+1)!} h^{n-1}$ in the expansion of $f(x+h)$

* In this connection Marx also speaks of the limit (see below , p.144).—Ed.

and, 2) as the first derivative of the derivative of n-th order. To all appearance, in the present manuscript, Marx still thinks, that Lagrange not only succeeded in deriving the second of these definitions from the first, but also, conversely, the first from the second, i.e., he could prove their equivalence. Later on, as we know (see : PV,80,§3) "Purely algebraic differential calculus"), Marx no more thought that Lagrange actually succeeded in putting the differential calculus upon a "purely algebraic " foundation.

At the beginning of point a) Marx once more writes out all the four equations cited above (PV, 136), in the following form (sheet 50,p. 46) :

$$1)\quad f(x+h)=f(x)+hf'(x)+\frac{1}{2}h^2f''(x)+\frac{1}{2\cdot3}h^3f'''(x)+\frac{1}{2\cdot3\cdot4}h^4f^{IV}(x)+\cdots$$

$$2)\quad f^{①}(x+h)=f'(x)+\frac{1}{2}hf''(x)+\frac{1}{2\cdot3}h^2f'''(x)+\frac{1}{2\cdot3\cdot4}h^3f^{IV}(x)+\cdots,$$

$$3)\quad f^{②}(x+h)=\frac{1}{2}f''(x)+\frac{1}{2\cdot3}hf'''(x)+\frac{1}{2\cdot3\cdot4}h^2f^{IV}(x)+\cdots,$$

$$4)\quad f^{③}(x+h)=\frac{1}{2\cdot3}f'''(x)+\frac{1}{2\cdot3\cdot4}hf^{IV}(x)+\cdots$$

(Here $f^{①}(x+h)$, $f^{②}(x+h)$, $f^{③}(x+h)$ have been used in a special sense. They designate respectively :

$$\frac{f(x+h)-f(x)}{h},\quad \frac{f(x+h)-f(x)-f'(x)h}{h^2}\text{ and}$$

$$\frac{f(x+h)-f(x)-f'(x)h-\frac{1}{2}f''(x)h^2}{h^3}\text{, see PV, 136 .)}$$

Having taken note of the fact that, afterwards from these equations the successive values of $\left(\frac{0}{0}\right)^1$, $\left(\frac{0}{0}\right)^2$, $\left(\frac{0}{0}\right)^3$ and $\left(\frac{0}{0}\right)^4$ were "developed", Marx continues (sheet 51, p. 47 according to Marx) as under :

Hence, here there is no algebraic ground to get astonished about the fact ,that in 1) *h has been assumed to be equal to 0* — and that is why *all the terms* containing h^2, h^3 etc. — in other words, containing the powers of h greater than h^1 — have vanished; that in equation 2) there again appeared, not only h, but also the extinct terms , and besides in a form, to which they were led in course of obtaining $\frac{dy}{dx}$ or $\left(\frac{0}{0}\right)^1$ from equation 1) ; and that the same holds good between equations 2) and 3) etc .

The supposition of $h=0$ in all the four equations can give, in each of them, only different results, since our $f(x)$, and that means also $f(x+h)$, is found in each of them in different conditions; all the four equations express f [functions of] $(x+h)$,but $f(x+h)$,$f^{①}(x+h)$, $f^{②}(x+h)$, $f^{③}(x+h)$ etc. are deduced one from the other under different conditions, and that is why these functions are different from each other.

After this six lines have been crossed out in the manuscript (sheet 51, p. 47 according to Marx). In them the following notations have been introduced for the "different y-s'' : y, y', y'', y'''etc. — through p,q,r etc., and a consideration of the example $y=ax^3$ began; however, apparently,

Marx did not plan doing that, and went over to a consideration of the second difficulty. Here he writes (sheet 51, p.47):

When in the first equation, which still contains the original function of x, and that is why also the original function $f(x+h)$, as a result of assuming h equal to zero ; in conclusion : in $\frac{dy}{dx}$ $\left(\text{or in } \frac{y_1 - y}{x_1 - x}\right)$ only the coefficient of h, now remains in its first power, however, freed from h [and] equal to $f'(x)$ — it is the first derived function of x ; the thing is that, this is the *limit* of the variation of x in that equation, in which the original function of x appears ; meanwhile the assumption that h is equal to zero gives at the same time the other limits in the derived equations or, what is the same thing, the other values of $\frac{0}{0}$. (*On the concept of limit see below* *.) The internal connection among the different equations, each of which starts from the result of the preceeding one, flows from their structure, their derivation, and since Lagrange elucidated it purely algebraically, no objection can be raised against it [111].

But if "no objection can be raised ", then the second objection also turns out to be settled. Actually, one may use the Lagrangian proof, that the coefficient of $\frac{1}{n!}h^n$ in the expansion of $f(x+h)$, in the series of powers of h, is the n-th derivative of $f(x)$ — for obtaining this derivative, not with the help of that n-th equality, on the left hand side of which the expressions $f^{(1)}(x+h)$, $f^{(2)}(x+h)$ etc. stand, but as the first derivative is obtained, from the expansion of $f(x+h)$ — but now already from those analogous expansions for $f'(x+h)$, $f''(x+h)$ etc., where $f'(x+h)$, $f''(x+h)$ etc., are the augmented values of the derived functions $f'(x)$, $f''(x)$, etc., in that very sense in which $f(x+h)$ is the augmented value of the function $f(x)$. (It is natural, that these expansions are no more to be written at the same time , since at first $f'(x)$, $f''(x)$, etc. should be found successively.) It is clear, that this is what Marx had in view, when immediately after the place cited above he wrote (sheets 51-52, pp. 47-48 according to Marx):

But, on the other hand, from here it follows that, since these different results

$$\left(\frac{0}{0}\right)^1, \left(\frac{0}{0}\right)^2, \left(\frac{0}{0}\right)^3, \left(\frac{0}{0}\right)^4$$

have as their source the connection among the equations, derived one after the other, then, namely, that is why, we can also operate otherwise, since this has already been established.

[Here what is being referred to is the Lagrangian proof, elucidating the connection among the coefficients of the expansion of $f(x+h)$ into a series of powers of h and the successive derived functions of $f(x)$, (see : Appendix p.332).]

[[Instead of having, for example, 4 equations for the 4 first differential coefficients at the same time, it is enough to have the result of the first, in order to obtain the data for the second, and in the same way — the result of the second, in order to develop the third from

* See , footnote in p.142. —Ed.

it, for, namely, because they are derived one after the other, the result which gives the first, must give the second etc...]]

[In the paragraph omitted below, briefly speaking the issue is this that : it is enough to consider the transition from the coefficient of h to consider the transition from the coefficient of h to the coefficient of h^2 since "after that everything is understandable all by itself for the coefficients of h^3, h^4 etc."]

[[$f'(x)$, the first *derived* function [in] x (as distinct from the original function [in] x, i.e., $f(x)$), in other words, the coefficient of h in the original equation, is the new starting point. Since x is a variable, it must permit a new increment within new boundaries. We shall proceed from this result $f'(x)$ and at first consider it as in independent function [in] x, not recalling, at first, its more distant relations with $f(x)$ and $f(x+h)$. Then we shall get:

$$y = f'(x),$$

$$y_1 = f'(x+h) = f'(x) + f''(x)h + \frac{1}{2}f'''(x)h^2 + \ldots,$$

$$\Delta y \text{ or } y_1 - y = f'(x+h) - f'(x) = f''(x)h + \frac{1}{2}f'''(x)h^2 + \ldots$$

Hence,

$$\frac{\Delta y}{\Delta x}\left(\text{or } \frac{y_1 - y}{x_1 - x}\right) = f''(x) + \frac{1}{2}f'''(x)h + \frac{1}{2\cdot 3}f^{IV}(x)h^2 + \cdots$$

Earlier the second line appeared as the second equation,the first derived equation beside the first.]]

[Here "earlier" indicates those places in the manuscript (sheet 39 and 41, pp. 35 and 37 according to Marx), which contain the notes of §248 of Boucharlat's book (p 171 of its 5th French edition), wherein the Lagrangian proof of the connection among the coefficients of the expansion $f(x+h)$ is given and for the successive derivatives of $f(x)$ the following equations are used :

$f(x+h) = f(x) + hf'(x) +$ terms *in* h^2, in h^3 etc.

$f'(x+h) = f'(x) + hf''(x) +$ terms *in* h^2, in h^3 etc.

$f''(x+h) = f''(x) + hf'''(x) +$ terms *in* h^2, in h^3 etc.

etc. etc. etc.]

$$\left(\frac{0}{0}\right)^2 \text{ or } \frac{dy}{dx} = f''(x) \text{ or } \left(\frac{y_1 - y}{h}\right).$$

Here Marx designated the transition to limit (as $h \to 0$), by placing within brackets an expression, which we would have placed after the sign of limit. Later on he specially stipulates it (see PV, 267).

Here symbols of the type $\left(\frac{0}{0}\right)^1$, $\left(\frac{0}{0}\right)^2$ etc. ere simply related to the successive derivatives.

Hence, now it is necessary to interpret this $\left(\frac{0}{0}\right)^2$ or $\frac{dy}{dx} = f''(x)$ in connection with the result of the first $f(x+h)$, which is its starting point. If it is considered all by itself, then, what was true for the first function [in] $x+h$, is absolutely true for it. The difference appears only then, when we consider it, not in an isolated way, but at the same time as the result, which we obtained as we proceeded from the result $f'(x)$ of the first equation.

Having written this, Marx continues (sheet 52, p.48 according to Marx):

Let us take as an example :

$a(x+h)^m = y_1$; i.e., $f(x) = ax^m$ and $y_1 = f(x+h) = a(x+h)^m$.

This example (sheets 52-63, pp. 48, 46, 44, 45-52, 52 according to Marx) is very convenient, because in it the expansion into a series according to the powers of h, is given by the binomial theorem of Newton, i.e., without the help of differential calculus. Here Marx verifies in detail, all that he has said above. Thus, for example, having shown sub 1), that the coefficient of h (i.e., max^{m-1}) is actually (the first) derivative of ax^m in the usual sense (i.e., it is the limit of the ratio $\frac{f(x+h)-f(x)}{h}$ as $h \to 0$), Marx writes further (sheets 53,p. 46, the second 46-th page according to Marx):

max^{m-1} is a pure function of x, wherein no h enters, just as it was the case with ax^m. In both of these expressions x is also the variable, and that is why it also admits of variation in the second, just as in the first. That is why, sub 1) we could just as well take max^{m-1} as the starting point, as $f(x)$, as we did in the case of ax^m.

After this, Marx goes through all these details, also for seeking the second derivative, specially stressing, that the second derivative of $f(x)$ — considered as the first derivative of $f'(x)$ — herein actually turns out to be the second derivative of $f(x)$ and that too in the Lagrangian sense (i.e., the coefficient of $h^2/2$).

In connection with this, and in addition to the foregoing, Marx dwells upon yet another definition of the second, that is, also of the higher order derivatives (and differentials) : through the differences of higher orders.

Some of the manuals at Marx's disposal are known to us. In them derivatives of higher orders have not been defined through finite differences of higher orders (usually this is not done in modern courses too). If it is done, as for example, in Lacroix's book (S.F.Lacroix, "Traité du calcul différentiel et du calcul intégral", vol.I-III, Paris, 1810-1819), then only in the section on "Finite differences", which Marx did not study at all. (Marx's notes on mathematical analysis — with the exception of those from the works of Newton, see above pp.127-131 — are related to the section entitled "Differential Calculus".) If we use that definition of the "second difference or $\Delta^2 y$", which to all appearance belongs to Marx himself, and which he introduced above (see p.136), then we get that $f''(x)$ is the limit of the ratio $\frac{2\Delta^2 y}{\Delta x}$ when $\Delta x \to 0$(here h is designated through Δx), meanwhile here for Marx (sheet 57, p. 47 according to Marx,second time "47" for Marx) $f''(x)$ is that very limit of the ratio $\frac{\Delta^2 y}{\Delta x^2}$, which is cited in the modern manuals, where $\Delta^2 y$ is the usual difference of second order. The calculations through which Marx lays the foundation of this conclusion, also indicate the same. These calculations — if carried out with more appropriate notations (Marx's notation for the augmented value of the derivative y' is inappropriate, because, ' was adopted to designate the derivative) — look as under :

Let us have the following notations (the symbol $\rightleftarrows$ reads : designates):

$y \rightleftarrows f(x)$,

$y_1 \rightleftarrows f(x+h)$, $\left|\begin{array}{l}\Delta y \rightleftarrows y_1 - y,\\ \Delta y_1 \rightleftarrows y_2 - y_1\end{array}\right|$ $\left|\begin{array}{l}\Delta^2 y \rightleftarrows \Delta y_1 - \Delta y\\ \hline \text{let } \Delta x = h \text{ be a}\end{array}\right|$

$y_2 \rightleftarrows f(x+2h)$ constant.

And let

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}. \qquad (1)$$

Let us substitute for the derivative $f'(x)$ its approximate ("pre-limit") value : $\frac{f(x+h) - f(x)}{h} = \frac{\Delta y}{\Delta x}$. Then analogously $f'(x+h)$ is substituted by

$\frac{f(x+2h) - f(x+h)}{h} = \frac{y_2 - y_1}{h} = \frac{\Delta y_1}{\Delta x}$ and the ratio $\frac{f'(x+h) - f'(x)}{h}$ by

$$\frac{\frac{\Delta y_1}{\Delta x} - \frac{\Delta y}{\Delta x}}{\Delta x} = \frac{\frac{\Delta y_1 - \Delta y}{\Delta x}}{\Delta x} = \frac{\frac{\Delta^2 y}{\Delta x}}{\Delta x} = \frac{\Delta^2 y}{\Delta x^2},$$

whence Marx draws the conclusion [see (1)] :

$$f''(x) = \lim_{\Delta x \to 0} \frac{\Delta^2 y}{\Delta x^2} \quad * .$$

Thus it is natural to think that here Marx used — not in the form of an extract, but a proper account (of) — some other source still unnoticed by us, where the differential calculus has been set forth in closer connection with the calculus of finite differences (as it was done, for example, in Euler, whose "Differential Calculus" even begins with a chapter on "The Finite Differences" ; see, Institutiones calculi differentialis cum ejus usu in analysi finitorum ac doctrina serierum, auctore Leonhardo Eulero, impensis Academiae imperialis scientiarum, Petropolitanae, 1755), (there is a Russian translation: L. Euler, "Differentsialnoe ischislenie", Moskva-Leningrad, 1949, ch. 1). It is true that, to all appearance, Marx could not get acquainted with Euler's book , though apparently, he intended to do so.

Later on (sheet 57-58, pp. 47-48 according to Marx) having done (excluding the definitions through finite differences), what was done for the second derivative, also for the derivatives of third and fourth orders, Marx still further concretises his example, assuming $m = 3$ in the formulae obtained by him. Namely, he writes (sheet 59,p. 49) :

For example, if instead of the indeterminate exponent or index m we substitute 3, then we shall get [....] .

In this example, Marx is able to complete all the calculations through to the end (upto obtaining the fourth derivative, equal to zero), which he does (sheets 59-63, pp. 49-52, once more 52) first by listing the formulae obtained earlier in a general form (in terms of m), and then by assuming in them $m=3$. Herein Marx especially highlights three circumstances : that 1) the initial function and all its derivatives are different functions (sheet 62,p. 52),

but all the terms, which later on develop independently, and are differentiated, are included in the initial function , so that the initial function already contains these derivatives in embryo.

2) The successive derivatives are not simply coefficients of $h, h^2, h^3, \ldots$ in the expansion of $f(x+h)$, but are distinguished from the coefficient of h^n by the multiplier $\frac{1}{1 \cdot 2 \cdot 3 \cdots n}$ (sheet 62,Marx's p. 52).

* In the modern courses of mathematical analysis (see, for example, F. Franklin, "Mathematical Analysis", Part I, Moscow, IL, 1950,§93, Finite Differences, pp.150-160) this is proved in a more general form and more strictly (though not constructively[111a]). —Ed.

3) While obtaining the coefficients of the expansion for $f(x+h)$ (if we do not have them yet) along the path of successive differentiation of the already obtained terms of the expansion, it should be remembered, that h is considered to be a constant (sheet 63,p. 52, Marx's second 52-nd page).

In other words, Marx knew beforehand that, the mistakes of calculation impeding understanding may be connected with the lack of attention towards the exact formulation of the theorem about the connection between the derived functions of $f(x)$ and the coefficients of expansion of $f(x+h)$ into a series according to the powers of h.

Following this, the notes on sheets 64-68 (pp. 53-57 according to Marx) have been struck out in pencil. They contain the following extracts:

extracts on Lagrange's method from Boucharlat's book, pp. 168-171 (of the 5th ed.)and, extracts on Taylor's theorem from Hind's book, pp. 83-85. He did not strike off the notes on sheets 69-77 (pp. 58-66 according to Marx). However, since they contain only the extracts from Hind's book, here we limit ourselves to indicating the corresponding pages of the book and the titles , under which these extracts have been cited by Marx.

Sheets 68-71 (pp. 57-60 according to Marx). Extracts from Hind's book, pp. 86-92. Marx's heading: "A. *Finding certain limits of Taylor's theorem in its application* "[112].

Sheets 71-73(pp. 60-62 according to Marx). Extracts from Hind's book, pp. 96-98, Marx's title :"B. *Further manipulations with Taylor's theorem*".

Sheets 73-77 (pp. 62-66 according to Marx). Extracts from Hind's book, pp. 92-96, under Marx's heading :"*C. Failure of Taylor's theorem*".

Sheets 78 and 79 (upper part) (pp. 67-68 according to Marx) contain his own observations. He specially marked them out on p.67 (sheet 78) by a line of the form⋿ in the left hand side. This section is being reproduced below. The title has been provided by us.

ON THE QUALITATIVE DIFFERENCE BETWEEN EXPRESSIONS OF THE TYPE $\frac{0}{0}$ IN ALGEBRA AND $\frac{dy}{dx}$ IN DIFFERENTIAL CALCULAS

Be it $\left(\frac{0}{0}\right)_1$, $\left(\frac{0}{0}\right)_2$ etc., i.e., that $\left(\frac{0}{0}\right)$, which we get in the first differentiation, and those $\left(\frac{0}{0}\right)_1$ etc., which appear as a result of the successive differentiations, the $\frac{0}{0}$ appears there, where x appears as the independent variable and y as the dependent variable; consequently, where it emerges from $\frac{\Delta y}{\Delta x}$ reduced to $\frac{dy}{dx}$, it is *qualitatively* different from [that] $\frac{0}{0}$, which we get there, where x is a constant magnitude, as in the ordinary algebra.

For example, if we have $\frac{x^2-a^2}{x-a}$, then for the latter we may also write $=\frac{(x-a)(x+a)}{x-a}$; if we assume that $x=a$, then $x^2-a^2=a^2-a^2$ and $x-a=a-a$; $\therefore \frac{x^2-a^2}{x-a}=\frac{0}{0}$; but it does not turn into $\frac{0}{0}$ because the denominator = 0, but rather because the numerator and the

denominator turn into 0 at the same time, when instead of x and x^2 we substitute their values a and a^2 [113] ... Meanwhile in

$$\frac{f(x+h)-f(x)}{x_1-x}=\frac{y_1-y}{x_1-x}$$

the value of the numerator is determined by the value of the denominator and is dependent on it. It is true, that I can say : if in the numerator h becomes equal to 0, then

$$f(x+h)-f(x)=f(x)-f(x),$$

and that is why = 0 ; but here I can assume that $h = 0$, only if $x_1-x=h=0$, i.e., if x_1-x becomes $=x-x=0$.

The numerator changes its magnitude, only if the denominator changes its own [114]...

On the other hand, in $f(x+h)-f(x)$, without preliminarily assuming $x_1-x=0$ or $x_1=x$, I can not assume that $h = 0$.

That is why, this qualitative inter-relation between dx and dy does not exist there, where the numerator is not a function of the variable x, but where x in the numerator and in the denominator is one and the same constant, though unknown and still indeterminate, but always a *constant* magnitude.

Further, $\frac{x^2-a^2}{x-a}=\frac{(x-a)(x+a)}{x-a}=x+a$. And when $x=a$, then that is why $\frac{x^2-a^2}{x-a}=2a$.

This $2a$ is not the limit of $\frac{x^2-a^2}{x-a}$ in the sense, as, for example, in $\frac{dy}{dx}=m$; $2a$ was obtained by simple division, as the actually obtained value of the fraction $\frac{x^2-a^2}{x-a}$.

It is a limit only in that sense, in which the actually obtained value of any numerical ratio is its limit.

Thus $\frac{6}{3}=2\cdot\frac{6}{3}$ is neither > nor < than 2, and in this sense every equality expresses some limit [115], and even for every constant quantity, like 3 etc. its limit coincides with its being. 3 is neither 2, nor 4, nor any other fraction in between 2 and 4, but is equal to 2 + 1, or 4 − 1.

$\frac{1}{3}$ is in itself its proper limit. If I express it in the form of a series, then

$$\begin{array}{r|l} 1 & 3 \\ 10 & \overline{0.33} \\ -\frac{9}{10} & \end{array}, \text{ then } \frac{1}{3}=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10{,}000}+\cdots$$

In this case $\frac{1}{3}$ becomes the limit of its infinite series.

The next two sheets 79-80 (pp. 68-69) have again been struck out by Marx. He gave them the heading :

Continuation of another note book (III, next to Kaufmann II) (last page).

These pages have already been referred to (see PV, 123). In addition to what has been said there we only note further that, here Marx puts a number of questions regarding the initial supposition of Lagrange. Having presupposed that $f(x+h)$ is equal to $f(x)+Ph$, why should Lagrange assume that P is a function not only of x, but also of h, represented in its turn in the form of $P=p+Qh$, where p is a function only of x, but Q — again of both x and h? Analogously why should it be valid also for Q etc., i.e., why should the Lagrangian series be infinite (unboundedly continuous)? And why must the same hold good for the series representing $f'(x+h)$, $f''(x+h)$ etc. ?

Here Marx's answer to these questions consists of the following : Lagrange's theorem has a general character, i.e., it should be applicable *not only* to such functions as $f(x)$, for which $f(x+h)$ is a polynomial of some determinate order relative to h, in particular, it must also be applicable to $f''(x+h)$ etc., owing to which the Lagrangian series also appears (in the general case) to be unboundedly continuous ("infinite").

This paragraph (though struck off by Marx) clearly belongs to Marx himself. It is not an extract from any source, not even a note, though it is connected with a desire to investigate the hazy ideas of Lagrange, consisting of contrapositing the "general instance" of an arbitrary function to some individual, "particular" function. That is why it would have been natural to cite this text here. However, in so far as, its beginning is situated in another note-book (manuscript 3888), and without this beginning (and what precedes it), the continuation of the text is incomprehensible, it will be adduced while describing manuscript 3888.

The last, 81-st sheet of manuscript 2763 contains only a heading : "*Continuation of the first inside cover (related to figure 1)*, which is, apparently, there in the second inside cover. Here manuscript 2763 comes to an end.

A NOTE BOOK CONTAINING NOTES ON THE DIFFERENTIAL CALCULUS ACCORDING TO THE BOOKS OF LACROIX, BOUCHARLAT, HIND AND HALL

S.U.N. 3888

Marx gave this note book the heading : *"III next to Kaufmann II"*, sheets 1-87. Language — German, in places French or English. Since the note book "Kaufmann II" (manuscript 3881) was used since March 1878, the date of this manuscript can not be placed earlier than 1878. Immediately under the heading of the note book (evidently, on its inner cover) there is a quotation from the book by Feller and Odermann, which runs (sheet 2) as under :

See p. 185 of the book *"Commercial Arithmetic"*, wherein it is said : "However, in order to calculate the net profit or net loss of an enterprise, it is necessary to take into account, also the *interest upon the invested capital* (he has in view the *capital, invested* in commodities etc.) *during the time, in course of which the latter can not be utilised"*.

The words within first brackets inscribed by Marx inside the words *"invested capital during the time"*, are not there in the book. Yet another small difference consists of this : Marx simply wrote *"to calculate"*, instead of "to be able to calculate" — which is there in the book.

On sheets 3-31 (above) (in Marx's numeration : 1-22, 22*a*, 23, 23*a*, 2-27) there are extracts from a source unknown to us : on the currencies of different countries, means of coinage, weighing ingots etc. It is still not clear, with what the above cited quotation is related.

After this, under the title : *"Insertion"*, on sheets 31-36 (pp. 27-32 in Marx's numbering) there is a note from two sections of the text-book by Feller and Odermann :

"A) *Partnership rule"*, pp. 144-151 of the text-book, sheets 31-35 of the manuscript. It is a rule for proportional division — "simple", which may be "direct" (directly proportional) and "inverse" (inversely proportional), and "compound" (compounded by some additional conditions imposed upon the given ratio).

This note consists mainly of the solution of problems, and contains almost no explanatory text from the book. The last problem solved therein (for illustrating the "compound" rule of partnership) reads: "4 mills must grind at one and the same time 2 000 tchetveriks of grain. How this grain may be divided among them, if mill A grinds 15 tchetveriks in 4 hours, B — 16 tchetveriks in 3 hours, C — 10 tchetveriks in 3 hours and D — 9 tchetveriks in 2 hours ? "(Feller and Odermann, p. 150). Marx adduces in his note both the solutions of this problem, consisting of a reduction to simple — direct and inverse — rules of partnership. (The first reduction is carried out with the help of the answer to the question : "How much grain does each of the mills grind in one hour ? "The second — by providing an answer to the question : "How many hours are needed by each of the mills, to grind one tchetverik of grain ?").

"B. Rule of mixture", i.e., the means for solving the problems of computing : 1) the mean value (of the unit of mixture according to the "weights" of the components and their quantities), 2) the numerical interrelation among the quantities of components of given "weights", necessary for obtaining a mixture of the required "weight". From this section Marx notes (sheets 35-36, pp. 31-32 in Marx's numbering) only a part, related to the problems of the first type (pp. 153 -154 of the book). Here, Marx's attention is especially drawn to a problem, in which the data about the consumption of tea in England during 1842-1846, is cited : the number of pounds of tea (consumed) and the (mean) price of one pound of tea in every year (is given) — the corresponding mean (amount and price) for 1 year is to be computed. In connection with this problem Marx not only cites the entire table of data and results, found in the text-book, but also writes down (within square brackets) a long comment, in which he analyses the dynamics of the process : the course of change

of the data over 5 years, in absolute numbers and percentages, and computes all of them. On the pages following this (on the lower part of sheet 36, sheets 37-39 ; pp. 32-35 in Marx's numbering) we find a paragraph about circulation of capital ; in terms of the subject matter this is related to the 2nd part of volume II of "Capital".

The properly mathematical part of this manuscript starts from sheet 40 (p.36 in Marx's numbering) and continues upto the end of the note book (sheet 87; in Marx's numbering 36-63, 66-73, 76-86, in the photo copy the last page is unnumbered). Almost the entire text of this manuscript is crossed out in pencil. It may be sub-divided into the following parts, the contents of which will be described below in brief. Of the comments of Marx, the smaller ones have been cited in the course of the description and the bigger ones have been specially separated and are being separately reproduced below. However, from the text of the description their reference points will be clear. Some additional references precede the text of these comments of Marx.

I. Sheets 40-41 (Marx's pp. 36-37). Extracts from the introduction to the big "Treatise" of Lacroix, which Marx begins by exactly mentioning the source, which he does only in case of such authors, whose name itself has a significance. In this manuscript he still mentions the names of Boucharlat and Hind (see below), but oftener only for criticising, and does so without mentioning the source. This part of the manuscript starts with a new page (sheet 40, Marx's p. 36), as under :

Lacroix : "Traité du calcul différentiel et du calcul intégral", t I, 1810.

Introduction.

The note consists of the following six points, corresponding to points 1-5, 10-11 of Lacroix, pp. 1-7, 13-14, but their content is given quite briefly. Since at issue here are the fundamental concepts of "function" and "limit", we shall describe it in greater detail, as compared to the other parts of the manuscript. The numbering of the points is according to Marx.

1) Marx gave this point the title : "*Function :1*". It contains only a quotation from Lacroix, cited by Marx within quotation marks. It is a definition of function as dependence (in exact German translation).

2) The second point also starts with some words which are translations from Lacroix's text (p. 2) (but here it is no more complete and, is without quotation marks) (sheet 40) :

Consideration of *indeterminate equations* led to a generalisation of the concept of function. If it is desired to express, that a certain quantity can not be assigned, without giving, *earlier, determinate values* to some other quantities, which could obtain infinite number of them [such values] in one and the same question, then the word, "function" is used for designating this dependence.

Marx notes this point in greater detail : here we have all the examples adduced by Lacroix, the definitions of *explicit* and *implicit* functions and the definition of *algebraic functions*, and comments (p. 3 of Lacroix), which reads (sheet 40) :

Among certain quantities *equations* are not demanded, if one of them is an implicit function of the remaining, it is enough, that its value is dependent upon their values; for example, in the circle *sine* is an implicit function of the *arc* — though no algebraic equation can express it, because when one of the two (magnitudes) is definite, so is the other, and conversely.

3) Infinite serieses and transcendental functions (like those inexpressible by a finite number of algebraic terms) (Lacroix, pp. 3-4), sheet 40.

4) Convergence and divergence of serieses (Lacroix, pp. 4-5), sheet 40.

A *series* does not always give the *value of the function*, to which it belongs; often it moves away from it [the function] with an increase in the number of terms.

Example :geometric progression (the series for $\frac{a}{a-x}$).

5) In order to be able to use the expansion into a series, it is enough to know *"the law, according to which its terms are formed"* (Lacroix, p. 5), sheet 41. But if the computation of the approximate value of a function, which is expandable in a series is at issue, then the convergence of the series should be carefully checked.

And further, in the same place :

...and these *calculations* may be fully relied upon, only if we are in a position to *indicate the limits of the difference* , which may occur between them and the true value.

Unbounded decrease of the "terms " of a series as the essential sign of its convergence. (And in Lacroix the discussion is about the "terms " of a series, not their absolute values, though in the next example — which Marx did not yet note — Lacroix proposes " to disengage from the sign " of just such numbers) .

6) Omitting all the examples cited (and discussed) by Lacroix , Marx goes over to points 10-11 of Lacroix (pp.13-14) and takes notes briefly under the title :*"Limit"*, sheet 41. Here (point 10) the concept of limit is introduced in the light of an example : it is about the search for the limit of the fraction $\frac{ax}{x+a}$ when $x \to \infty$.

First, the difference between a and this fraction is computed; then Marx writes (sheet 41) :

This difference $\frac{a^2}{x+a}$ becomes the smaller, the greater x is ; [it] may be made smaller than any given magnitude; hence, the proposed fraction may be made as close to a as you please, i.e., a is the limit of the function $\frac{ax}{x+a}$, relative to indefinite augmentation of x.

(There is no general definition of limit in Lacroix's "Treatise". On Lacroix's concept of limit, see Appendix, pp 309-311). From point 11 Marx took down that part , where it is said that , if the ratio ax : ($ax+x^2$) is to be reduced to its "simplest expression" $\frac{a}{a+x}$, then it turns out that, as x diminishes, not only does this expression more and more approach one , but also turns exactly into 1 , when $x = 0$. Here a question has been raised. Marx states it as follows :

But could ax, $ax+x^2$ — since they have turned into 0 — still have a determinate relation?

Marx wrote down also the answer to this question. It consists of this: "the ratio $\frac{a}{a+x}$ of the quantities ax and $ax+x^2$ can not only attain unity , when in it we put $x = 0$, but can also surpass it, when we assume that x is negative" (Lacroix , p.14) .

With this Marx's notes from the "Introduction" of Lacroix's "Treatise" comes to an end.

II. sheets 42-66 (pp. 38-62 in Marx's numbering). Extracts from the English translation (1828) of the third French edition of Baucharlat's book, points 3-72, pp. 2-45. This part of the manuscript contains a number of Marx's own observations, the text of which is being reproduced below. It has been divided by him into the following 20 points. (Titles of the points, found in the manuscript, are being reproduced here within quotation marks. Number of the points and the pages of the English edition of Boucharlat's book are being indicated by putting them within brackets. In the end the sheet number of the manuscript is being given.

1) The derivative and the differential of the function $y=x^3$ (points 3-4, pp. 2-4). For Marx's critical comments on Boucharlat, see PV, 160-161.

2) "*The differential of* x = [*is equal to*] dx", (points 9-10, pp. 5- 6). After this title : "*proved in the following strange way* ". Sheet 42 (see Appendix, pp 328-329).

3) The method of finding the derivative as the limit of the ratio $\frac{f(x+h)-f(x)}{h}$ (when "$h = 0$") by expanding $f(x+h)$ into a series according to the powers of h (point 13, p. 7). Sheet 42 .

4) Application of this method to the differentiation of two or more functions (point 14, p. 8). Sheets 42-43.

5) Differentiation of the quotient $\frac{z}{y}$ (point 16, p. 9). Sheet 43.

6) Derivative of the power x^m (through the equality $\frac{d \cdot (xyztu\dots)}{xyztu\dots} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} + \frac{dt}{t} + \frac{du}{u} + \dots$and the assumption in it of $x = y = z = t = u = \dots$, and altogether there are m of them). Extension to fractional and negative powers. Another method of differentiation of power (through the expansion of $(x+h)^m$ according to Newton's binomial formula) (points 17, 19 , 20, pp. 9-12) .Sheets 43-44 (top).

7) "*On the principle of the method of indeterminate coefficients* " ("third foot note to the English edition of Boucharlat's book, pp. 364-365). Sheet 44.

8) Differentiation of composite function (points 26-29, p. 14-17). Sheets 44-45.

9) "*Successive differentiation*"(point 30, pp.17-18). Sheets 45- 46(top)

10) "*MacLaurin's theorem* "(points 31-33, pp. 18-21). Sheets 46-47.

11) Definition of "transcendental quantities "(point 35, p. 22). The entire text of this point, in the English translation of Boucharlat's book, consists of the words: "*Transcendental quantities [are] such as are affected by variable indices, logarithms, sines, cosines etc.*"; Marx adduces it in full (save the word "are", which we have put within brackets)., and underlines it. Sheet 48.

12) "To differentiate a^{x^m} (points 36-37, p.p 21-24). Sheets 48-49.

13) Differentiation of logarithm (point 38, p. 24). Sheets 49-50.

14) "*The arc is greater than the sine, and less than the tangent* "(points 39-40, pp. 24-25). These are the first words of point 39. In point 40 the issue is : the limit of the ratio $\frac{\sin x}{x}$ (when '$x = 0$"). Sheet 50. (See , Appendix, pp. 309.)

15) "*Differential of the sine, whose arc is* x"(points 41-43, pp. 25-27). Sheets 50-51.

16) "*Differential of* $\cos x$" (point 44, p. 27). "*Differential of* $\tan x$" (point 45, pp. 27-28). "*Differential of* $\cot x$ *and* $\sec x$" (points 46-47, p. 28). "*Differentiai of the cosecant* x" (point 48, p. 28). "*Differential of versinus* "(point 49, p. 28). Sheets 51-52.

17) Compendium of formulae for the differentials of trigonometric functions (composed by Marx). Sheet 52.

18)"*Taylor's theorem* " (points 52-60, pp 30-35). Contains two long comments of Marx : *a*) about Boucharlat's lemma (see, PV, 161), b) comparison of Taylor's and MacLaurin's theorems (see, PV, 161-163). Sheets 53-57.

19) "*On the differentiation of equations of 2 variables* " (points 61-68, pp. 35-41). Here the issue is differentiation of the implicit function (including successive) and complete differential. Sheets 57-62.

After this there are three insertions in the manuscript (on sheet 62) under the titles : "*7a) ad p. 40, after 7, also 7 a* ", "*7b)*" *and* "*7c)*". Here Marx takes notes from the points 22-23, 25, the pp.

12-13, omitted earlier. 7a) is related to the differentiation of sum, 7b) — to the role of the constant as a multiplier (in differentiation), 7c) the same for the constant as an item. Here Marx pays special attention to a general comment of Boucharlat, which he expresses (mainly in German) as under :

The method for the process of differentiation, consisting, *at first of finding the value of* y_1, and then from there to obtaining $\frac{y_1 - y}{h}$ and, then passing on to the limit by making $h = 0$, is very *cumbersome* , when the issue is differentiation of *a* quantity which contains several terms. However, if we can differentiate each term separately, then a simpler procedure is possible owing to the theorem : the differential of a sum of functions is equal to the sum of the differentials of those functions.

(Proof of the theorem — which is there in the notes — is given in Boucharlat's book with the help of the expansion of the values of the terms at the "point" $x + h$ in the series according to the powers of h.)

20) "*Method of tangents (i.e., differential expressions of tangents, subtangents, normals, and subnormals of curves)*" (points 69-72, pp. 41-45), with Marx's critical comments. Sheets 63-66 (top).

Here Marx did not take notes from the succeeding sections, devoted to the other problems of differential geometry, the methods of "revealing" the indeterminancies, the problems of maximum and minimum, and also the different methods of laying the foundations of differential calculus.

III. Sheets 66-75 (pp. 62-63, 66-73 in Marx's numbering, by mistake "66" instead of "64"). Extracts from ch. V of Hind's book : section "III. Development of functions etc.", points 69-82, pp. 68-98, with a few omissions. The extracts are related to the theorems of Taylor and MacLaurin and they proceed in the following order (the numbers within square brackets are ours) :

[1] Sheet 66 (p. 62 of Marx). MacLaurin's formula, regarding which Marx writes :

Putting u in place of y the *formula of MacLaurin* :

$$u = (u) + \left(\frac{du}{dx}\right)\frac{x}{1} + \left(\frac{d^2u}{dx^2}\right)\frac{x^2}{1 \cdot 2} + \left(\frac{d^3u}{dx^3}\right)\frac{x^3}{1 \cdot 2 \cdot 3} + \dots$$

Hind puts for this

$$u = u_0 + u_1\frac{x}{1} + u_2\frac{x^2}{1 \cdot 2} + u_3\frac{x_3}{1 \cdot 2 \cdot 3} + \dots$$

[[where the absurd u_1, u_2 etc. in place of u' , u'' , u''' may give rise to misunderstandings, since these symbols have an entirely different meaning in operations with the finite differences]].

He observes first of all that, $u = f(x)$ may be expanded according to MacLaurin's formula, only if *u represents a function of x, which can be expressed in ascending and integral powers of x.*

The words underlined by Marx (here italicised) are important for understanding the subsequent text of the manuscript.

[2] Sheets 67-68, Marx's pp. 63, 66. Examples of applicability of MacLaurin's formula ($u = \sqrt{2x - 1}$, $\log x$). Methods for obtaining the expansions in certain cases of applicability of MacLaurin's theorem, by representing the functions of u in the form of $u = x^k v$, where v is a function of x expandable according to MacLaurin's theorem. Examples :

1) $u = \sqrt{x-x^2} = \sqrt{x}\sqrt{1-x}$, $k = \frac{1}{2}$, $v = \sqrt{1-x}$;

2) n implicit functions given by the equation $au^3 - ux^3 - ax^3 = 0$, $k = 1$, v is a function definable by the equation $av^3 - xv - a = 0$ (Hind, point 69, example 9 ; points 70-71, corollary 1-2; examples 1-2, pp. 74-76).

[3] Sheet 69, Marx's p. 67. On obtaining the expansion into a powered series without the help of MacLaurin's theorem : differentiation of a series with indeterminate coefficients.

Example 1 — expansion of the function a^x "by the method of indeterminate coefficients" (Hind, point 72, corrolary 3, pp. 76-77, Marx did not write the remaining 4 examples of this point). Substitution of $x = \frac{1}{2}$ for obtaining the expansion in diminishing powers of x (ibid , second time point 72, cor. 4, pp. 81-82).

[4] Sheets 69 (bottom)-70, Marx's pp. 67-68. Formulation of Taylor's theorem in Hind's book. Marx begins this section with the words (sheet 69, bottom) :

Hind expresses *Taylor 's theorem* as follows :

Thus it is clear that Marx does not ascribe this formulation to Taylor himself, though later on he sometimes so writes, as though he thinks, that if not the formulation, then in any case the proof of Taylor's theorem, presented in the books of Hind and Boucharlat, belongs to Taylor himself.

In connection with Hind's words, that "if a function does not have a rational algebraic form, then the number of its differential coefficients [derivatives] is infinitely large" (that is why the series obtained is also"unboundedly continuous"), Marx writes after the words "if it *does not have an algebraic form*" :

[[i.e., always, when [they are] *transcendental functions (exponential, logarithmic, or trigonometrical*) since the nature of these functions does not permit their expression *by means of algebraic expressions of finite number of terms*]].

Unbounded continuity of a series is illustrated by the example of expansion of $\sin(x+h)$ into Taylor's series .

Other examples and the proof of Taylor's theorem (analogous to the proof given in Boucharlat's book, earlier Marx took detailed notes of it and analysed it critically, see II, 18) ; and here Marx does not note the corresponding observations contained in point 74 of Hind's book.

The section comes to an end with the words (sheet 70, Marx's p. 68) :

So long as the result of application of Taylor's theorem is considered only as an analytical transformation, irrespective of *whether the number of terms are finite or infinte,* or else, if the value of the function is required to be ascertained in any particular state of the principal variable, then it becomes important to find out the *limit of the quantity, which is omitted by stopping at any assigned term of the development,* for example, whether the value of the omitted quantity will be greater or lesser, than the value of the limiting quantity, at which we stopped. (For example, will the first term be greater than the sum of all the rest ?)

In Hind's book the corresponding text follows immediately after the comment on the "unbounded continuity" of Taylor's series, mentioned above (point 75, cor. 1, pp. 86-87). All the English words in this extract belong to Hind ; on the whole it does convey the text of the book exactly, though in the latter words(after those underlined by him) Marx isolated and mentioned the task, which the author attempts to solve later on .

This section of the manuscript contains both the formulations of Taylor's theorem : [found] in pp. 83 and 84 (point 74) of Hind's book, example 3 from this point (p. 86) and, in the beginning of point 75 (pp. 86-87).

[5] Sheets 70 (bottom, under a line) -74 (top), Marx's pp. 68-72.

Cases of inapplicability of Taylor's theorem to the function $f(x+h)$, considered according to Lagrange, i.e:, as possible only for some particular values of x. Example :

$$u = x^2 - \sqrt{x-a}$$

when $x = a$ (Hind, point 77, pp. 92-93, upto example 2). Sheets 70-71.

Proof of the fact that in the "general" cases the expansion into Taylor's series cannot contain fractional and negative powers of h (according to Lagrange) (Hind, point 78, p, 94). Sheets 71-72.

Certain signs, permitting recognition of the existence of such special values of x, for which $f(x+h)$ is not expandable into Taylor's series (Hind, points 79-80, p. 94 (bottom) -96). Sheets 72-73.

Taylor's series giving the expression for the increment of a function through its successive differentials : $\Delta u = \frac{du}{1} + \frac{d^2u}{1 \cdot 2} + \frac{d^3u}{1 \cdot 2 \cdot 3} + \cdots$(Hind, point 81, pp. 96-97). Use of such an expression for the increment (in the instance, when the expansion for $f(x+dx)$ according to the powers of dx, is already known) for finding out the successive derivatives of u in terms of x. Example 1 : $u = x^m$, where the expansion for $(x+dx)^m$ is obtained according to the binomial theorem. Sheets 73-74.

After this having written the heading (sheet 74, Marx's p. 72) :

Development of functions of 2 or more independent variables $u = f(x, y, z \text{ etc.})$ or $f(u, x, y, z \text{ etc.}) = 0$ (Hind, ch. XII, point 256, p. 370), Marx, however, did not take notes from this section.

IV. Here Marx returns backward, to the fundamental question, which interests him: about the nature of the two methods of differentiating, with which Boucharlat started ((sec II, 1) and 3)) and the meaning of the symbols of differential calculus. Marx's own comments (sheets 74-75, Marx's pp. 72-73) summing up the "first" method, which has been placed in the manuscript (sheet 74) after the number "1)", is being reproducted below (see, PV, 164).

Marx got acquainted with the "second", i.e., Lagrange's, method enunciated as per Poisson, at first in Hall's book (Th. G. Hall, A treatise on the differential and integral calculus, 5th ed. (with us), London, 1852). Having taken notes (sheets 75-79(top) , pp. 73, 76-79,"76" — Marx's slip of pen) from points 6-10, 13, pp. 2-8 of Hall, containing examples of expansion of $f(x+h)$ into a series of ascending integral positive powers of h and, Lagrange's "proof" of such expansibility, by the method of indeterminate indices of power (through the representation of $f(x+h+h)$ once as $f((x+h)+h)$ and next time as $f(x+2h)$), Marx sums up the "second" method (sheet 79, p. 79 of Marx). This comment of Marx is also being reproduced below (see p. 166). The entirety of this part of the note is placed under the title (sheet 75, Marx's p. 73) :

2) *In place of the method indicated above, the method of Poisson, etc.*

Going over the rules of differentiation, which must facilitate the task of differentiating and "make it a simple algebraic operation" (Hall, p. 7), Hall still considers preliminarily, the example of differentiation of the function $u = \frac{a+x}{b+x}$(point 13), directly using the definition of the derivative as the coefficient of the first power of h in the expansion of $f(x+h)$ and, from there he goes over to its definition as the limit of the ratio $\frac{f(x+h)-f(x)}{h}$. Marx wrote down this example in full (with all the calculations), calling herein the second definition the "first" (sheets 78-79, Marx's pp. 78-79). (Herein he obtains the expansion of $\dfrac{1}{1+\dfrac{h}{b+x}}$ by "angular" division .)

Here Marx stops taking notes from the beginning of Hall's book (apparently, he was in need of it only to sum up the essence of the "second "method of differentiation — the method of Lagrange). Further (sheets 80-81, Marx's pp. 80-81) on he takes notes from the beginning of chapter VIII of this book (points 95-96, pp. 87-88) under the title (sheet 80, p. 80) : "*Functions of two or more variables and implicit functions*" with the instruction "(*cf. p. 53 sqq*)". Here (see II, 19) it is possible that Marx turned to this chapter, because while propagating Lagrange's method Hall wrote in the preface to this book (p.V), that if for the functions of one variable both the methods(of limits and of Lagrange) are of equal value, then for the functions of two (or more) variables the expansion of $f(y+k,\ x+h)$ specifically gives only the complete differential. (Of course Hall had in view the simplicity of the formal transformations.) However, considerable simplicity — and especially clarity — was not obtained, and, perhaps, that is why Marx did not take notes from beyond the beginning of this section, where the expansion of $f(x+h,\ y+k)$ is obtained from the expansion of $f(x+h,\ y)$ when y turns into $y+k$.

V. This part of the manuscript (sheets 81 (bottom) -87, pp. 81-86 and one unnumbered, according to Marx) contains a short extract from Sauri's book and a continuation of the notes from Boucharlat (the same English translation of 1828). Marx summed it up with his comments.

Extracts from Sauri (sheets 81-82, Marx's pp. 81-82) are related to the third volume of this book (Paris, 1778), pp. 3 and 11-12, and contains :

a) differentiation of the product xy (sheet 81, bottom). Here, speaking about the rejection of the product of dx and dy as an infinitesimal of higher order in comparison with dx and dy, Marx adds to Sauri's text :"*according to Leibnitz*" ;

b) successive differentiation and then, conversely, the integration of the functions $y = x^m$ and xy (in the cases, when dx is either a constant or it is not). Sheet 82.

To the extracts from Boucharlat, Marx gave the heading (sheet 82) :

"*Failure [s] of Taylor's theorem (continuation of p.69)*".

Here, the continuation of III, [5], sheet 71 is being referred to. Marx sub-divided these extracts into 5 points, indicated by the Roman numerals and these are, in their turn, sometimes subdivided into sub-points. We too shall be speaking of their contents under these numerals ; numbers of the points and pages of the source have been indicated according to the English translation of Boucharlat, and in the end the corresponding sheets of the manuscript have been indicated. In the source the corresponding text is printed in brevier. With this the section entitled "Differential Calculus" in Boucharlat's Book comes to an end.

I. 1)-2). The instance, when for some value a of the variable x the radical vanished in $f(a)$, but is retained in $f(a+h)$ (points 253-254, p. 162.). Sheet 82.

3) The general proof, that if there is a fractional power of h in the expansion for $f(a+h)$, then all the derivatives of $f(x)$, starting from some, turn into infinity when $x = a$ (point 255,. p 163). Sheets 83-84.

II. 1)-2). The method of obtaining the expansion for $f(a+h)$ in such cases, not according to Taylor's theorem (but by a substitution of $x+h$ in place of x in $f(x)$ and an application, for example, of Newton's binomial theorem). Example :

$$f(x) = 2ax - x^2 + a\sqrt{x^2 - a^2}.$$

To find out the expansion of $f(a+h)$ (points 256-257, pp. 164-165). Sheets 84-85.

This point of the note comes to an end with Marx's words (sheet 85):

It comes to this, that the application of Taylor's theorem is feasible only upto the n-th term [[if at all]] through the application of the *expedient means of ordinary algebra,* supplemented ; this has nothing in common with differential calculus.

III. Lagrange proved that (point 258, p.166 ; sheet 85) : the development of $f(x+h)$ can not contain *terms with fractional powers of* h, so long as x, as it happens in a general development, remains *indeterminate*, i.e., does not obtain some particular value a. Here, as in the entire text of this part, the proof is noted in full : *upto obtaining a contradiction with the assumption*, as writes Marx.

In this extract Marx very briefly explains the text of the source, where it has only been said : "when x remains indeterminate".

IV. The same for the negative powers of h (point 259, pp. 166-167). Sheet. 86.

V. The same for terms of the type $A \log h$ (point 260, p. 167). Sheet 86.

Drawing a line after this, Marx writes further under the number "VI" (sheet 86, p. 86 according to Marx) :

VI. *In general in the Lagrangian algebraic deduction of Taylor's formula or, still more generally, in the deduction of his*

$$f(x+h) = f(x) + p\,h + q\,h^2 + r\,h^3 + \dots$$

from the very beginning the cases which latter on appear as "the cases of inapplicability" of Taylor's theorem are excluded. (This besides the fact , *that it has already been noted* especially sub III (p.85) and sub IV and V on this page *).

a) In Lagrange, who obtains *algebraically*. [[and not by applying the differential calculus]]

$$f(x+h) = f(x) + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1\cdot 2} + \frac{d^3y}{dx^3}\frac{h^3}{1\cdot 2\cdot 3} + \cdots$$

$\frac{dy}{dx}$ is nothing other than a symbol of the *operation* , *by which the coefficient of h is obtained* in the development of $f(x+h)$; and once this coefficient is found, the expressions $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ etc. mean only this that by repetition of the same process the *coefficients* of h^2, h^3 etc. will be found, but only after — in consequence of these successive processes — the *second term is again and again* reduced to a term *affected by the first power of h* ; so that we require only to know by the rules of algebra, what $\frac{dy}{dx}$ ought to be for each function. If, for instances, it were asked what $\frac{dy}{dx}$ is for the function x^m, then we should have developed $(x+h)^m$ by the binomial theory, which gives

$$x^m + mx^{m-1}h + \dots;$$

hence, as $\frac{dy}{dx}$ must indicate the coefficient of the first power of h in the development, we should have

$$\frac{dy}{dx} = m\,x^{m-1}.$$

If we are asked further, what $\frac{dy}{dx}$ will be for $m\,x^{m-1}$ [[hence what $\frac{d^2y}{dx^2}$ will be for x^m]], we shall develop again by the binomial theory :

*Evidently what is being referred to here is carrying the proof forward right upto the contradiction with the "assumption", which is also considered to be excluding the special cases ; p.85 is also sheet 85. — Ed.

$m(x+h)^{m-1} = m x^{m-1} + m(m-1) x^{m-2} h + \ldots$

That is why, $\frac{dy}{dx}$ for $m x^{m-1}$, or $\frac{d^2y}{dx^2}$ for $x^m = m(m-1)x^{m-2}$.

Thus the whole thing is reduced to being able to find out, *by analytical processes, the development of the different sorts of functions which algebra can present*.

In fact with this keenly summed up comment — saying that in essence Lagrange proved nothing, as he assumed what he wanted to prove — this manuscript comes to an end. Sheet No. 87 (the unnumbered page) is full of calculations without any order and words, and it is many times crossed out (by oval curves).

Now we present Marx's own comments, mentioned in the description of the manuscript. The first of them is related to the very beginning of Boucharlat's book (see II, 1) and contains Marx's critical comments, related to the mode of introducing the concept of differential in this book (sheet 42, Marx's p. 38).

ON THE CONCEPT OF DIFFERENTIAL ACCORDING TO BOUCHARLAT

$$\begin{array}{l|l} y = x^3, & y_1 = x^3 + 3x^2 h + 3xh^2 + h^3, \\ y_1 = (x+h)^3, & y_1 - y = 3x^2 h + 3xh^2 + h^3, \end{array}$$

$$\therefore \frac{y_1 - y}{h} = 3x^2 + 3xh + h^2;$$

if h diminishes to 0, then $\frac{y_1 - y}{h} = 3x^2$; hence $3x^2$ *is the limit* to which $\frac{y_1 - y}{h}$ tends as h goes on diminishing; but then also $\frac{y_1 - y}{h} = \frac{0}{0}$, or $\frac{\text{increment of the function } y}{\text{increment of the variable } x} = \frac{0}{0}$; hence $\frac{0}{0} = 3x^2$.

This is still in accordance with common algebra, as $\frac{0}{0}$ can be equal to any quantity. But as in $\frac{0}{0}$ all trace of the function as well as of the variable x has vanished, so in place of $\frac{0}{0}$ the expression $\frac{dy}{dx}$ is substituted, *reminding us that the function was y and the variable x*; dy and dx are *evanescent* quantities; $\frac{dy}{dx} = 3x^2$; $\frac{dy}{dx}$, or rather *its value* $3x^2$ is the *differential coefficient of the function y*.

Having written this Marx comments (on p. 38):

Introduction of *evanescent quantities* and of their ratio $\frac{dy}{dx}$ instead of $\frac{0}{0}$ no more belongs to algebra; but more than that: though $\frac{dy}{dx}$ "is the *symbol* which represents the limit $3x^2$"

and therefore "*dx ought properly to be always placed under dy*", nevertheless, "in order to facilitate operations in algebra" we treat $\frac{dy}{dx}$ as a common fraction and $\frac{dy}{dx} = 3x^2$ as a common equation; and thus by clearing the equation of its denominator — [we] obtain $dy = 3x^2\,dx$, this expression — obtained in a rather equivocous way — [is] then called *the differential of the function y* [116].

The following comment of Marx (which he later on struck off by a vertical line) is taken from II, 18), and is related to Boucharlat's lemma, with the help of which he (Boucharlat) proved Taylor's theorem. It is on sheet 53 (Marx's p. 49).

ON THE LEMMA OF BOUCHARLAT

b) [[Quite a new element [is] here introduced, as compared to what precedes. Till now, if $y = f(x)$, and we got

$$y_1 = f(x + h),$$

[then] x was not [the] only variable, but also *its increment h*. It is true [that] through the operations, the latter was treated so far as [a] constant, as a given magnitude, because otherwise, f.i. $(x + h)^2$ could not be treated as a binomial, [and] we could not have $x^2 + 2xh + h^2$; but x itself appears in this formula, for the given moment, as [a] constant. It is separated as an independent magnitude from h^2, f.i. Besides x is virtually a fluent, but it becomes only a real fluent in the moment it generates a fluxion. On the other hand h, after it has performed its business as a "constant" magnitude throughout the binomial operation, is immediately treated as a variable, it becomes not only 0 in $\frac{0}{0}$ (as the denominator of $\frac{0}{0}$), but it figures as an evanescent quantity $\frac{0}{0}$ being transformed in $\frac{dy}{dx} = \frac{y_1 - y}{h}$ (when h become evanescent) and it is only as a ratio of

$$\frac{\text{evanescent [increment of the] function } y}{\text{evanescent increment } h \text{ (increment of } x\text{)}},$$

that this ratio finds its value in a *coefficient* free of all differential quantities. But in what now follows, and upon which the theorem of Taylor [is] founded, we proceed to a till now unknown dilemma.

Either x is considered as variable, and its increment h as constant, or x is considered as constant and its increment h as variable — in order to prove, that it is the *same whether we start from the one view or the other* [117]. The question seems rather to be whether we had laid any foundation in the precedent development, which would allow us to put such a dilemma!]]

A COMPARISON OF THE THEOREMS OF TAYLOR AND MACLAURIN

The following comment of Marx, devoted to a comparison of the theorems of Taylor and MacLaurin is there on sheets 55-56 (Marx's pp. 51-52).

[[The difference between it [Taylor's theorem] and MacLaurin's theorem :

1) MacLaurin starts from $y = f(x)$. Taylor starts from $y_1 = f(x+h)$ [118].

2) *MacLaurin* arranges the function $f(x)$ according to the powers of x :

$$y = A + Bx + Cx^2 + Dx^3 + \cdots$$

That equation is successively differentiated in *respect* to x ; the differential coefficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc. so found (i.e., the values on the other side (as $\frac{dy}{dx} = B + 2Cx + \text{etc.}$)) are then reduced to their expression when x becomes 0 ; so y becomes (y), $\frac{dy}{dx}$ becomes $\left(\frac{dy}{dx}\right)$ etc. and these values, *together with the numerical coefficients* substituted for the indeterminate coefficients in the original equation. Thus for instance in the third differentiation, if [we] put $x = 0$, we get $\left(\frac{d^2y}{dx^2}\right) = 2C$, and so $C = \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)$, also $D = \left(\frac{d^3y}{dx^3}\right)\frac{1}{1\cdot 2\cdot 3} \cdots$, while in the original development of y, the factors x, x^2, x^3 etc., the development of x remains unchanged and reappears with the differential coefficients.

3) On the other side, *Taylor* [119] starts not from [the function] y , or $y = f(x)$ [120], but with the function y_1 or $y_1 = f(x+h)$.

The indeterminate coefficients A etc. (unknown functions of x) are found by differentiating the primitive development of

$$f(x+h) \text{ or } y_1 = y + Ah + Bh^2 + Ch^3 + \cdots$$

where the factor x does not appear.

In MacLaurin the values of A, B etc. in [the] form of differential coefficients, [are] found by *successive differentiations* of the *first equation,* arranged according to the powers of x, but *Taylor* proceeds differently; he *differentiates first (once) in respect of h,* and then in respect of x, so that he gets :

$$1)\ \frac{dy_1}{dh} = \cdots \text{ and } \quad 2)\ \frac{dy_1}{dx} = \cdots ;$$

as these two expressions are equal, according to b) (p. 49), the *coefficients of the same power of h, found in the two different ways, are equated,* and then $A = \frac{dy}{dx}$ gives by substitution, the *differential expressions*, consisting of successive ascending differentiations *without further processes of differentiation* having been recurred to ; the powers of h, h^2, h^3 etc. play the same part as x, x^2 etc. in MacLaurin's theorem; x does as little appear for itself as a coefficient as h in MacLaurin's theorem ; the numerical coefficients like $\frac{1}{2}$, $\frac{1}{2\cdot 3}$, $\frac{1}{2\cdot 3\cdot 4}$ found by MacLaurin through differentiation and taking the limit are found by Taylor, through the equation of the two *different expressions found for the coefficients of the same powers of h.*]]

MacLaurin :

$$y \text{ (or } f(x)) = y + \left(\frac{dy}{dx}\right) x + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right) x^2 + \frac{1}{2\cdot 3}\left(\frac{d^3y}{dx^3}\right) x^3 + \cdots$$

Taylor :

$$y_1 \text{ (or } f(x+h)) = y + \frac{dy}{dx} h + \frac{d^2y}{dx^2}\frac{h^2}{1\cdot 2} + \frac{d^3y}{dx^3}\frac{h^3}{1\cdot 2\cdot 3} + \cdots$$

On the other side of the equation, $f(x)$ could stand instead of y as the *first term*, since

$$y_1 = f(x+h), \quad y = f(x).$$

MacLaurin's theorem is deduced from that of *Taylor*. According to the latter, if everywhere instead of y we put $f(x)$, then we shall get :

$$f(x+h)) = f(x) + \frac{df(x)}{dx} h + \frac{d^2f(x)}{dx^2}\frac{h^2}{1\cdot 2} + \frac{d^3f(x)}{dx^3}\frac{h^3}{1\cdot 2\cdot 3} + \cdots$$

If *we make* $x = 0$ *and represent, like MacLaurin, with the help of brackets, the values of the different coefficients of* x, *when* x *becomes 0, then* the formula of Taylor becomes :

$$f(h) = (f(x)) + \left(\frac{d\,f(x)}{dx}\right) h + \left(\frac{d^2 f(x)}{dx^2}\right)\frac{h^2}{1\cdot 2} + \cdots$$

h enters into $f(h)$, as x entered into $f(x)$; here we can put x for h, because through this change nothing is altered in the differential coefficients ; then

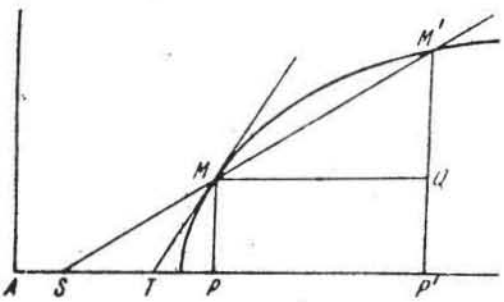
$$f(x) = (f(x)) + \left(\frac{d\,f(x)}{dx}\right) x + \left(\frac{d^2 f(x)}{dx^2}\right)\frac{x^2}{1\cdot 2} + \cdots,$$

which is MacLaurin's theorem.

MacLaurin's formula, that of successive differentiation generalised ; Taylor's, more general, *formula for the development of different functions in the form of a series .*

On sheet 64 (Marx's p. 60) after an extract from Boucharlat (See II, 20), related to the problem of constructing tangent to a curve, we find the following comments of Marx :

THE PROBLEM OF TANGENT : TWO DIFFERENT METHODS OF SOLUTION



[[This second method, where Taylor's theorem is applied, is more complex and more tedious than the first one ; apparently, in it is circumvented a difficulty, consisting of the fact

that the arc MM' *coincides with the chord* MM' [see figure] and that is why there appears a triangle, one of the sides of which is in fact the arc; meanwhile it is clear that when increment of the abscissa diminishes, then M' comes closer to M, till the ordinate $M'P'$ coincides with MP, and hence, the secant MM' *also turns out to be a mere continuation* of the tangent TM, i.e., SP too coincides with PT. On the other hand this evasion is only apparent, for the entire ruse is reduced to a similarity of the 2 triangles, and since the two sides of the auxiliary triangle dx and dy are smaller than a point, so under such circumstances there is no need to stand on ceremony to assume that the chord coincides with the arc and vice versa. This apart, in the first method too, comparison in pairs arises only in application to both the cathetus; and phantasy may assign the character of the hypotenuses [121].]]

TWO DIFFERENT METHODS OF DIFFERENTIATION

Marx describes the first method on sheets 74-75 (Marx's pp. 72-73). for explanations see : p. 157.

THE "FIRST" METHOD : "OF LIMITS"

1) At the foundation, or at the starting point, namely, to *find out the differential coefficient*, we started with the following method:

I) $y = f(x)$;

II) $y_1 = f(x+h)$;

If, for instance

$$y = f(x) = ax^2 ,$$

then

$$y_1 = f(x+h) = a\,(x+h)^2 \;;$$

hence

$$y_1 = ax^2 + 2ahx + ah^2 .$$

and

III) $y_1 - y = 2ahx + ah^2$.

Dividing both the sides by h, we obtained

IV) $\dfrac{y_1 - y}{h} = 2ax + ah$.

$y_1 - y$ is equal to the difference of y_1 (increment to y_1 over y), hence $= \Delta y$;

$x + h = x_1$; h is equal to the difference of x_1, equal to Δx (i.e. = the excess in x_1 over x);

hence we can write instead of $\dfrac{y_1 - y}{h} = 2ax + ah$:

V) $\dfrac{\Delta x}{\Delta y} = 2ax + ah.$

The left hand side of this equation expresses the *ratio of the finite difference of the function y to the finite difference of the independent variable x.* Now assuming $h = 0$, we shall get :

VI) $\frac{0}{0} = 2ax$.

It still does not go beyond the limits of ordinary algebra. If, for instance, we have : $\frac{x^2 - a^2}{x - a}$, then it is equal to $\frac{(x + a)(x - a)}{x - a}$.

That is $\frac{x^2 - a^2}{x - a} = x + a$, since on the other hand

$$(x + a)\frac{(x - a)}{(x - a)} = (x + a) \cdot 1.$$

Assuming on both the sides $x = a$, we shall get $\frac{0}{0} = 2a$. Since $\Delta x = h$, so, if $h = 0$, $\Delta x = \theta$; and as y becomes y_1 only in consequence of x increasing by h, so $y_1 = y$, when $h = \theta$, i.e., when $x + h = x$; hence $\Delta x = 0$, $\Delta y = 0$. In this form nothing [is] to be done with the equation, which contains not even the trace of a function, or of the principal variable ; $\frac{0}{0}$ expresses [the fact] that both the differences Δy and Δx have disappeared, but we want to fix the character of the factors that have disappeared ; we want to fix them as evanescent (in the negation, the character of that which is negated) and that is why we assume $\frac{dy}{dx}$ in place of $\frac{0}{0}$; instead of Δ we write d, instead of the difference its diminutive, the *differential.* Hence :

VII) $\frac{dy}{dx} = 2ax$.

This shows firstly, that the terms, which compose the ratio are evanescent, and that they have in fact disappeared or become $= \frac{0}{0}$, as soon as we have $2ax$.

$2ax$ is therefore the limit of their variations.

This differential coefficient has therefore two expressions, the one showing the movement $\frac{dy}{dx}$, the other showing its value, its *limit.*

What, after the operations are performed, disappears is $\left.\frac{dy}{dx}\right\} = 0$; and there would be only an error in the calculus, if they were not removed.

The only difficulty is therefore the dialectic[al] notion of fixing a ratio between evanescent quantities, and when this *has done its duty, the ratio* $\left(\frac{0}{0}\right)$ *disappears also in the result of the calculus.*

THE "SECOND" METHOD : OF LAGRANGE

On this comment of Marx, see the description on p. 157 It is on sheet 79, Marx's page number is the same.

The importance of this method appears in the latter analytical operations, and not in the initial ones, since we have

$$u_1 = f(x+h) = u + Ah + Bh^2 + Ch^3 + \cdots,$$

the latter operation, at first consists in $\dfrac{u_1 - u}{h} = \dfrac{du}{dx} = A$, if we assume that $h = 0$; if we do not assume $h = 0$, then, in order to get rid of $Bh^2 + Ch^3 + \cdots$, it is taken as a small vanishing [magnitude] in respect of A, meanwhile, here such a representation is absolutely superfluous. But it is important, that when $u = f(x)$ and x turns into $x + h$, [u becomes] $= u_1$, then

$$u_1 - u = f(x+h) - f(x),$$

and this is the *difference between the functions of $x + h$ and of x, which must depend upon h, the increment of x*, i.e., the difference $u_1 - u$, is represented by a series of the form

$$Ah + Bh^2 + Ch^3 + \text{ etc., i.e.,}$$
$$u_1 = u + Ah + Bh^2 + Ch^3 + \ldots,$$

or

$$u_1 = f(x) + Ah + Bh^2 + Ch^3 + \ldots,$$

where the *powers of h ascend*; that these *powers have only positive indices* (and are integers, not fractions); that the first term must be $u = f(x)$; and that A, *the coefficient of the first power of h, is the first differential coefficient and Ah ($= Adx$)the first term of the difference between u_1 and u* or *between $f(x + h)$ and $f(x)$.*

The fundamental task of the *differential calculus is to find out the values of the coefficients A, B, C, etc.*; that is why *a differential is the second term of the expansion $f(x + h)$.*

Sheets 80-81. Extracts from the same book of *Hall*, §§ 95-96, pp. 87-88, related to the expansion of functions of certain variables into Taylor's series.

Sheets 81-82. Extracts from *Sauri*, volume III, pp. 3, 11-12, on the differentiation of product and on repetitive differentiation.

Sheets 82-87. Extracts from *Boucharlat*, §§ 255-260, pp. 176-180, related to the cases of inapplicability of Taylor's formula.

In Boucharlat's book this is the last section of the differential calculus.

THE NOTE BOOK "ALGEBRA I"

S. U. N. 3932

This note book contains notes on algebra (on the general theory of the equations of higher orders), taken mainly from Lacroix's "Elements of Algebra". (We have the 11-th French edition of this "Eléments d'algèbre" à l'usage de l'école centrale des quatre-nations, par S.F. Lacroix; Paris, 1815.) It is possible that Marx had at his disposal a quite exact English translation of this book. Marx took these notes in German, in places we find English and French words and phrases. There are 93 sheets in this note book. This note contains a number of Marx's own lengthy comments, from which it is clear, that in it Marx collected materials devoted to the search for the algebraic roots of the differential calculus. That is why it may be held, that this manuscript belongs to the second half of the 70s, when Marx's characteristic point of view about the nature of symbolic differential calculus began to take shape.

The structure of this conspectus is quite complex. Marx sub-divided it into five parts, numbered them with Roman numerals and supplied their headings. In consonance with this, we sub-divided our description of this manuscript into five parts and gave them the titles provided in the manuscript.

Marx did not number the first, title sheet. On it is written only : *"Algebra I"*.

Then follows that part of the note book, to which Marx gave the title : *"I. General theory of equation[s]"*.

This part occupies the sheets 2-18 (in Marx's numbering pp. 1-17). Marx begins straight off by taking notes from § 178 (pp. 246-247) of Lacroix's book, wherein begins the section on the "General Theory of Equations". However, after a few lines, in connection with a reference to § 109 of Lacroix's book, containing a short enunciation of the method of solving the quadratic equations, by adding to its left hand side, till a full square [is formed], and having finished the conspectus of § 178, Marx turns generally to the section on quadratic equations in Lacroix's book. Marx gave point 2) of his conspectus (sheets 2-4, pp. 1-3 according to Marx), the title : *"Roots of the equations of second power"*. Here the questions about the number of roots of the quadratic equations and the signs of the roots have been considered (§ 106, p.156, with reference to the "Algebra" of Emily (Emmanuel) Develey, which Lacroix cites in this paragraph); here the following issues have also been considered : *"when a root of the equations of the second power may become imaginary "* (§ 114, pp.166-167 of Lacroix's book); if $x = a$ is a root of the equation $x^2 + px = q$, then the other root is $x = -a - p$ (§ 116, pp. 168-169). After the quadratic equations Marx went over (sheets 4-8, pp. 3-7) to the section on equations with two terms, in Lacroix's book (§§ 156-159, pp. 222-228). Here he took specially elaborate notes from § 159 (pp. 225-228), devoted to the roots of the m-th power from unity.

Afterwards, following Lacroix, Marx went over (sheets 8-9, pp. 7- 8) to "the equations which can be solved as equations of second power" (§§ 160-162, pp.228-231), i.e., to the equations of the form $x^{2m} + px^m = q$.

Marx omits the section "On the Calculus of Radicals" (§§ 163-171, pp.231-239), but specially dwells upon the section entitled. "Comments on certain special cases of the Calculus of Radicals" (§§ 172-174, pp. 239-243). Here (sheets 9-10, pp. 8-9) his attention turns to Lacroix's comment about the so-called "paradoxes", connected with the formal transfer of the rules of operations with the roots of real numbers on to the roots of imaginary numbers, in particular to the transformation.

$$\sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)\cdot(-1)} = \sqrt{1} = \pm 1,$$

in connection with which he cites Lacroix's reference to Bézout, according to whom, "when we do not know how the square a^2 was formed, and we seek its root, then we must consider both $+a$

and $-a$, but when we know beforehand, which of these two quantities was multiplied with itself in order to form a^2, then we must take namely, it"(pp. 239-240), Marx comments (Sheet 9) :

According to *Lacroix*, this explanation of Bézout is enough for the elimination of the difficulties in such particular instances; concerning the other instances, says he, sufficient elucidation is given only by the *properties of two-termed equations.*

Here itself is cited an example of the transformation: $\sqrt[4]{a}\cdot\sqrt{-1}=\sqrt[4]{a}\cdot\sqrt[4]{(-1)^2}=\sqrt[4]{a}\cdot\sqrt[4]{+1}=$ $=\sqrt[4]{a}$, which is wittingly false, since if $\sqrt[4]{a}$ is a real number, then the imaginary number $\sqrt[4]{a}\cdot\sqrt{-1}$ turns out to be equal to the real number $\sqrt[4]{a}$. In this connection Marx mentions Lacroix's explanation, consisting of an instruction to the effect, that in the general instance such transformations introduce superfluous roots : there are only two quadratic roots of one, the roots of the fourth power are four [in number].

Next section of the conspectus (sheets 10-11, pp.9-10) is devoted to the operations with fractional indices (in Lacroix's book it is §§ 175-177, pp 243-246). Here Marx especially singles out Lacroix's comments on the significance of the designations of radicals, introduced by Descartes, with the help of the fractional indices of power. Here Marx writes (sheet 11, Marx's p.10) :

Calculation of the roots with signs requires a special analysis and it is clumsy, since the sign $\sqrt{}$, expressing the radicals, has no connection with the operation, through which they are obtained. Replacement of this notation by that of the *fractional indices of power*, is a great contribution of Descartes. It facilitates all operations by its *analogy with the integral indices of power* and makes applicable to them the rules, which are applicable to the calculations of the latter.

Only after this long digression (sheets 2-11, pp. 1-10) does Marx return (sheet 12, p.11) to the general theory of equations, with which he began his note book.

Here he takes notes (sheets 12 -18, pp. 11-17) from the section on the general theory of equations (§§ 179-184, pp. 248-256 of Lacroix's book), where the issues are : the number of divisors of the first power, which an equation may have ; formation of an equation through the multiplication of its simple divisors ; the relation between the roots and the coefficients of an equation (the fundamental theorem of algebra about the existence of roots is only mentioned, but not proved). With this, the first part of this note book of Marx comes to an end.

Marx takes notes, from the next section of Lacroix's book, devoted to the elimination of the unknowns from equations of power greater than one, only in the fifth part of this note book of his. Here (sheets 18-27, pp. 17-26) Marx goes over to that part, which he entitled :

"*II. The first elementary appearance of* $\frac{a}{0}=\infty$ *and of* $\frac{0}{0}$ *in ordinary algebra.*"

This part contains a selection of extracts from Lacroix's "Elements of Algebra", where the special cases of the equations of first and second powers, leading to expressions of the form $\frac{a}{0}$ and $\frac{0}{0}$ are investigated in the light of the examples about the problem of two couriers (pp. 97-104, quoted from Lacroix's book) and Clairaut's problem about the point, equally illumined by two different sources of light (ibid, pp 174-180). In connection with the emergence of the sign ∞, a place from Euler's "Elements of Algebra" (Lyons edition, 1795, § 293, p. 227) is cited, where the equality $\frac{1}{0}=\infty$ is "established" by Euler, through an expansion of the fraction $\frac{1}{1-a}$ into an

infinite series, when $a = 1$. This extract is accompanied by the following comments of Marx. The first comment (sheet 21, p. 20), which Marx placed in a box, is being reproduced below :

It should be noted, that in this example from the most elementary algebra, *the difference* $b - c$ *in the denominator all the while diminishes, as b remains constant, and c all the time increases*[122]. The matter is entirely different with $\frac{f(x+h)-f(x)}{h}$ or with $\frac{y_1 - y}{x_1 - x}$. Actually, $x + h = x_1$ hence $x_1 - x = h$; here x remains a constant, but h all the while diminishes, that is why x_1 also diminishes.

$\frac{y_1 - y}{x_1 - x} = \frac{\Delta y}{\Delta x}$; gives $\frac{dy}{dx}$ in its minimum value; but this transformation of Δx into dx, and that is why, of Δy into dy, takes place when x remains constant. That is why, it does not matter at all, as to whether we have :

1) $\frac{\Delta y}{\Delta x} = a$; where on the right hand side, the variable x has completely vanished ($= x^0$),

or

2) $\frac{\Delta y}{\Delta x} = f'(x) + \frac{1}{2} f''(x)\, h + \cdots$

In the second case, assuming $h = 0$ we get

$$\frac{dy}{dx} = f'(x) \;;$$

for instance, $\frac{dy}{dx} = m\, x^{m-1}$. In both the sides only h changes, while x is not affected, and that is why it remains a constant during this operation. $\frac{1}{2} f''(x)\, h +$ etc. vanish, since the multiplier h turns into zero, but $f'(x)$does not change, because it does not contain h ; on the other hand, in

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h} = \frac{y_1 - y}{x_1 - x}$$

x *also* remains *unchanged* ; the difference $x_1 - x$ becomes $= 0$, since x_1 turns into x, i.e., since $h = 0$. That is why, in

1), for example, $f(x) = ax$,

$$f(x+h) = a\,(x+h) = ax + ah,$$

hence,

$$f(x+h) - f(x) \text{ or } y_1 - y = ax + ah - ax = ah,$$

whence,

$$\frac{y_1 - y}{h} = a \text{ or } \frac{y_1 - y}{x_1 - x} \text{ or } \frac{\Delta y}{\Delta x} = a\,.$$

In the right hand side there is no function of x, there is only the constant a ; this in no way obstructs the perception of what happens with $\frac{y_1 - y}{x_1 - x}$, when $x_1 - x$ constantly diminishes, owing to the fact that h diminishes, i.e., x_1 constantly approximates to x. Herein x remains a constant ; finally we get dx in place of Δx, and *that is why* also dy in place of Δy, whence $\frac{dy}{dx} = a$ instead of $\frac{\Delta y}{\Delta x} = a$. No change occurs in the right hand side, since a is a constant magnitude and, that means, it does not contain x ; but the same thing happened, when $\frac{\Delta y}{\Delta x}$ turned into $\frac{dy}{dx} = f'(x)$, though here in the right hand side [we have] a function of x ; however, the transformation of Δx into dx assumes only a change in x_1, i.e., in the element h of x_1, while x remains a constant [123].

Marx's second comment (sheet 23, p.22) is related to such an instance of the problem of two couriers, when they start from one point ($a = 0$) and move in one direction with the same speed ($b = c$). In application to the indeterminancy $\frac{0}{0}$ obtained in this case, Marx writes :

Thus, here, the expression $\frac{0}{0}$, for the values of x and, of y equal to it, is *only the symbol of an indeterminate quantity*. In other cases, when the expression $\frac{0}{0}$ has *other origin*, this symbol also acquires another meaning.

The following comment of Marx (sheet 23, p.22) is related to Euler's work.

[[In his "Elements of Algebra" *Euler* says about the expression $\frac{m}{0} = \infty$: it would have been a mistake to think, that an infinitely big number can not increase [124]. (Earlier he said, that ∞ is obtained by dividing 1 by 0, since

$$\frac{1}{0} = \frac{1}{1-1} = 1 + 1 + 1 + \cdots \textit{ in infinitum.})$$

Since $\frac{1}{0}$ stands for an infinitely big number, and $\frac{2}{0}$ is, beyond doubt, doubled $\frac{1}{0}$, namely $= \frac{2 \cdot 1}{0}$, so it is evident, that a number, *even if it is infinitely large*, can nevertheless become 2,3 or x times bigger.

Here, first of all, it should be noted, that $\frac{2}{0}$ (or any other numerator with 0 as its denominator), expanded into a series, is exactly that, what $\frac{1}{0}$ is, because $\frac{2}{0} = \frac{2}{2-2} = 1 + 1 + 1 + 1 + \cdots$ *in infinitum*.

Consequently, since the terms in the right hand side are equal, so $\frac{2}{0} = \frac{1}{0}$.

This apart, since

$$\frac{1}{0}=\frac{1}{1-1} \text{ and } \frac{1}{1-1}=\frac{2\cdot 1}{2\cdot(1-1)}=\frac{2}{2-2},$$

so again

$$\frac{2}{0}=\frac{1}{0}.$$

∞ may be represented with the same success, as with the series from unity like

$$\frac{1}{0}=\frac{1}{1-1}=1+1+1+\cdots,$$

through an infinite series of numbers, growing in any given ratio. Though, herein, a determinate part of one infinite series may be equal to $\frac{1}{2}$, $\frac{1}{3}$ etc., of the determinate part of another infinite series, but neither the first nor the second determinate part, is in *any proportion to the entire infinite series,* and in this case only this much may be said, that the serieses march to infinity in various ratios.]]

Marx's heading of the third part of this note book (sheets 27-34, pp. 26-33) :

"III. Rudiments of infinite serieses."

In its turn, this section is subdivided into two parts. First part : "A) *As a preparation for this, to begin with the approximate calculation of roots*" (sheets 27-30, pp. 26-29). Selection of extracts from Lacroix's "Elements of Algebra", §§ 98-104, pp.145-152. This is part of the chapter entitled "On the equations of second power with one unknown", in which is proved the theorem : "The whole numbers, which are not squares, have neither whole, nor fractional roots" and, the following questions are enunciated : "What is incommensurability or irrationality", "How are the radicals, of the roots to be extracted, designated by signs", "Method of approximate calculation of roots", "Method of abbreviated extraction of roots by division", "Method of its (of the process of extraction of roots) unbounded continuation by ordinary fractions", "Method of obtaining, as far as possible, simpler approximate values of roots from fractions, whose terms are not squares."

Second part : "*B) Infinite serieses*" (pp. 31-34). Selection of extracts from the same book of Lacroix, §§ 235-237, pp. 326-331. Concluding paragraphs from the chapter entitled, "On proportions and progressions", wherein, "The division of m by $m-1$, continued unboundedly" and "The cases in which this quotient *converges* and can be taken as the approximate value of the fraction $\frac{m}{m-1}$", are considered. Here, a short excerpt from the big "Treatise" of Lacroix — its Introduction, pp. 3-6 — is also included.

On p.30 of the note book, there are two short comments of Marx :

$\sqrt{2}$, found with the help of successive division, turns out to be an expression through infinitely continuing *approximation of the roots,* all of which are ordinary fractions. Thus, the extraction of irrational roots, with the help of the extraction of roots from successive remainders, leads to an infinite series.

Already in the case of a *simple approximation of the ordinary fraction by decimals,* we get an infinite expression, just as in the case of the *irrational roots.*

The fourth part of this note book (sheets 35-38, pp. 34-37) contains a short note of Marx, reproduced below (it was first published in Russian, in the journal "Voprosy Filosofii", 1958, No. 11, 89-95).

This note is based upon informations contained in : A) Lacroix's "Treatise on differential and integral calculus" (Paris, 1810, vol.1. pp. 2-4) and, B) Euler's "Elements of Algebra "(Part Two, chapters I and II).

"IV. On the concept of function".

A) If a problem is generally determinable[125], then for its determination *as many equations* are required, as are the *unknown quantities* to be sought. That is why, all those problems, in which the number of equations given are as many, as are the unknown quantities[126] present, belong to the realm of *determinate analysis.*

If a problem does not furnish as many equations, as are the number of unknown quantities [there], then some of the latter must remain indeterminate and [[the discussion will now enter into these]], they are determined by us *arbitrarily.* That is why such problems are called *indeterminate* and [they] constitute the subject matter, of a special division of algebra, of the *indeterminate analysis.*

Since in such cases instead of one or more unknown quantities some arbitrary numbers are taken, the problem permits of different solutions. On the other hand, upon such problems, often conditions are imposed, so that the numbers sought for become *integral and positive* or at least *rational* ; thereby the number of possible solutions are often reduced to very few; sometimes their number is infinite, and it is difficult to ascertain [them]; sometimes a solution is not at all possible.

a) *To find out two positive and whole numbers, whose sum* = 10.

We have before us the problem :

1) $x + y = 10$, 2) $x = 10 - y$,

where y is restricted only to this, that it must be *an integral and positive number*.

If we assume that $y = 10$, then we would get $x = 10 - 10 = 0$; but $x = 0$ is excluded, for x too must be an integral and positive number. Thus, the value of $y = 10$ also falls through ; the values of y, which we have the right to try, are possible only within the bounds of 1 to 9. Such limits are possible for y, thanks to the given conditions of the very problem.

On the other hand, already from the first assumption we see ; that $y = 10$ — wherein x turns into 0 — is excluded.

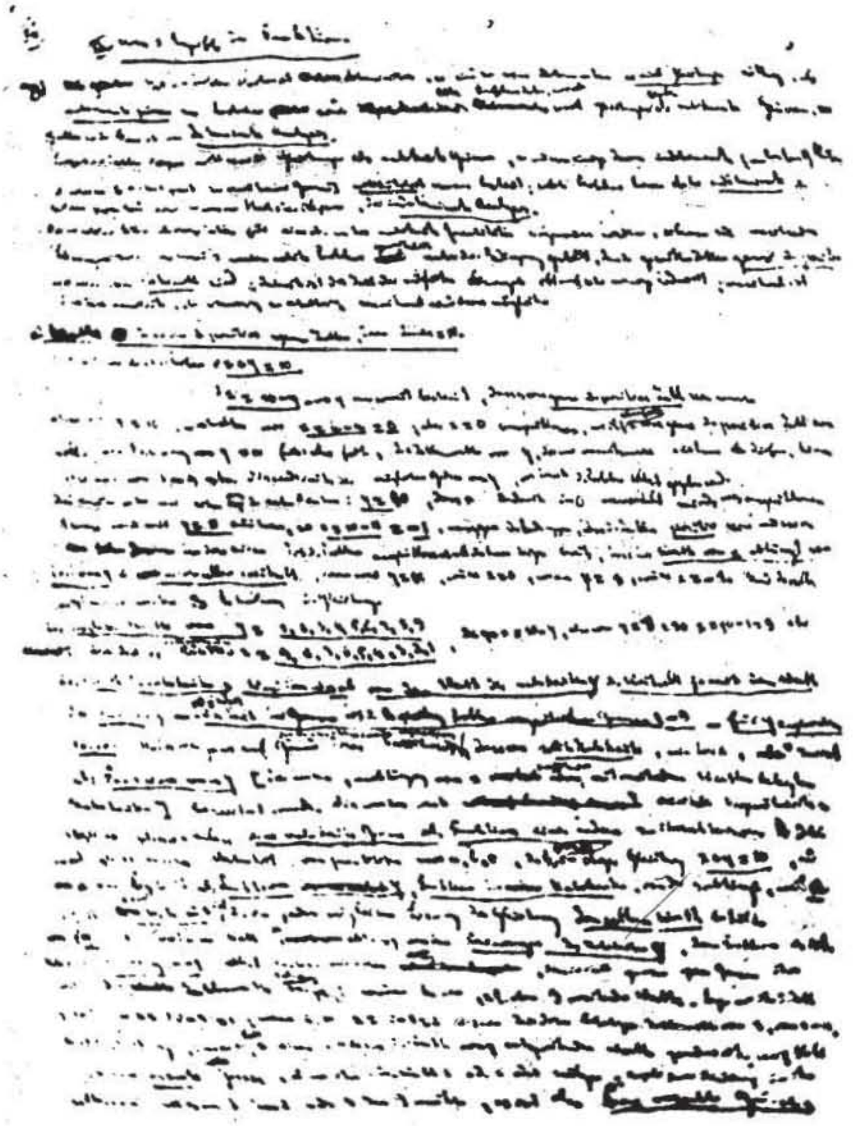
Exactly in the same way if we put $y = 11$, then

$$x = 10 - 11 = -1,$$

which contradicts the condition that, the numbers must be *positive.*

But both of these assumptions, excluded by the problem, show, that *the value of x depends upon the value of y and changes along with the latter* ; for when $y = 10$, then $x = 0$, and when $y = 11$, then $x = -1$. Further operation with the equation leads to the same thing. The *possible values of* $y = 1, 2, 3, 4, 5, 6, 7, 8, 9$; but in that case the corresponding values of $x = 9, 8, 7, 6, 5, 4, 3, 2, 1$, since $x = 10 - y$, owing to which, when $y = 1$ then $x = 10 - 1 = 9$ etc.

Hence, here the value of the unknown x depends upon the value of unknown y and it always changes depending upon the arbitrarily assigned values of y, but never going beyond the bounds of 1-9, prescribed by the problem. And, namely, on the basis of this interrelationship in the indeterminate equations, one of *these unknowns*, like x here, was at first called *a function*



"On the concept of function". Photocopy of a page from the note book "Algebra I".

of y [[of the other unknown, successively obtaining different values independently of x]]. In ordinary algebra this was the first occasion of characterizing *one unknown as the function of another*. Herein, from the very beginning we abstract from such quantities as a, b, c, for instance from 10 in the aforementioned $x + y = 10$, and define x only as a *function* [of] y, a *function of that unknown*, upon which it depends, for a or 10 is *already determined* and retains *one and the same value* in every possible solution of this problem.

$f(y)$ or x changes its value depending upon the *changes of the unknown* y, of which it is a function.

But *changes* of y itself consist of this, that within known limits different numerical values may be arbitrarily assigned to it, for instance, the above [mentioned] 9 different values. If, for example, we assign it the value 9, then $x = 10 - 9 = 1$; if 8, then $x = 10 - 8 = 2$ etc. Each of these arbitrary numerical values of y, from 1 to 9, solves the equation, permitting for x a value, corresponding to a value assumed for y ; but whether we assign y the value 1, or 8, or 3, herein y always remains a *constant ;* in y itself, no change takes place, which would turn it into 2 from 1 or into 9 from 8 ; hence, it is *not a variable*, though — within determinate limits — we may change its value at will. Similarly, in the expression $\frac{m}{m-1}$, where m is *not an unknown, as distinct from y*, [and where] we *vary* the value of m at will , however, owing to this m never becomes *a variable* ; it is merely an *indeterminate constant*, which, namely, owing to this, may obtain arbitrary, and [besides] any arbitrary, numerical value. If we put $m = 1$, then $\frac{m}{m-1} = \frac{1}{0} = \infty$; if we assign m the numerical value 3, then $\frac{m}{m-1} = \frac{3}{2}$ etc. Exactly in the same way the *unknown* y in the equation $x = 10 - y$ differs from 10, not due to the fact that 10 is a constant and y a variable, but because 10 is the *determinate* constant value of a magnitude, which remains determinate in every possible solution of this equation, whereas y is *indeterminate,* but is always also the *constant value of a magnitude*, and that is why in the solution of the equation it may be determinately varied as 1, 2, etc.

That is why, here, the *independent unknown* y is not a *variable,* in the same measure as m in the expression $\frac{m}{m-1}$; it is *not determined* in the same way, as m is not determined in respect of numerical arithmetical values ; it differs from m owing to [the fact], 1) that in an algebraic expression m, as distinct from y, is not an *unknown* ; 2) that the determination of the value of some other *unknown* x, does not depend upon the determination of the value of m. But if we had the equation

$$x = \frac{m}{m-1},$$

then the value of x would depend upon the different numerical values, which we would assign to m.

Thus, we see that the *concept of function*, as it initially emerged in *indeterminate analysis*, still had a very limited meaning, applicable only to definite forms of equations.

If now we return to the solution of the equation $x = 10 - y$, then

$y = 1, 2, 3, 4, 5, 6, 7, 8, 9,$

$x = 9, 8, 7, 6, 5, 4, 3, 2, 1.$

The last four values of $y = 6, 7, 8, 9$ gives us for x, its corresponding first values : 4, 3, 2, 1.

Hence the equalities are :

$6 + 4 = 10,\ 7 + 3 = 10,\ 8 + 2 = 10,\ 9 + 1 = 10.$

But we get the same 4 equalities when

$y = 1, 2, 3, 4,$ and $x = 9, 8, 6, 5.$

Thus the problem in fact admits of only 5 *different* solutions :

$y = 1, 2, 3, 4, 5,$ $\qquad x = 9, 8, 7, 6, 5.$

If we had, for instance, $x = \frac{y}{y-1}$, then the solution would not have differed from that of $[x =]\ \frac{m}{m-1}$; the condition that x has to be a positive whole number, would make the problem difficult, but would change nothing in the character of the equation.

b) If *instead of one*, as above, two *equations* are given, then the problem may be *indeterminate*, only if these two equations contain more than two *unknowns*.

This type of problem [[where only the equations of first power are proposed]] is met with in the ordinary *elementary text-books of arithmetic* and is soived with the help of the so-called *Regula Coeci (the Rule or Position of False).*

For example, 30 persons, men, women and children, spent 50 Sh. in a tavern, wherein every man spent 3 Sh., woman — 2 Sh., and child — 1 Sh. *How many men, women and children were there* ?

Let the *number of men* be $= p$, of women $= q$ and, of children $= r$; then we get

1) $p + q + r = 30$; 2) $3p + 2q + r = 50$;

from here we are required to find out p, q and r in *whole and positive* numbers.

Equation 1) gives us $r = 30 - p - q$; hence,

$$p + q < 30 ;$$

having put the value of r in the 2nd equation, we shall get

$$3p + 2q + 30 - p - q = 50 ;$$

that is

$$2p + q + 30 = 50 ;$$

hence

$$q = 20 - 2p,\ p + q = 20 - p < 30 .$$

From the equation $q = 20 - 2p$ it follows, that if $p = 10$, then $q = 20 - 20 = 0$; that is why, had we taken a number > 10 in place of p, then q would have been negative.

For example, if $p = 11$, then $p + q = 20 - p$ turns into $11 + q = 20 - 11$, or $q = 20 - 22 = -2$ This is excluded. Hence, in place of p we can take all those numbers, which are not > 10.

Remembering, that $p + q < 30$, and $q = 20 - 2p$ [[and that, *that is why*, when $p = 0$, $q = 20$]], we then get 11 answers :

$p = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10,$

$q = 20, 18, 16, 14, 12, 10, 8, 6, 4, 2, 0,$

$r = 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20.$

If we exclude

I) $p = 0, q = 20, r = 10,$

II) $p = 10, q = 0, r = 20,$

i.e., the first and the last solutions, where sub I) $p = 0$, and sub II) $q = 0$, then there will remain 9 solutions.

B) 1) Thus, initially the concept of *function* was restricted to those *unknowns* in the *indeterminate equations* (i.e., in the equations whose number was less than the number of unknowns entering into them), *whose value depend* upon the values of the other unknowns and *that is why change with different* [[either entirely arbitrary, *or* arbitrary *within certain bounds*, determined by the very problem]] *values*, assigned to the other unknowns.

For instance,

$$y = ax^2 + bx + c\,;$$

here y is a function of x ; in $y = axz + bx^2 + cz^2$, y is a function of x and z. In the equations

a) $x^3 + y^3 = axy$, b) $x^3 + y^3 + z^3 = axz + byz + cxy$,

1) x, y and z are, relative to b), mutual functions one of the others, 2) [in the first two equations] y is an *explicit function* of x or of x and z, since its value is given, when the values of x and z are determined, but in b) y is an *implicit function* of x and z, since even when they are known, the algebraic equation still remains to be solved for the determination of y.

Thus, the concept of *function*, as it was obtained from indeterminate equations, consisted of the following : if the desire was to express [the fact] that *a certain magnitude is indeterminable,* without *prior assignment of determinate values to other magnitudes,* which could obtain indeterminate number of such values in one and the same problem, then the word *function* was used for designating that *dependence.*

2) Later on the concept of *function* was *generalised* ; it was extended to every *algebraic unknown,* whose dependence upon indeterminate magnitudes may be expressed by an *algebraic equation*. Such functions were called *algebraic*. Algebraic functions always contain only a determinate number of terms, when these functions are expressed in their characteristic form. But, as we saw, *proper fractions*[127] can be expanded only into infiniteserieses ; and the concept of *function* was transferred also upon the latter, and thereby the path towards the *transcendental functions* was paved ; such are the *logarithms,* which can be expressed only by an infinite number of roots [128], as well as the *sines* and *consines*, if they are to be expressed in terms of their arcs.

C) *Further generalisation* [consists of this, that it] *does not demand equations* among some magnitudes, for one [of them] to be an *implicit function* of the others ; it is enough that its value depends upon that of the others. For instance, in the circle the sin is an implicit function of the arc, though no algebraic equation can express this, since one of the two is determined, if the other is determined and vice versa. (Here we digress from the radius, because a definite arc is not at issue [129].)

D) The concept of *function* is further developed and it obtained greater importance thanks to the Cartesian application of algebra to geometry, i.e., owing to the analytical or higher geometry. The unknown quantities x, y etc. turn into *variables*, and the *known* — into *constants.*

The function of a variable is another variable, whose value changes along with that of the first, i.e., *depends* upon it. It has this in common with the functions in indeterminate equations : when a particular value of that variable is given, of which it is a function, then it assumes a *corresponding determinate value.*

Marx gave the following title to the final, fifth part of this manuscript (sheets 39-93, pp. 38-76, 74 (instead of 77-89)) :

"*V. Elimination of unknowns form equations of power greater than one*".

Contentwise it is related, first of all, to the chapter bearing the same heading in Lacroix's "Elements of Algebra"(§§ 185-196, pp. 257-271). However, since in the text of this chapter there are references to the previous chapters, from which Marx did not take notes earlier, here the conspectus begins with these earlier chapters. And since, later on, Marx did not confine himself to the chapter on the elimination of unknowns, but included in his notes the two succeeding chapters from Lacroix's book — he divided the entire text [of this part] in its turn into five parts, numbered them with the Latin letters A-E and provided the titles :

"*A. For the equations of first power*",

"*B. Finding out the greatest common measure*",

"*C. Elimination of unknowns from equations of power greater than one*",

"*D. Finding out the rational and multiple roots of numerical equations*",

"*E. Approximate solution of numerical equations*".

Part "A" (sheets 39-47, pp. 38-46) contains notes from two earlier chapters of Lacroix's book, devoted to : 1) solution of a system of linear equations by successive elimination of the unknowns (§§ 78-82, pp. 114-123), and 2) the general formulae for the solutions of systems of linear equations (§§ 83-89, pp. 123-134).

Part "B" (sheets 47-52, pp. 46-51) contains extracts from the chapter entitled "On algebraic fractions" (§§ 48-50, pp. 67-76) of the same book of Lacroix, related to the Euclidean algorithm for finding out the greatest common measure of two polynomials.

In part "C", having taken notes in points 1)-6) (sheets 52-56, pp. 51-55) from the beginning (§§ 185-190, 257-262) of the same chapter of Lacroix's book, to which the whole of section V should have been devoted and which contains the general method of finding out the resolvents of the two equations

$$f(x,y)=0 \quad \text{and} \quad g(x,y)=0$$

through the method of finding out the greatest common measure of the polynomials $f(x,y)$ and $g(x,y)$, right upto that particular example, the analysis of which Lacroix premised by an enunciation of the general method, Marx writes (sheet 57) :

7) the method sub 6) *applied to particular equations*, can *also* be applied *to general equations*.

After this generalisation follows (sheets 57-58, pp. 56-57) the insertion adduced below, devoted to a short statement of the general theory of equations interpreted also through the apparatus of differential calculus. It appears that MacLaurin's "Treatise of Algebra" served as the source of this insertion by Marx.

ON THE GENERAL THEORY OF EQUATIONS

[[We note at first :

α) A general equation like

$$x^n + P x^{n-1} + Q x^{n-2} + R x^{n-3} + \ldots + Tx + U = 0,$$

has the form $f(x) = 0$.

If we consider such a polynomial expression, not as an equation, but as — its first side [L.H.S.] — a function of x, then, when x assumes a determinate value, for example, a, [we get] $f(a)$, $f'(a)$, $f''(a)$, $\cdots$, $f^{(n)}(a)$; *the function* x[130] *varies for the various values of* x. When x [assumes] a particular value a, and this particular value a turns $f(x)$ into 0, i.e., $f(a) = 0$, ... then the value a satisfies the equation, solves it, or is its root.

The investigation into the roots of the equation $f(x) = 0$, coincides with the expansion of the polynomial $f(x)$ into its factors, as the search for some root a of the equation is determined by the *corresponding factor* $(x - a)$ *of the polynomial* and *vice versa*. This is proved by the fact that, Taylor's series is always applicable to such a polynomial.

We have

$f(x) = f(a + (x - a)) = f(a) + (x - a) f'(a) + \ldots + (x - a)^n$.

That is why, when $f(a) = 0$, i.e., when a is *a root*, then $x - a$ is *a factor*; and conversely, when $x - a$ is *a factor*, then $f(a) = 0$ and a is *a root*.

β) *Every equation has so many roots, as many it has powers.*

$f(x) = 0$ always has one root[131], i.e., always has a factor of the form $(x - a)$, *which divides f(x) without a remainder.*

The quotient $\frac{f(x)}{x-a}$ has the same form as $f(x)$, but in power it is lower only by one. Hence, it must have a factor of the form $\frac{f(x)}{(x-a_1)(x-a_2)}$, and the quotient of this division is a polynomial of power $n - 2$. Operating further in the same mode we shall finally get a quotient, where x no more has any power ($x^0 = 1$), hence,

$$\frac{f(x)}{(x-a_1)(x-a_2)\ldots(x-a_n)} = 1,$$

i.e.,

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_n).$$

a_1, a_2, ..., a_n are the roots of the equation $f(x) = 0$, and no other magnitude can be its roots, in fact, if we substitute some other magnitude Q for x, then

$$f(Q) = (Q - a_1)(Q - a_2) \dots (Q - a_n),$$

which is not = 0, i. e., Q can not be a root.

γ) *Connection between the coefficients of an equation and its roots.*

Let the roots be $a, b, c, \dots, l$; then if we have a general equation of n-th power, then

$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = (x - a)(x - b)(x - c) \dots (x - l) = x^n - x^{n-1}(a + b + c + d + \dots) +$
$+ x^{n-2}(ab + ac + \dots + bc + \dots) - x^{n-3}(abc + acd + \dots) + \dots + x(-1)^n abc \dots l.$

If we designate by Σ the sum of all the expressions, analogous to that, before which this symbol is put, then

$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = x^n - x^{n-1}\Sigma(a) + x^{n-2}\Sigma(ab) - x^{n-3}\Sigma(abc) +$
$\dots + x(-1)^n abc \dots l.$

Thus, in general : $(-1)^r p_r =$ *the sum of all products of r roots.*]]

Having finished this insertion, Marx continues (sheets 58-65, pp. 57-64) the notes from the chapter on the elimination of unknowns from Lacroix's book (§§ 191-196, pp. 262-271), after which he goes over to part V of his conspectus (sheets 66-82, pp.65-76, 74-78), related to the search for the rational and multiple roots of numerical equations. This part contains notes from the chapter bearing the same heading, of Lacroix's book (§§ 197-210, pp. 271-288).

Sheets 74-78 (pp. 73-76, 74) contain the following comments of Marx, related to the connections between algebra and differential calculus :

ON THE CONNECTIONS BETWEEN ALGEBRA AND DIFFERENTIAL CALCULUS

[[1) We note first, that the general equation, which is the starting point, is

$x^m + P x^{m-1} + Q x^{m-2} + \dots + T x + U = 0,$

or $f(x)$ or $y =$ etc.

Having written the equation [132] in reverse order, we shall get

$f(x)$ or $y = U + T x + \dots + Q x^{m-2} + P x^{m-1} + x^m,$

which could be written as :

$y = A + B x + C x^2 + \dots + P x^{m-2} + Q x^{m-1} + \dots,$

such that this $[Q x^{m-1}]$ is here only a general term.

Thus, the latter equation, from which *MacLaurin* proceeds [133], is nothing but the general equation of algebra with *one unknown*, written in the reverse order, because here we need the ascending order of powers. For the rest the difference consists only in this, that the series for an equation with power m contains $(m + 1)$ terms, whereas here we have an infinite series [134].

2) As regards the deduction of equations from the proposed or initial equation[135], the procedure is exactly the same, as in the differential calculus, in fact that is how *MacLaurin* comes forth, *successively differentiating the initial equation* $y = \dots$. Thus, this method represents a translation from the language of algebra, into the language of differential calculus.

3) *Lacroix* says, that equation 2) or (A) [[the first derivative is obtained by putting x for a in

$$m\,a^{m-1} + (m-1)\,P\,a^{m-2} + (m-2)\,Q\,a^{m-3} + \ldots + T = 0,$$

which is permissible, since a is one of the values of x]], is such that

$$m\,x^{m-1} + (m-1)\,P\,x^{m-2} + (m-2)\,Q\,x^{m-3} + \ldots + T = 0$$ [136]

is *directly deducible* from the initial :

$$x^m + P\,x^{m-1} + Q\,x^{m-2} + \ldots + T\,x + U = 0\,,$$

and besides such that : multiply every term with the index of that power of x, which it contains, and then diminish this index by one. Just as the equality A = 0 is directly deduced from V = 0, so is the equality B = 0 deduced from A = 0, by repeating the same method. Analogously C = 0 from B = 0 etc. [137].

It is not apparent, as to what permits *on the basis of ordinary algebra to change* with the help of arbitrary division or multiplication, each separate term of the equation individually, independently of its other terms. For example, from x^m, I make $m\,x^{m-1}$, multiplying x^m by m and by dividing that by x; in fact, $m\,x^{m-1} = m\dfrac{x^m}{x}$; hence, thereby I change the numerical value of this term, which I would be allowed [to do], according to algebra, only under the condition, that I *would multiply all the terms by m and divide them by x* ; then, instead of $P\,x^{m-1}$, I would get as the second term

$$\frac{m\,P\,x^{m-1}}{x} = m\,P\,x^{m-2}\ ,$$

but by no means $(m-1)\,P\,x^{m-2}$ etc.

Thus, this method comes forth as Hysteron Proteron [138], in so far as the differential method has been silently assumed. [[This comment is false, as Lacroix does not assert, that he obtains an identical equation with the help of this method, but only that in this way the power of the equation is lowered by one. However, he knew it only from the differential calculus.]]

The proof should have been along opposite lines, so that the differential operation was obtained from the algebraic.

I) $\therefore\ x^m + P\,x^{m-1} + Q\,x^{m-2} + R\,x^{m-3} + \ldots + T\,x + U = 0\,;$

assuming $x = a + y$, we shall get

$$(a+y)^m + P\,(a+y)^{m-1} + Q\,(a+y)^{m-2} + \ldots,$$

hence,

$$(\mathrm{Ia})\ \left\{\begin{array}{ll}
1) & a^m + m\,a^{m-1}\,y + \dfrac{m\,(m-1)}{1\cdot 2}\,a^{m-2}\,y^2 + \cdots + y^m + \\[2ex]
2) & +P\,a^{m-1} + (m-1)\,P\,a^{m-2}\,y + \dfrac{(m-1)\,(m-2)}{1\cdot 2}\,P\,a^{m-3}\,y^2 + \cdots + \\[2ex]
3) & +Q\,a^{m-2} + (m-2)\,Q\,a^{m-3}\,y + \dfrac{(m-2)\,(m-3)}{1\cdot 2}\,Q\,a^{m-4}\,y^2 + \cdots + \\[2ex]
4) & +R\,a^{m-3} + (m-3)\,R\,a^{m-4}\,y + \dfrac{(m-3)\,(m-4)}{1\cdot 2}\,R\,a^{m-5}\,y^2 + \cdots + \\
 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
 & +T\,a + T\,y + \\
 & +U = 0\,.
\end{array}\right.$$

Read *horizontally*, in these expressions we have :

1) $a^m + m\,a^{m-1}\,y$ + etc.. the binomial expansion for $(a + y)^m$, substituted in place of x^m;

2) the binomial expansion for $P\,(a + y)^{m-1}$;

3) the same for $Q\,(a + y)^{m-2}$;

4) the same for $R\,(a + y)^{m-3}$;

....

5) the same for $T\,(a + y)$ instead of the term Tx in the initial equation. Further, I have the right to place U at the end of the first vertical series, since by analogy with the other terms U is the same coefficient of y^0 (= 1).

Hence, these results are obtained by a *simple application of the binomial theorem,* when the monomial x is replaced by the binomial $(y + a)$. And these results, read *vertically*, give us as the first series $a^m + P\,a^{m-1} + Q\,a^{m-2} + R\,a^{m-3} + \ldots + T\,a + U$, which in fact coincides with equation 1), only when in place of x is substituted one of its value = a.

Since a is a value of x [wherein $f(x) = 0$], this series vanishes by itself or = 0. Hence, what remains, is

$$\text{II)}\quad \left\{ \begin{array}{lllll} 1) & m\,a^{m-1}\,y & + \dfrac{m\,(m-1)}{2}\,a^{m-2}\,y^2 + \cdots + y^m + \\ 2) & +\,(m-1)\,P\,a^{m-2}\,y + & \dfrac{(m-1)\,(m-2)}{2}\,P\,a^{m-3}\,y^2 + \cdots + \\ 3) & +\,(m-2)\,Q\,a^{m-3}\,y + & \dfrac{(m-2)\,(m-3)}{2}\,Q\,a^{m-4}\,y^2 + \cdots + \\ 4) & +\,(m-3)\,R\,a^{m-4}\,y + & \dfrac{(m-3)\,(m-4)}{2}\,R\,a^{m-5}\,y^2 + \cdots + \\ & \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ & +\,T\,y = 0\,. & \end{array} \right.$$

If I divide all these terms by y[139],then I shall get :

$$\text{IIa)}\quad \left\{ \begin{array}{lllll} 1) & m\,a^{m-1} & + \dfrac{m\,(m-1)}{2}\,a^{m-2}\,y + \cdots + y^{m-1} + \\ 2) & +\,(m-1)\,P\,a^{m-2} + & \dfrac{(m-1)\,(m-2)}{2}\,P\,a^{m-3}\,y + \cdots + \\ 3) & +\,(m-2)\,Q\,a^{m-3} + & \dfrac{(m-2)\,(m-3)}{2}\,R\,a^{m-4}\,y + \cdots + \\ 4) & +\,(m-3)\,R\,a^{m-4} + & \dfrac{(m-3)\,(m-4)}{2}\,R\,a^{m-5}\,y + \cdots + \\ & \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ & +\,T = 0\,. & \end{array} \right.$$

Since here the first vertical series

$m\,a^{m-1} + (m-1)\,P\,a^{m-2} + (m-2)\,Q\,a^{m-3} + (m-3)\,R\,a^{m-4} + \ldots + T$

does not contain y, so it must be $= 0$, in so far as its sum together with the other vertical serieses $= 0$[140], i.e., it $= 0$; independently of the value of y.

Hence, I have,

$m\, a^{m-1} + (m-1)\, P\, a^{m-2} + (m-2)\, Q\, a^{m-3} + (m-3)\, R\, a^{m-4} + \ldots + T = 0.$

Now comparing the first derivative of equation 1) with this, I find :

I) or (V) =

$= a^m + P\, a^{m-1} + Q\, a^{m-2} + R\, a^{m-3} + \ldots + Ta + U = 0,$

II) or (A) =

$= m\, a^{m-1} + (m-1)\, P\, a^{m-2} + (m-2)\, Q\, a^{m-3} + (m-3)\, R\, a^{m-4} + \ldots + T = 0.$

Here substituting again in I) and II) a by x, which can be done, since a is one of the values of x, we shall get :

I) or (V) =

$= x^m + P\, x^{m-1} + Q\, x^{m-2} + R\, x^{m-3} + \ldots + Tx + U = 0,$

II) or (A) =

$= m\, x^{m-1} + (m-1)\, P\, x^{m-2} + (m-2)\, Q\, x^{m-3} + (m-3)\, R\, x^{m-4} + \ldots + T = 0.$

A comparison of these two equations shows, that in obtaining (A) from (V) x^m is multiplied by the index of its power m and from this index itself 1 is subtracted ; thereby x^m turns into $m\, x^{m-1}$. I treat the remaining terms in the same way; for example, I multiply $P\, x^{m-1}$ by the index of the power of $x = m - 1$; $(m-1)\, P\, x^{m-1}$ is obtained ; then subtracting 1 from the index, I finally get $(m-1)\, P\, x^{m-1-1} = (m-1)\, P\, x^{m-2}$ etc. In the same way Tx turns into T, when I multiply it by the index of the power of $x = 1$, and subtract 1 from the index of its power, thus, getting $1 \cdot T\, x^{1-1} = T x^0 = T$. Finally $U = U \cdot x^0$ vanishes, when I multiply it by the index of the power of $x = 0$.

In obtaining this result I do not start from the fact that I can so operate, but from this, that A = 0 is deduced from V = 0 upon a strict algebraic foundation ; it shows, that I could have directly acted like that [141]. Now, concerning the deduction of B = 0 from A = 0, in so far as A = 0, i.e., A vanishes, we get as remainder :

$$\text{III)} \left\{ \begin{array}{l} 1)\ m\,(m-1)\, a^{m-2} y + \cdots + y^{m-1} + \\ 2)\ + \dfrac{(m-1)(m-2)}{2} P\, a^{m-3} y + \cdots + \\ 3)\ + \dfrac{(m-2)(m-3)}{2} Q\, a^{m-4} y + \cdots + \\ 4)\ + \dfrac{(m-3)(m-4)}{2} R\, a^{m-5} y + \cdots + \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \quad + \left[\dfrac{2S}{2} y\right] = 0 . \end{array} \right.$$

Since all the terms [of the first column] of the left hand side, [which] = 0, has 2 as denominator, I can multiply the entire equation by 2 and thus remove this denominator. Further, since all the terms contain y as the general coefficient, I can divide the entire equation by y and thus remove y [from the first column]. Then I get III or (B) =

$$= m(m-1)a^{m-2} + (m-1)(m-2)Pa^{m-3} + (m-2)(m-3)Qa^{m-4} +$$

$$+ (m-3)(m-4)Ra^{m-5} + \ldots + [2 \cdot 1 \cdot S].$$

Again substituting in this equation a by x and by comparing it with equation II) or (A), we shall get :

$$(A) = mx^{m-1} + (m-1)Px^{m-2} + (m-2)Qx^{m-3} + (m-3)Rx^{m-4} + \ldots + T,$$

$$(B) = m(m-1)x^{m-2} + (m-1)(m-2)Px^{m-3} + (m-2)(m-3)Qx^{m-4} +$$

$$+ (m-3)(m-4)Rx^{m-5} + \cdots + [+ 2 \cdot 1 \cdot S].$$

This comparison shows, that B = 0 is deduced from A = 0, in the same way as A = 0 from V = 0 [142]. mx^{m-1} is transformed into $m(m-1)x^{m-2}$, i.e., mx^{m-1} is multiplied by the index of its power $m-1$, and this index of power is itself diminished by 1, i.e., x^{m-1} is divided by x ; we get :

$$m(m-1)\frac{x^{m-1}}{x} \text{ or } m(m-1)x^{m-1-1} = m(m-1)x^{m-2},$$

and so with each of the following terms. [2S] vanishes, in so far as its coefficient is x^0; it vanishes upon being multiplied by 0, i.e., by the index 0 of x etc.

Hence, the method of *successive differentiation,* applied in *MacLaurin's* theorem, is also borrowed from ordinary algebra, just like the *general* form of the function x, from which he sets off and which is the *general algebraic equation with one unknown*, differing only in this, that instead of a determinate equation [here] appears a polynomial expression of the general function of x in the form of an *infinite series.*

[[Whether I consider the expression

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U,$$

as equal to 0, or only as a function of x independently of its equality with 0, the essence of the matter is not altered by this. In both the cases the issue is only about the general *polynomial expression* of this equation [143].]]

That MacLaurin *inverts this series*, i.e., writes it, beginning not with the first term, but with the last, is also not an arbitrary and artificial mode, but is simply based upon the binomial theorem.

If I put a binomial with one unknown in its simplest form, i.e., $x + a$ in an indeterminate power — where x, as well as a may, in their turn, represent whatever polynomial expansions you like — then, if I make x the first term, and a the second, I obtain $(x+a)^m$, and here the expression takes place according to x, meanwhile a figures only as the multiplier a, a^2 etc. in different derived functions of x^m ; and, conversely, if I make a the first term, and x the next, i.e., write $(a+x)^m$, then I expand a^m with its derived expressions $m\,a^{m-1}$...

In differential calculus, when we start from the first polynomial expression, the function of x, then all further derivatives can generally figure only as derived functions of the variable x.

Instead of assuming $x = a + y$, as in algebra, here, at first $(x + a)$ is expanded and then it is assumed that $a = 0$ [144], which leads to the same result, since in the algebraic deduction y is afterwards algebraically removed through successive division of both the sides of the equation by y [145].

Separate terms of the algebraic equations give us at the same time the general proof of this, that the next derived function of x^m is equal to $m x^{m-1}$, that of $m x^{m-1}$ is equal to $m(m-1)x^{m-2}$ etc., i.e., in essence, [the proof] of successive differentiation.

Initial Equation of MacLaurin :

$y = A + Bx + Cx^2 + Dx^3 + \cdots,$

That of Taylor :

$y_1 = y + Ah + Bh^2 + Ch^3 + Dh^4 + \cdots$

In both the cases the issue is the determination of the indeterminate coefficients A, B, C etc ; in the first case they are *constants*, as in the algebraic derivation of the expressions V, A etc., $a^m + Pa^{m-1} + \cdots$, $m a^{m-1} + (m-1)Pa^{m-2} + \cdots$ are , serieses according to the powers of the given value a of the variable x ; in the second case A, B, C etc. are indeterminate *functions of the variable* x ; but here again [we have] an analogy with algebra. For the solution of the general equations with two unknowns, we reduce them to the form :

1) $x^m + Px^{m-1} + Qx^{m-2} + \cdots + Tx + U = 0,$

2) $x^n + P_1x^{n-1} + Q_1x^{n-2} + \cdots + Y_1x + Z_1 = 0\,;$

from them x is eliminated, and [for this] we are required to find out the coefficients p, q [etc.], p_1, q_1 etc., containing the functions of the second unknown y, entering into the final equation [146] ; only x is to be replaced by h, so that instead of x^m + etc. the first equation turned into

$f(x)$ (or y) $+ Px + Qx^2 + \cdots$ [147] .]]

The last part of the note-book "E", on sheets 83-93 (pp. 79-89), is the conspectus of the chapter on approximate solution of numerical equations from Lacroix's "Elements of Algebra" (§§ 211-222, pp. 289-312). With this chapter, the enunciation of the general theory of algebraic equations comes to an end, in Lacroix's book. The next chapter of the book is entitled :"On proportions and progressions", and Marx begins his note-book "Algebra II" with its examination (see, manuscript 3933).

THE NOTE BOOK "ALGEBRA II"

S.U.N. 3933

Marx gave this note book the heading "Algebra II". It consists of 92 sheets (pp. 1-67,48-72 in Marx's numbering). It is a continuation of the note book "Algebra I"(manuscript 3932). Contentwise, it has been sub-divided into three parts : in the first, the notes taken from Lacroix's "Elements of Algebra" have been completed; the second is devoted., specially, to Newton's binomial theorem and to the questions of combinatorics, acquaintance with which is assumed in its proof; in the third, notes have been taken from MacLaurin's "Treatise of Algebra".

The contents of these three parts are as follows, in brief:

I. Sheets 1-25. Sections VI-VII contain notes taken from the chapters:"On proportions and progressions", "Theory of exponentials and logarithms", and "Questions related to interest on money", of Lacroix's book, with insertions from other sources.

II. Sheets 26-27. This part of the manuscript, after the first two pages (of Section VIII), is devoted to "variation" (linear dependence of one magnitude upon another, on its square, on its roots, upon the product of others etc). It is no mere conspectus, but a systematization of a large amount of material collected from the most diverse sources.At first (sheets 27-38) — as a preparation towards the apparatus, necessary for Newton's binomial theorem — questions of combinatorics : finite sets of objects ("Combinations"), different modes of forming combinations (making full lists of combinations of a determinate type) and counting the number of combinations of different types without a preliminary construction of their lists, are considered.

Then follows (sheets 38-68) the section under the heading :"C) *The binomial theorem*". Here, at first (sub-section I, sheets 38- 40) materials, testifying to the empirical emergence of the theorem are adduced. After that (sub-section II, sheets 41-51) its proof is given for the integral positive index n of the power of a binomial (with the help of combinatorics). Finally(in sub-section III, sheets 51-68) under the heading "*General binomial theorem*", materials related to the generalisation of the theorem for fractional and negative index n of the power of a binomial, and to the application of the generalised theorem for calculation of roots and expansion into serieses, have been collected.

Here, amidst the sources appear the classical works of Euler and MacLaurin (in the notes form these works Marx always mentions the names of their authors), as well as a large number of various text-books on algebra — English and German (apparently, Marx did not think, that it was necessary to remember the surnames of their authors). Among them there are such authors, whose names could not be ascertained.

III. Sheets 68-92. Notes taken from MacLaurin's "Treatise of Algebra", chapter XIV of the first part, and from the first five chapters of the second part(for the continuation of this conspectus, see manuscript 3934). Here the following questions have been considered : commensurability and incommensurability (Euclid's algorithm), "the number of roots, which an equation of any power may have", symmetric functions, the number of positive (and, correspondingly, negative) roots of an equation (Descartes' rule of signs); here special attention has been paid to the question of multiple roots of an equation, in so far as this question is connected with the emergence of derived functions in algebra . This note contains a large number of Marx's own comments.

Now we shall give a detailed description of this manuscript.

Sheet 1 (Marx did not number it). On the title page we read : "*Algebra II*".

Sheets 2-9 (Marx's 1-8). "*VI. Proportions and Progressions*" . Notes from the chapter bearing the same title, of the same book by Lacroix (§§ 223-236, pp. 312-317).

Summing up an examination of the different types of derived proportions, Marx briefly formulated the conclusion (sheet 3), at which Lacroix arrived, as under :

4) What has been stated in points 1),2),3), in fact contains an extract of the theory of proportion ; *the entire doctrine is superfluous*, since *for every proportion, an equation corresponding to it may be substituted.* A special consideration of *proportions* is still useful, only in so far as it provides an easy transition to progressions.

While taking notes from the section on progressions, Marx underlined those places from Lacroix, where the discussion is about"infinite continuation of a series"(p. 326) about that fact, that " the expansion

$$1+\frac{1}{m}+\frac{1}{m^2}\text{ etc.}$$

can be considered as the value of the fraction $\frac{m}{m-1}$ every time, when it is convergent" (pp. 327-328)[148], and that convergence takes place, only when $m>1$. "In all other cases[of continuous division of m by m-1] the remainders should not be neglected, since by their constant increase they prove, that the quotients move off more and more form the true value"(p.328).

Sheets 9-25 (Marx's 8-24). "*VII . Exponential magnitudes and logarithms.*" This section of the note book begins with notes from the chapter on the same theme of Lacroix's book.Taking notes from §§238-250 (pp. 331-346), Marx numbered them (including also a large comment on pp. 337-344) by the numerals 1-13. It is devoted to the arithmetic complement of logarithm. § 248 (pp. 342-345 ; § 12, sheets 13-14 in Marx's manuscript) — Marx's conspectus of this topic is incomplete. The conspectus comes to an end with the following extract(sheet 14) from Lacroix (p. 344):

Thus, by this operation we turn subtraction into addition, using, instead of the number to be subtracted, its *arithmetic complement.*

After this Marx wrote :"[[*Further on this, in latter* §]] ". In connection with this comment, see below"sheet 25, bottom" ; PV,188.

In § 13(sheets 14-16), devoted to the mode of transition from one system of logarithms to any other, Marx alternates the conspectus of the corresponding paragraph (§ 250) of Lacroix with that from the book : J.Hind, "The Elements of Plane and Spherical Trigonometry", 3rd ed., Cambridge, 1837, ch VII," The Calculation of Logarithms and the Construction of Mathematical Tables ", §§ 162-177. Here Marx's notes do not strictly follow Hind's text. In these notes the discussion is about the different systems of logarithms: decimal (which is called the "common system of logarithms or that of Briggs"), and natural, which was identified — as was still the case, usually in most of the manuals of 19th century — with that of Napier. § 14 (sheets 16-19) under the title "*calculation based on the Napierian system* ", begins with the following comment of Marx:

a) The *starting point* is the exponential equation $y = a^x$, and the problem is, first of all, to express a^x in terms [i.e., positive and integral powers] of a (the base) and x, i.e., the base and its exponent, which is the logarithm of y. In order to carry out this trick — *on the basis of ordinary algebra* — it is necessary first of all, to turn the monomial a^x into a *binomial* ; and since every *magnitude = itself* + 1–1, so nothing prevents [us] from writing instead of the monomial a^x, the binomial $(a+1-1)^x$ or,what is the same, $\{1+(a-1)\}^x$, where 1 is the first term of the binomial and $(a-1)$ is the second. Thus we obtain for a^x, *a series in ascending*

positive integral powers of x, by applying the binomial theorem. The problem is solved by the method of indeterminate coefficients and their determination with the help of two different expressions for the expansion in series of one and the same function; the latter itself is based, here, upon the fact that $f(x) \cdot f(z) = f(x+z)$[149].

After this Marx takes detailed notes from the same chapter of Hind's book, now from the very beginning, i.e., §§ 160-165, pp. 154-158. This entire point 14 of the manuscript has been sub-divided into three points : a), b), c).

In point a), after the extract adduced here, there is a deduction of the "exponential theorem", which is usual for the majority of courses on analysis and algebra, of the first half of 19th century :

$$a^x = 1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \frac{A^3 x^3}{1 \cdot 2 \cdot 3} + \ldots + \frac{A^p x^p}{1 \cdot 2 \cdot 3 \cdots p} + \cdots$$

(where $A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \cdots$)through the method of indeterminate coefficients (see, editor's note [149]).

Point b) (sheet 18) carries the heading (given by Marx) : "*b) To deduce from the equation $y = a^x$ an expression for x (for the logarithm) in terms of a and y*".

This expression is obtained in the form of a quotient of two logarithmic serieses :

$$\frac{L}{A} = \frac{(y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \cdots}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \cdots}$$

After this it is said that :

This expression for x has no practical value for arithmetical calculation of *x, save the case, when both the serieses* — in the numerator and in the denominator — *are convergent.*

Point c) (sheets 18-19) carries the heading : "*c) Calculation of the numerical value of the base of the Napierian system of logarithms*". This point contains the definition of the base e of the natural logarithms, as such a value of a, wherein $A = 1$; and, further, it contains a calculation of e with the help of the sum of ten terms of the series for e, thus obtained from the "exponential theorem". Marx mentioned this calculation in full,with a mistake in the 6th decimal place: $e = 2 \cdot 71828276$. Presence of the same mistake in the book by Hind, quoted above, finally solves the question — which remained open for a long time — of the unknown source of the sheets 14-19 of manuscript 3933.

The issues in § 15 are (sheet 19) : negative logarithms, logarithms of zero and of negative numbers. The beginning of this paragraph is the conspectus from § 251 of Lacroix's "Elements of Algebra" (p. 346). In this paragraph Lacroix explains (it is now commonplace) the meaning of the words "logarithm of zero is equal to negative infinity", which " we see in many tables ". Having written down this explanation, Marx writes further :

Usually it is argued like this :

$$0 = \frac{1}{\infty} = \frac{1}{a^\infty} = a^{-\infty}.$$

After this Marx continues :

The equation

$$-y = a^x \text{ or } y = -a^x$$

can not be satisfied either by a positive or by a negative value of x; that is why *logarithms of negative magnitudes* can not exist in a system, which has *a real magnitude* as its *base* ; they are, owing to this, *imaginary*.

The source of both these places (they are not there in Lacroix), is the same text book on trigonometry by Hind, mentioned above; now it is chapter IV, " The Nature and Properties of Logarithms", its very beginning : §§ 89-90, pp. 67-68. However, the beginning of chapter XI (pp. 239-240) of another book by the same author,"The Elements of Algebra", 4th ed. Cambridge, 1839, is also contentwise very close to these paragraphs.

After this, on sheets 19-20 (Marx's 18-19), the notes from chapter VII come to an end. This is the chapter on "Exponential Magnitudes and Logarithms "in Lacroix's book .

Sheets 20-25 (Marx's 19-24). These are notes from the next, the last, chapter of Lacroix's book (§§ 256-262,pp.349-358), under the heading :"*16) The theory of geometric progressions and that of logarithms applied to the problems of interest on money*", devoted to these applications.

The conspectus comes to an end with the following comments [150] (sheet 25) :

If n becomes *infinite*, then $a = Ar$ and $A = \frac{a}{r}$; then a becomes *a perpetuity* ; A is the *present value* of this *perpetuity*. If we represent the general expression for A in the following form :

$$A = \frac{a}{r}\left\{1 - \frac{1}{(1+r)^n}\right\} = \frac{a}{r} - \frac{a}{r(1+r)^n},$$

then we shall get the *difference* between $\frac{a}{r}$, *the present value of a perpetuity* a and the [present] *value of an annuity, payable yearly* [for $n < \infty$ years, in instalments $= a$].

Leases are often concluded for a term *of 99 years* ; if we put this number and assume that increase $= 5\%$, i.e., $r = \frac{1}{20}$, then $A = 20a\left(1 - \frac{1}{125}\right)$ is obtained as the *present value of a lease for* 99 *years*; its price differs from the present value of a perpetuity only by $\frac{1}{125}$ [in yearly instalments] of the same amount [151].

Sheet 25(Marx's 24) , bottom. " 17) *Addition to* §12, p. *12 and 13 on arithmetic complement* ". However, this addition, contemplated earlier by Marx(see, p.186), remains unwritten even here.

All the same, apparently, Marx did not renounce his intention to return to this question, since part of the page under this heading and the whole of next page of the note book remains blank: in the photocopies there is no page marked 25 by Marx.

Sheets 26-28 (Marx's 26-67, 48). For the whole of this part of the manuscript (excluding sheet 68) Marx's numerations, and archival numerations coincide. That is why, hence forth , upto sheet 67, only the archival numeration will be mentioned. Section VIII contains materials related to combinatorics and Newton's binomial theorem. On the top of sheet 26 we read :

"*VIII . (Continuation of the theory of equations see note book I, Algebra)*".

Sheets 26-27. "A) *Variation*". Under "Variation" here he considered : the change of some magnitudes proportional to others, their product, quotient, square root of their products etc. The symbol $\propto$ is introduced for the designating the fact that y changes in proportion to x, and it is suggested that the expression $y \propto x$ be read as:" y varies as x".

Properties of variation , like:

if $y \propto x$ and $x \propto z$, then $y \propto z$;

if $y \propto x$ and $x \propto z$, then $x \propto \sqrt{yz}$;

and others, are proved. Examples from geometry, commercial arithmetic (interests on capital) etc., are cited. One of the most probable sources [of this part of the manuscript] appears to be the book : H.Goodwin, "An Elementary Course of Mathematics", Cambridge, 4th ed., 1853 [152].

The beginning of the section on "variation" in reality reads as follows :

A) *Variation*. If a *magnitude y depends upon another x*, such that, when x changes its value, a correspondingly proportional change of value takes place in *y*, then it is said, that *y varies as x*, and for this the symbol $\propto$ is used, i.e., $y \propto x$.

For instance, in Euclid VI, 1: "*Triangles and parallelograms of the same height correspond to each other, as do their bases.*" If we double the base of a triangle or of a parallelogram, whose height remains the same, then we shall double their area,and their area changes in the ratio in which we change the base . *That is why, the area varies as the base, for a given height.*

In goodwin (§93, p.59) we read :

"*Variation*, § 93. If a magnitude y depends upon another x, such that, when x change [its] value, the value of y changes in the same proportion, then it is said, that y *varies directly* as x or in brief, *varies* as x.

For example, we know from Euclid VI,1 that if we double the base of a triangle, keeping the height same, then we shall double the area, and that, in whatever proportion we change the base, the area changes in the same proportion, hence, we must say, that (for a given height) the area of a triangle *varies as its base*. The phrase "y varies as x" is written as : $y \propto x$".

The theorems in Goodwin's course are also contentwise close to this. However, there is a discrepancy between the text of the manuscript and that of Goodwin's book. That is why, one may think, that either Goodwin's book was, in fact, not the source book for Marx, or, that along with it, Marx had yet other books at his disposal. According to the searches conducted in the libraries of England , the most probable other source in this context may be : Th. G. Hall, "The Elements of Algebra", 3rd ed. , Cambridge, 1850, §§ 125-127, pp. 149-152, chapter IX, "Ratio, Proportion, Variation and Inequalities".

Sheets (27-38). "B) *Permutations, Combinations and Variations*".

It is a conspectus of materials on combinatorics, at least from two sources, the identity of which could not be established with full certainty. Judging by the terminology used by Marx, the conspectus is in the main based on an English source. The coincidence in the sequence of exposition, modes of proofs, terminology and notations, provides greater ground to assume, that this source was the same book by Hall, chapter XIII, "Permutations and Combinations", pp. 209-214. However, within these notes Marx makes a big insertion (on sheets 31-38) from some other, apparently German, source; for our surmises on which, see below (pp.191-192).

This note begins with the empirical examination of permutations from two, three, four and five letters. Then follows (point 2) the definition of the concept of variation of n elements taken r at

a time ; and in point 3) their number is calculated inductively, for which the symbol nVr is introduced (in Hall it is $^{n}V_{r}$). Then the number of permutations from n elements is obtained as $n\,V\,n$. Then (in point 4), permutations with repetitions are considered. It has been shown that the number of such permutations from n elements, among which we find α magnitudes of "one type", β of another, γ — of a third, is

$$\frac{n\cdots 2\cdot 1}{\alpha\cdots 2\cdot 1\times\beta\cdots 2\cdot 1\times\gamma\cdots 2\cdot 1}$$

(in Marx's writings, the "factorial" sign is not there; however, it is not there even in the text book by Potts, published in 1880).

In point 5) the combinations from n elements taken r at a time, are considered, for the number of which the sign $n\,C\,r$ is introduced, and it is shown that

$$nCr=\frac{nVr}{rVr}=nC(n-r).$$

In the next insertion, containing new points 1)-5), at first the concepts of *element, form or complex* and, *class of forms* have been introduced. Here in the manuscript we read (sheet 31):

Things, which are in a definite order and are to be united into a group in this or that way, are called *elements*, since only the *order* in which they appear, is of importance, and not their magnitude or peculiarity. Each of such union of some elements, is called a *form* or *complex*. A *class of forms* is determined by the *number of its elements,* thus , for example, [the form] 2 3 1 4 5 is of a class *higher* than 2 3 5 4.

Since the properties of the elements are unimportant , so any element is designated by a numeral or a letter of the alphabet, and the problem is raised in this way :

The aim of the investigations into permutations, combinations and variations is to :

a) establish *strict rules for the generation of forms* ;

b) *determine the number of these forms*, without listing their totality.

To solve the first of these problems, the forms are put in *lexicographic* order (i.e., in the alphabetic order, herein the numerals 0, 1, 2...., 9 constitute the alphabet), if the differences of their classes are not considered, and in *arithmetic* [order] — in the contrary case. (By "arithmetic " is meant the arrangement of forms in the ascending order of their classes, i.e., of the number of elements in a form; herein, the forms entering into every class, are arranged within it lexicographically.)

Point 2) is devoted to the solution of both the tasks mentioned above, for ordinary permutations and for permutations with repetitions. Here the symbols like $P\,(1,2,3)$ and $P(a,b,c,c)$ are used and a certain extension of the concept of permutation is spoken of, regarding which Marx writes :

The concept of *permutation* has been widened — however, it coincides with the *well known case of variation* — when from given elements, by all possible means, some *determinate set* of them is to be chosen and the latter is to be placed in all possible orders, for example, for obtaining *from n elements*, all possible arrangements, taking *two at a time.*

In point 3) combinations are considered. Symbols like the following are used :

$$\overset{3}{C}\,(1,2,3,4,5)\,,\ \overset{3}{C}(1,2,3,4,5)^{\,3}\,,\ \overset{4}{C}\,(1,2,2,3,3,3,4,),$$

among which the first designates combinations from five elements, taken three at a time, the second — combinations from the same five elements, taken three at a time, with all possible

repetitions (for example, 111,122,124 and others), the third — combinations from four elements 1,2,3,4 taken 4 at a time, besides the numeral 2 is permitted to be repeated not more than twice, and the numeral 3 — not more than thrice. The conspectus contains : examples of making full lists of all possible combinations of this or that type; the formula for the number of combinations from x elements, taken m at a time; the formula for the number of combinations with repetitions, from n elements, taken m at a time, under the condition that every element may be repeated any number of times, not greater than m ; and notes on this, that the number of such combinations with repetitions is equal to C^{m}_{n+m-1} (in our notation).

In point 4) variations have been examined, again, with repetitions or without. The symbols used are analogous to those used for the combinations. We find notations like:

$$\overset{3}{V}(1,2,3,4,)\ ,\ \overset{4}{V}(1,2,3)^4 .$$

All forms corresponding to the first notation are listed. It is proved that, the number of all possible variations with repetitions from n elements, taken m at a time, is equal to n^m. To put in modern language, the question of formation of the direct product of some (finite) set of elements, has been specially considered and the direct product of the sets ("serieses") : (1, 2,3,), (1, 2) , (1, 2, 3, 4,) have been fully listed . It has been proved , that the number of elements of the direct product is equal to the product of the number of elements "of the factors". It has been noted, in particular, that if all the, m "factors" have the same number of (n) elements, then the number of elements of their direct product, i.e., n^m, coincides with the number of variations with repetitions from n elements taken m at a time, obtained earlier.

The (last) point 5) is devoted to the various modes of forming a given sum by choosing some (of the given) "addenda", which, herein, may or may not be repeated, and by composing lists of the corresponding combinations and variations "of the addenda". An algorithm has been adduced, answering the question of solvability or unsolvability of the problem and permitting easy listing of all its possible solutions in a definite order. Here the following symbolic notations are found :

$$9\overset{3}{C}\ (1,2,3,4,5),\quad 9\overset{3}{C}\ (1,2,3,4,5)^{3},\ 9\overset{3}{V}\ (1,2,3,4,5)\,,9\overset{3}{V}\ (1,2,3,4,5)^{3} .$$

Here 9 is the given sum of 1, 2, 3, 4, 5 — the permissible addenda, whose number must not be greater than three and which (in cases, when 3 stands above the right hand side bracket) may not be repeated more than three times.

In this insertion, there are no English words; contentwise it is close to the ideas wide-spread within the German school of combinatorial analysis (Hindenburg, Eschenbach, Rothe, Kramp and others) of the first half of 19th century. And this provides the basis to assume, that the source of this insertion is, to all appearence, German. However, Marx could have at his disposal only such German sources, as were available in England. The corresponding search in the British Museum and in the other libraries of England has not yet yielded a conclusive result.However, closest resemblance to the text of the insertion contained in manuscript 3933, is observed in the following books:

1. B. Thibaut,"Grundriss der allgemeinen Arithmetik order Analysis zum Gebrauch bei akademischen Vorlesungen "("Outlines of general arithmetic or analysis for use in academic lectures"), Göttingen, 1809. Chapter 2 of this book : "First principles of the studies on combinations" — contains a discussion on *elements, forms, classes of forms,* their lexicographic and arithmetic order and contains the notations

$\overset{4}{C}$ (1, 1 ,1 , 2, 2, 2, 3) , $\overset{3}{C}$ (1, 1 ,1 ,2, 2, 2, 3, 3 ,3, 4, 4, 4, 5, 5, 5), as well as algorithms, similar to those mentioned in the manuscript, for obtaining all possible combinations of this or that type, and calculation of their number(without a preliminary compilation of these very combinations).

2. Fr.W.Spehr, "Vollständiger Lehrbegriff der reinen Kombinationslehre, mit Anwendungen derselben auf Analysis und Wahrscheinlichkeitsrechnung" ("A complete academic conception of the pure studies on combinations, with their application in analysis and calculus of probabilities"), Braunschweig, 1824, 2nd ed. , 1840.

This book is written on the basis of the ideas of Thibaut.The author specially stipulates, the merits of Thibaut's notations and, in essence, considers those very concepts (and in the same order), which were mentioned in connection with Thibaut's book

However, in spite of this quite significant similarity of the material and notations, both the books differ from Marx's text, as regards the style of exposition, as well as in respect of the examples.That is why, the question of the source of the latter [i.e., of Marx's text] may still be considered unsolved.

Having finished this insertion, Marx summed up the noted material on combinatorics , commenting first of all (sheet 37),that :

In order not to confuse these 3:*Permutations, Combinations and Variations*, *Permutations* should be *restricted to Variations of all elements from a given complex of them*; and it would be the best thing to consider *Variation* along with *Permutation* and after that *Combination* [153].

After this, Marx once more briefly listed all those results mentioned earlier, which are connected with the number of permutations, variations and combinations(with repetitions or without). After mentioning the formula for the number of permutations with repetitions: a times the elements of one type, b times that of the other and c times — of a third, of the form :

$$\frac{n(n-1)\cdots 3\cdot 2\cdot 1}{a\cdots 2\cdot 1\times b\cdots 2\cdot 1\times c\cdots 2\cdot 1},$$

Marx makes the following insertion:

[[In the exposition on pp. 31(end) and 32 the *concept of permutation* "is extended" to prepare the "proof" for determinate instances of variation. This is nonsense [154].]]

Sheets 38-68. "*C) The binomial theorem*". These notes were taken from a number of sources, one of them (apparently the same one, to which belongs the insertion on combinatorics, pp. 31-38) could not be established.This part of the note book consists of three sections; Marx indicated them by the Roman numerals I), II) and III).

Section I) (sheets 38-41) carries no heading . It contains notes taken from : MacLaurin's " Treatise of Algebra", §§ 42-47, pp. 38-42. In this case Marx clearly mentions that he is using its 6th edition(London,1796). This edition fully corresponds to the first edition of the book at our disposal .The conspectus begins with the following words of Marx:

I)1) The empirical origin of this theorem is explicitly observable in *MacLaurin's* Algebra; in fact, the latter is the first commentary(reinforced by proofs) on Newton's "*Arithmetica Universalis*", where in application to the most difficult things, results have been brought forth without explanations , often in an inaccessible form, not developed and, without proof.

This entire section of this conspectus is not an extract from MacLaurin, but Marx's own exposition of the noted material, wherein Marx specially notes the circumstance : that MacLaurin draws general conclusions, not on the basis of proofs,but merely with the help of inductive generalisation of observations, related to the expansion

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 .$$

Marx stresses that the character of these observations is heuristic and comments (on the method of searching the coefficient of the subsequent term according to the coefficient of the previous term) (sheet 39) :

In the second term the coefficient = 6. This simple comparison of the second term with the first would only show, that the coefficient of the second term= *the power of the first, since we have* a^6, *it is* $6(a^5b)$ *for the second term.* But if we compare the third term with the second, then the second term= $6(a^5b)$; the third term= $15\,(a^4b^2)$.

Adducing further, as regards the observations, the rule of MacLaurin, suggesting — with the aim of finding out the coefficient of any term according to the coefficient of the previous one — that we " divide the coefficient of the previous term by the index of power of the given term and multiply the quotient obtained by the index of power of a in the same term, increased by one", Marx comments, that it would have been simpler to multiply the quotient obtained, by the index of power of a in the previous term. (The words cited within quotation marks have been borrowed by us from MacLaurin's "Algebra", p.40, where too, they have been put within quotes. Apparently this gives Marx the occassion to think that they do not belong to MacLaurin, but to Newton himself, in so far as here, as elsewhere , whenever MacLaurin's text is found within quotation marks, Marx in his conspectus mentions Newton.In such cases he writes : "*according to Newton*", "*in Newton*", "*i.e., Newton*".)

Concerning the next paragraph, where MacLaurin obtains the general binomial theorem of Newton, by the simple extention of a rule, verified for the expansion of the sixth power of the binomial, to the instance of an arbitrary index of power m, Marx writes:

Now this empirical finding is generalised.

Marx makes an analogons comment in connection with this , that the number of terms of the expansion $(a + b)^m$ herein turns out to be $m + 1$. He writes:

This latter is also, in fact, obtained only by generalising the example $(a + b)^6$.

Lower down, Marx once more stresses this when he "sums up" as under(sheet 40):

2) We saw above, that in MacLaurin (i.e., in Newton) the expansion of the binomial — in connection with the investigation into the coefficients — into the 2nd,3rd, n-th power, gives 2+1, 3+1, generally $n + 1$ terms ; this simple generalisation gives:

$$(x + a)^2 = x^2 + 2ax + a^2 \ , \ (x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 ;$$

$(x + a)^2$ gives 3 terms, $(x + a)^3$ gives 4, and *that is why* $(x + a)^n$ gives $n + 1$ terms.

Stating the general formulation of the Newtonian binomial theorem according to MacLaurin, Marx observes that in it the general term is not written :

.... as given in MacLaurin, *without the general term.*

Going over to the generalised rule for the expansion in power of polynomials Marx writes:

Newton at once applied the *binomial theorem* [[which was still only a generalised empirical expression,obtained, when instead of 6, for instance, m is put]] to the *polynomials.*

In connection with the explanation of this rule in the light of the example of expansion in square of the three terms $a + b + c$, as the binomial $([a + b] + c)$, Marx observes:

It could also have been expanded as $(a + [b + c])^2$.

After this there is an insertion by Marx (sheet 40). Its source is no more the "Treatise of Algebra" of MacLaurin. Here Marx writes :

This, in general, is the *first elementary exposition* according to Newton. To it, in essence, when it *again* appears in the *elementary form* under the rubric of involution * or rise in power, nothing has been added;[it has] since then been only somewhat generalised; suppose, for instance, the expansion for $(x + y)^6$ is to be obtained, then:

$$\begin{array}{l}
x^6 \ldots x^5 \ldots x^4 \ldots x^3 \ldots x^2 \ldots x^1 \ldots x^0 \ (\textit{powers of } x), \\
y^0 \ldots y^1 \ldots y^2 \ldots y^3 \ldots y^4 \ldots y^5 \ldots y^6 \ (\textit{powers of } y), \\
1 \ldots 6 \ldots 15 \ldots 20 \ldots 15 \ldots 6 \ldots 1 \ (\textit{coefficients}), \\
\hline
x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.
\end{array}$$

In general the n-th power of $(a \pm b)$ or $(a \pm b)^n =$

$$= a^n \pm \frac{n}{1} a^{n-1} b + \frac{n}{1} \cdot \frac{n-1}{2} a^{n-2} b^2 \pm \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3} b^3 + \cdots \pm b^n.$$

If the terms of a binomial have *coefficients* like, for example, $(2a + 3b)^4$, then here they may be raised to that power, in which the term containing the given coefficient is raised.

Hence,

$$\begin{array}{c}
16a^4 \ldots 8a^3 \ldots 4a^2 \ldots 2a^1 \ldots (2a)^0 \\
(3b)^0 \ldots 3b^1 \ldots 9b^2 \ldots 27b^3 \ldots 81b^4 \\
1 \ldots 4 \ldots 6 \ldots 4 \ldots 1 \\
\hline
16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4 .
\end{array}$$

Here it is clear that having taken notes from the old "Treatise" of MacLaurin, Marx had to turn to a source closer to his time, in order to draw the conclusion that, since then little has changed in the elementary exposition of the question. Apparently, this source was the book : Hall, "The Elements of Algebra" London, 1840, where at the end of the section on involution there is both the algorithm for raising the binomial in power, mentioned by Marx, and its application to the example $(2a + 3b)^4$.

Marx gave section II. (sheets 41-51) the title: "*II. The binomial theorem for positive and integral indices of power*". This conspectus is from the second of those two (apparently German) sources, to which the conspectus on combinatorics belongs. The greater part of this section is devoted to the *proof* of the binomial theorem, in which, a) the concepts : of "form", "classes of form ", "order of forms", combinations and variations "with a definite sum" , and b) the notations:

$$\overset{n}{V}(1, 2)^n \ , \ \overset{n}{C}(1, 2)^n,$$

*Here is a slip of pen in the manuscript : instead of "involution", there is "evolution". In MacLaurin, the rise in power is called "involution", and the opposite operation — extraction of roots — "evolution". —Ed.

have been used; we have already met with these and others in the notes on combinatorics (see pp.190-192). The new notations : $\overset{1}{C}, \overset{2}{C}, \ldots, \overset{n}{C}$ have also been introduced, for the sum of all possible products, corresponding to all possible combinations from n elements, taken respectively one, two $\cdots n$ at a time. Here, the elements are numbers or alphabetical expressions of ordinary algebra; here the combinations of elements have also been considered as products. The proof begins with a consideration of products of two, three etc. factors, represented as the sum of an arbitrary number of addenda. (In every factor) these addenda are designated by the numerals 1, 2, 3, ... With the help of the formation of "variational forms" — direct products of those sets of numerals, by which the addenda are numbered in the factors — the terms of the product are put in a definite order. An example of such ordering and of the corresponding table of all"variational forms", "realising" the terms of the product : $(a + bx + cx^2)(d + x)(fx - gx^2 + hx^3)$, have been adduced by Marx in full. Latter on the products of some more special types are considered. At first, it is explained (with the help of the construction of "variational forms" from the numerals 1,2 and, their "realizations"), that every product of the type

$$(x + a)(x + b)(x + c) \cdots (x + k)(x + l)$$

is equal to (if n is the number of factors)

$$x^n + \overset{1}{C}x^{n-1} + \overset{2}{C}x^{n-2} + \cdots + \overset{n-2}{C}x^{n-2} + \cdots + \overset{n-1}{C}x + \overset{n}{C}.$$

Then, the assumption that all the secondary terms in the factors are mutually equal, gives the binomial theorem.

The usual mode of constructing the tables of binomial coefficients is later on obtained with the help of multiplication of both the sides of the equality

$$(1 + x)^n = 1 + Ax + Bx^2 + Cx^3 + \cdots + Mx^n, \text{ by } (1 + x).$$

In a latter part of the notes, where we find new notations $n\overset{r}{B}\,(r = 0, 1, \cdots, n)$ for the number of combinations of n items taken r at a time *, i.e., for the binomial coefficients, Marx writes (sheet 48):

C) In further "commercial" elaboration the *powers of both the terms of the binomial*, like a^n, b^n figure as *dignitaries*, and their *coefficients*, like $\frac{n}{1}$, $\frac{n(n-1)}{2}$ etc.as *binomial coefficients.*

In this connection Marx makes the following comment :

[[We note here, that the *functions* with x^n, along with the *derived functions* or *differential coefficients* deduced from them, in the differential calculus, include within themselves only a *part of these binomial coefficients*, for example

$$\frac{d(x^n)}{dx} = nx^{n-1};$$

here the differential coefficient or the first derivative includes within itself the entire *binomial coefficient* $\frac{n}{1}$, since the latter = n,i.e., the numerical value of the fractional coefficient = $\frac{1}{1}$; where the latter appears independently, as already in f'', i.e., in the 2nd differential

*such notations, and the entire stock of formulae connected with them, contained in this section of the manuscript, are there in the book: B.Thibaut, "Grundriss der allgemeinen Arithmetic.......", Göttingen, 1809,pp. 44 and afterwards. — Ed.

coefficient, there only the coefficient derived from the index of power of the function enters into the derivative, but not the numerical fractional coefficient accompanying it. Thus

$$\frac{d^2(x^n)}{dx^2} \quad \text{or} \quad \frac{d(n\,x^{n-1})}{dx} = n(n-1)\,x^{n-2} = f''(x^n)$$

does not include within itself $\frac{1}{1\cdot 2}$; rather it * would be $\frac{1}{1\cdot 2} f''$, as initially wrote Lagrange ; thus, not including $\frac{1}{1\cdot 2}$ in f'' or $\frac{d^2y}{dx^2}\left(\frac{h}{1\cdot 2}\right)$ where the fractional coefficient appears as the denominator of the second term (with) h, as in Taylor's theorem.]]

This (last) part of section II) contains the usual theorems about the properties of binomial expansions (that $n\overset{r}{B} = n\overset{n-r}{B}$, that $n\overset{r}{B} + n\overset{r+1}{B} = (n+1)\overset{r+1}{B}$; that the sum of the binomial coefficients for the binomial of power n is equal to 2^n; that about raising the power of complex numbers according to the binomial theorem, and others). Marx gave section III(sheets 51-68) the title: *"General binomial theorem"*. It consists of three parts : A) B) and ad B). Part A) (sheets 51-57) begins with the following words, related to its content.

That what was obtained for the *binomials with integral and positive indices*, holds good also for the binomials *with negative and fractional indices* [[and the *imaginaries*, from which *here we digressed initially*]]. But *in these cases the series becomes infinite.*

The source of this part of this conspectus, could not be finally established. Apparently, here, again some English text book on algebra has been used.

Extention of the binomial theorem in the case of the fractional and negative indices of power, is carried out with the help of the method of indeterminate coefficients, to which at first the point 2) of Marx's conspectus was devoted. Here the proof of univocality of the notion of a function in the form of a powered series coincides almost word for word (right up to a complete identity in notations), with the one found in the book : Hall, "The Elements of Algebra", London, 1850,ch.VIII "The binomial theorem", p.211.

Point 3a) is devoted to the application of this method when $n = \frac{1}{2}$ (In Hall's book this example is found in p.223.)

Namely, it is assumed that

$$\sqrt{1+x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$$

Raising both the sides of this equality to their square, given, then after, the possibility of obtaining a system of equations, determining the coefficients $A, B, C, \cdots$, it is consequently observed that they coincide exactly with those which were obtained when the binomial theorem was extended to the case of $n = \frac{1}{2}$. Having noted, that (sheet 52) :

thus this method may be applied to the binomial with *negative indices*,

Marx went over in his conspectus (point 3b)), to the proof of the theorem in the general case of fractional or negative (integral and fractional) n. Herein at first it is assumed that

$$(1+x)^n = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + etc., \qquad (1)$$

* That is, the expression $\frac{n(n-1)}{1\cdot 2}\,x^{n-2}$. —Ed.

where the coefficients depend only on n,but do not depend on x; then in this equality x is substituted by $x+y$, which gives

$$[1+(x+y)]^n = A + B(x+y) + C(x+y)^2 + D(x+y)^3 + E(x+y)^4 + F(x+y)^5 + etc. \qquad (2)$$

In the right hand part the binomial $(x+y)$ is found only in integral positive powers, for which the binomial theorem is already proved. Using this, the author of the manual from which Marx took notes, permits himself (though herein the coefficients take the form of infinite serieses) to expand the right hand side of the equality (2) according to the powers of y. Then having represented the left hand side in the form of $[(1+x)+y]^n$, he obtains the possibility of using the method of indeterminate coefficients, to show, that all the indeterminate coefficients, A, B, C, D, $\cdots$ are expressed through the second coefficient (through B) exactly in the same way as it happens in the binomial theorem for the integral and positive index n. Finally, with the help of the same method of indeterminate coefficients it is proved that, just as for the fractional, so also for the negative index n, the equality $B = n$ holds good.

Part (A) comes to an end with the application of the binomial theorem to the extraction of roots of numbers and, to the modes of hastening the convergence of the serieses obtained therein. Thus, with this aim $\sqrt{2}$ is represented as $\frac{1}{5}\sqrt{50}$, i.e., as $\frac{7}{5}\left(1+\frac{1}{49}\right)^{\frac{1}{2}}$. Since here all the ordinary fractions are substituted by decimal fractions, in his notes Marx makes an insertion (placed within a box, in sheets 56-57), devoted to the theory of decimal fractions ; contentwise and notationwise it is very close to §§ 38-43 (pp. 27-29) and, later on, to § 107 (pp. 72-73) of the book: H. Goodwin,"An Elementary Course of Mathematics", Cambridge, 4th ed., 1853.

Regarding the question of the source of section III, it is essential to note that this section, devoted to Newton's binomial theorem, happens to be a general systematisation of a large amount of material culled from different books. It is clear that, Marx did not accidentally 1) begin with a non-strict deduction of this theorem (from MacLaurin's "Algebra"), reduced to a simple empirical generalisation of the observations related to the cases when $n = 2,3,\ldots,6$ (where n is the index of power of the binomial); 2) locate after that, a proof, in which it is obtained conversely, as a particular instance of a more general theorem (about the product of binomials of the type $(x+a_i)$, $1 \leq i \leq n$) and which thus, convincingly reveals the reason of its validity ; 3) later on go over to the various modes of extending the theorem to the instances of fractional and negative n, encountering therein the problematique connected with the infinite powered serieses and the modes of calculating their "sum"; and 4) in spite of his disaffection to arithmetic — about which Marx said that he "never felt at home" with it — having met with questions of computational mathematics, he did not resent the labour of searching for the materials, connected with the algorithms for operating with the decimal fractions, and made a special insertion in his notes, regarding these algorithms.

It stands to reason, all by itself, that the questions of convergence, and of hastening the convergence, of an infinite numerical series, are also to be met with in Marx's conspectus, in connection with the generalisation of the binomial theorem.

With the help of this theorem approximate roots of any kind may be extracted. But the series must be so constructed, that it *converged.* It means, that the series, which the binomial theorem gives for the unknown root, converges, i.e., [the sum] of the first term + the 2nd, 3rd etc. successive terms of the series constantly approximates to the root sought, and can be brought as close as you wish to this quantity, such that taken in absolute magnitude the mistake becomes smaller than any given positive quantity, when a sufficient number of terms of the series are taken into consideration.

However Marx's conspectus contains no criterion for convergence. This, in spite of the fact that he examined a large number of different text books on algebra, and "complete" courses of mathematics: English, German and French, which could be obtained in England in the 60-70s of the last century. However, he did not find a text book, where questions of this sort were discussed. Meanwhile, Marx used such manuals, as the highly specialised "Complément" of Lacroix to his course on algebra. It is this manual that Marx used for the next point of his manuscript.

Part B) (sheets 57-63) consists of five points. The first among them is devoted to Euler's generalisation of the binomial theorem for fractional and negative indices of power. Its source is Lacroix's book :"Complément des éléments d'algèbre", 4th ed., Paris, 1817, § 79, pp. 159-163.

It starts with the words (sheet 57) :

"1) *Euler gave* the following *general proof of the binomial theorem*".

As is well known, this proof is based on this : that if the series

$$1+mz+\frac{m(m-1)}{1\cdot 2}z^2+\text{etc.}$$

is considered as a new function of m, then the equality

$$f(m)\cdot f(n)=f(m+n)$$

will express its characteristic property. In Euler the proof of this property is restricted to the following words : "Formation of the terms of this product [of the series for $f(m)$ multiplied by the series for $f(n)$]must remain the same, irrespective of what sort of numbers are represented by the letters m and n : integrals or, as you wish ". On this Lacroix comments, that such a proof did not satisfy many mathematicians. In the 7th (posthumous) edition of 1863, after this (pp 151-153) an inductive proof was adduced ; it says that : the coefficients of the product of the serieses for $f(m)$ and $f(n)$ are actually equal to the coefficients of the series for $f(m+n)$. In the 4th ed., used by Marx, on p.160, instead of this proof a short explanation of Euler's idea is given, which reads in Marx's conspectus, as under.

Having designated through P the product of the serieses

$$f(m)=1+mz+\frac{m(m-1)}{1\cdot 2}z^2+\dots,\ f(n)=1+nz+\frac{n(n-1)}{1\cdot 2}z^2+\dots,$$

Marx writes (sheet 57) :

This product $=P$. Expanded according to the powers of z, it may be represented by the series

$$P=1+Az+Bz^2+Cz^3+\cdots$$

The coefficient A, B, etc., of any term of this series depends upon the mode of *joining the terms of both the factors*, from the first to that, which contains that very power of z, since these are the only terms, which participate in the formation of the terms of the product under consideration.*The mode of joining these terms does not change*, independently of *this or that value of m and n* ; and that is why, if [the terms of the product] are known in one case, where m and n are *determinate numbers*, then they are unknown in *all the other.*

Having noted, that from this follows the necessary coincidence of the coefficients of the series for the product of $f(m)$ and $f(n)$, with the coefficients of the series for $f(m+n)$, Marx completed the exposition of Euler's proof according to Lacroix. Then Marx made a summary of this entire material, by enunciating it once more, in a shorter form (sheets 58-60). He has put this summary within square brackets.

The following points 2) and 3) (sheet 60) of the conspectus are devoted to the instances of irrational and imaginary indices of power. The source of these points could not be established (these instances have not been considered in Lacroix). However, this source belongs entirely at the level of 18th century mathematics. Thus the proof of $(1+z)^m = f(m)$ when m is an imaginary number (and where $f(m)$ is the binomial series $1+\frac{m}{1}z+\frac{m(m-1)}{1\cdot 2}z^2+\ldots$), is based in it upon the following reasoning :

when m is integral and positive we have

$$f(m)^m = \underbrace{f(m)\cdot f(m)\cdots f(m)}_{m\ times} = \underbrace{f(m+m+\cdots+m)}_{m\ times}$$

i.e.,
$$f(m)^m = f(m^2)\,.$$

According to the author, imaginary numbers are mere imaginary objects. That is why there is nothing to prevent us from imagining that, such a rule holds good also for them . But then we shall have (since for negative m the theorem is already proved)

$$f(\sqrt{-1})^{\sqrt{-1}} = f(\sqrt{-1}^{\,2}) = f(-1) = (1+z)^{-1} = (1+z)^{\sqrt{-1}\cdot\sqrt{-1}}\,,$$
$$\text{i.e } f(\sqrt{-1})^{\sqrt{-1}} = (1+z)^{\sqrt{-1}\cdot\sqrt{-1}}\,.$$

By , equally formally, raising the power of both the sides of the equality by $\frac{1}{\sqrt{-1}}$, we shall get

$$f(\sqrt{-1})^{\frac{\sqrt{-1}}{\sqrt{-1}}} = (1+z)^{\frac{\sqrt{-1}\cdot\sqrt{-1}}{\sqrt{-1}}} = (1+z)^{\sqrt{-1}}\,,$$

or

$$(1+z)^{\sqrt{-1}} = f(\sqrt{-1}),$$

i.e., the theorm is valid for $m=\sqrt{-1}$. Here Marx abruptly stopped taking notes. Apparently, he did not think that it was necessary to take down this kind of a "proof", further.

Marx gave his point 4)(sheets 60-62) the heading :" 4) *Transformation of the binomial series for fractional or negative indices of power*". It is devoted to the elucidation of the types of coefficients of an expansion and, to the calculations simplyfying their transformation.

In point 5) (sheet 63) entitled :"5) *Lambert's formula for approximation*", Lambert's method, of calculating roots of numbers with the help of successive approximation, is enunciated. An analogous exposition is found in the courses of :Hind ,"The Elements of Algebra", 4th ed., 1839,§259, pp. 230-231 and, Hall, "The Elements of Algebra", 1840, §165, pp. 225-226. However, it appears that none of these courses served as the source of Marx's conspectus here : Marx's notations and calculations are different.

The last part of section III under the title : "*Ad. B) Something elementary from Euler on binomial theorem* "(sheets 63-68) consists of two points. In point 1) Marx took notes from §§358,359(pp. 117-119), chapter XI of Euler's "Elements of Algebra", devoted to the algorithm of calculating the coefficients of the expansion for $(a+b)^n$, when n is integral and positive. Here Marx also reproduces the corresponding translator's foot-note (on pp. 118-119) from the 1822 English edition of the book : "Elements of Algebra by Leonhard Euler, translated from the French with two notes of M. Bernoulli... and the additions of M.de la Grange. 3rd edition, by the Rev. John Hewlett..., to which is prefixed a memoir of the life and character of Euler, by the late Francis Horner, Esq., M.P., London, 1822". That is why, we may think that it was this edition which Marx had at his disposal.

The translator's foot-note on pp. 118-119 is related to Euler's mode of representing the coefficient of $a^p b^{n-p}$ in the form of $\frac{n!}{p!(n-p)!}$, based on the fact that the term $a^p b^{n-p}$ contains n letters, out of which $n!$ permutations may be made, such that, all permutations related only to the letters a (altogether there are p of these letters) or only to the letters b (of them altogether there are $n-p$), give one and the same result. On this score the translator comments, that it is better to use the formula for the number of combinations from n elements taken p at a time, i..e., $\frac{n(n-1)\ldots(n-p+1)}{p!}$; then he illustrated the application of this formula for calculating the binomial coefficients for n equal to 7 and 4.

Having taken down this comment and then representing the expansion for $(a+b)^n$ in the form: $a^n + na^{n-1}\frac{b}{1} + n(n-1)a^{n-2}\frac{b^2}{1 \cdot 2} + n(n-1)(n-2)a^{n-3}\frac{b^3}{1 \cdot 2 \cdot 3}$ + etc., Marx commented (sheet 66):

And so after this in *Taylor's theorem* there appears as the derived functions of x^n:

$$nx^{n-1},\ n(n-1)x^{n-2},\ n(n-1)(n-2)x^{n-3}, \ldots,$$

meanwhile $\frac{h}{1}, \frac{h^2}{1 \cdot 2}, \frac{h^3}{1 \cdot 2 \cdot 3}$ etc. figure instead of the 2nd term h, i.e., the denominators, to be more precise, the fractions $\frac{1}{1}, \frac{1}{1 \cdot 2}, \frac{1}{1 \cdot 2 \cdot 3}$ etc. appear as the *numerical coefficients of* [the powers of] h.

Point 2) has the title : " 2) *Irrational powers* ". It contains notes taken from §§361-369 of chapter XII (" On representing the irrational powers by infinite serieses") and §§370,373, in part §375 of chapter XIII (" On the expansion of negative powers") of the same book of Euler. Marx sub-divided this point, in its turn, into points α), β) and γ).

In point α), $\sqrt{a+b}$ is expanded according to Newton's binomial theorem (§364) in the form of

$$\sqrt{a+b} = \sqrt{a} + \frac{1}{2}b\frac{\sqrt{a}}{a} - \frac{1}{2}\cdot\frac{1}{4}b^2\frac{\sqrt{a}}{a^2} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}b^3\frac{\sqrt{a}}{a^3} - \frac{1}{2}\frac{1}{4}\cdot\frac{3}{6}\cdot\frac{5}{8}b^4\frac{\sqrt{a}}{a^4} + \text{etc.} \qquad (1)$$

In point β) is considered the case, when a is the square of a rational number c, i.e., when in the right hand side of the equality (1) "there is no sign of root": all terms of the series are rational (§365).Euler's example, wherein the first two terms of this series are used for approximate calculation of $\sqrt{6}$ (by the method of successive approximation), was noted by Marx in all its details, right upto the obtaining, for this root, of the approximate value $\frac{4801}{1960}$, the square of which is greater than the number 6 only by $\frac{1}{3841600}$ (§366). This note also contains an extension of this method for approximate calculation of the roots of any power n, where n is integral and positive, and a reference — marked by a half box ⌐ — to the general method of approximate calculation of roots, published by Halley in the "*Philosophical Transactions*" of 1694. (This reference is found in the foot-note, with which Euler brought his chapter XII to an end.)

Marx completed his point β) (sheet 68, Marx's p. 48) with the following comment :

Stating all this in the "Elements of Algebra", Euler does not specially take up the expansion of $(a+b)^{\frac{m}{n}}$ without which he can manage, since $(a+b)^{\frac{m}{n}} = \sqrt[n]{(a+b)^m}$, here he

expands $(a+b)^m$ and $\sqrt[n]{(a+b)}$; on the contrary, as we saw, his theoretical deduction provides a special proof for the case, when $(a+b)^m$ or $f(m) = f\left(\frac{p}{q}\right)$.

Point γ) is related to chapter XIII of Euler's book, i.e., to the binomial theorem for the negative index of power. Here the coefficients of the expansion for $(a+b)^{-3}$ are computed (§373) and, a comment is made to the effect, that the multiplication of the right hand side of the equality

$$\frac{1}{(a+b)^3} = \frac{1}{a^3} - 3\frac{b}{a^4} + 6\frac{b^2}{a^5} - 10\frac{b^3}{a^6} + 15\frac{b^4}{a^7} - 21\frac{b^5}{a^8} + 28\frac{b^6}{a^9}\dots \qquad (2)$$

by $(a+b)^3$ actually gives the result 1 (as it should be, if equality (2) is valid, since $\frac{1}{(a+b)^3}\cdot(a+b)^3 = 1$).

With this, the section of manuscript 3933 devoted to the binomial theorem comes to its end . On the last sheet of this section, Marx wrote 48 instead of 68; apparentlay it is a case of slip of pen. That is why the numeration of the sheets that follow differ from that of Marx by + 20.

Sheet 69 (Marx's sheet 49) begins with the following sentences, contentwise related only to one of the later pages (sheet 73) of the text:

[Pure] *imaginary quantities. 2 imaginary quantities, being mutually multiplied,* give *a real quantity;* but *an imaginary , multiplied by a real one, always gives an imaginary quantity.*

The entire remaining text of manuscript 3933 (sheets 68-92) is a note taken from MacLaurin's "Treatise of Algebra". Marx gave it the title : "*From MacLaurin's Algebra*".

The conspectus begins with chapter XIV of the aforementioned book (from the previous chapters, devoted to the algebraic operations with polynomials and to the solution of equations of first and second degree, Marx took notes only of the aforementioned §§42-47, related to the binomial theorem). Point 1) of this note bears the title "Of surds", i.e., the same as that of this entire chapter of MacLaurin.These notes have been taken from §§92-109 (pp. 94-104) and §117(p.109) (in Marx's notes — pp.49-51, sheets 69-71) . Here the issues are : the concepts of commensurability and incommensurability; Euclid's algorithm for searching the greatest common measure; divisibility of numbers; that the quadratic or cubic root of a whole number can only be a whole or irrational number; commensurability in power and complex surds; and the existence of such particular instances, in which the products of complex surds are rational.

MacLaurin based the proof, that if the algorithm for seeking the greatest common measure of two magnitudes a and b ($b < a$)does not come to an end, then these magnitudes are incommensurable, upon the well known proposition 1 of book X of Euclid's "Elements" . (It is well known, that this proposition — according to which, if from a magnitude more than half is subtracted, from the remainder more than its half etc., then upon continuing this process, we may obtain as remainder a magnitude, smaller than any magnitude given in advance — lies at the base of the method of exhaustion, the ancient fore-runner of the method of limits).

Namely, MacLaurin shows at first, that if $a = bm + c$ (where $0 < c < b$ and, m is a whole number), then $bm > c$,owing to which

$\frac{1}{2}a = \frac{1}{2}bm + \frac{1}{2}c > \frac{1}{2}c + \frac{1}{2}c = c$, i.e., that $c < \frac{1}{2}a$ or, in other words, that the successive remainders in Euclid's algorithm so diminish, that each of them turns out to be smaller than half

of the *pre-previous* one. Hence MacLaurin concludes , that the remnants become smaller than any magnitude given in advance.

In this context , there is no reference to Euclid in MacLaurin. Marx gives this reference in his notes. He writes (sheet 69) :

Subtracting from any magnitude, more than its half, from the rest more than its half etc., we shall arrive at a remainder < any assignable magnitude (Euclid).

In connection with what he read, Marx made the comment (which he placed within a box), wherein he stressed that, the inequality $c < \frac{1}{2}a$ follows from the fact, that $c < b$ and $m \geq 1$. In addition, Marx adduced a further argument, in defence of this, that if b is not the common measure for a and b, then the inequality $c < \frac{1}{2}a$ is strict:

But had it been the case that $c = \frac{a}{2}$, then $a - mb = \frac{a}{2}, a - \frac{a}{2} = mb$; hence , $\frac{a}{2} = mb$ or $a = 2\,mb$, which contradicts what been assumed.

Marx did not take notes from §§110-116, related to formal transformations of the expressions containing radicals. Having taken notes from §117 — in which the problem raised is : to seek for a given complex expression, containing radicals, an expression of the same type, which being multiplied by the given, gives a product, not containing radicals — Marx writes in brackets : "*continuation later*".However, this continuation is not to be found in the later part of this note : Marx did not take notes from the paragraphs containing the solution of the problem raised, for certain particular instances.

Then follows the notes from the first five chapters of MacLaurin's "Algebra", under the title:"*Theory of equations according to MacLaurin*". The contents of these chapters and of the corresponding sections of Marx's conspectus are as follows.

The issues in chapter I are :"obtaining" equations of higher degrees by multiplying some equations of the first two degrees; solution of equations of higher orders as an inverse problem consisting of representing them in the form of products of the equations of the first two degrees; and "the number of roots, which an equation of any power may have".

Chapter II is devoted to the question of connection of the number of changes in the signs of coefficients of an equation with the number of its positive and negative(real) roots (*Descartes' rule of signs*) and of the other dependencies between its roots and coefficients .

Chapter III is about the transformation of equations and the removal of their intermediate terms.

In chapter IV the search for the multiple roots of an equation, and in chapter V the boundaries of the roots of an equation, have been discussed.

In this note book the conspectus of chapter V does not come to an end. It continues in a new note book (see below manuscript 3934).

Marx took notes from chapter I (§§1-11,pp. 131-138) in full (sheets 71-73, pp. 51-53 in Marx's numeration). MacLaurin begins it (§2) with the construction of an equation according to its roots, noting therein, that if all the roots are equal, then the problem is reduced to what has already been

considered : to *involution*, i.e., to raising the binomial in power; the opposite task of seeking the roots of an equation obtained, also turns out to be a case of what has already been considered : of *evolution*, i.e., of extracting the root. However, if the roots of an equation are equal, then we find ourselves facing a new inverse task: solution of equations. In his notes Marx puts it this way (sheet 71,Marx's p.51) :

Or else, if the *equations multiplied are different*, the equations generated thereby are not powers [of binomials]; that is why, *their expansion into the simple equation*, from which they arose, is an operation *different from extraction of roots* and, it has its own name : *resolution of equations* (resolution expresses the same thing as *solution*).

In this connection Marx notes below (sheet 72, Marx's p. 52) within square brackets:

[[Here it should be noted , that in application to *equations* we choose a path, opposite to that along which we went, *expanding the binomial theorem* (and the theory of combinations, upon which it is based) ; in the later case we start with the binomials with different second terms, in order to go over[to those] , whose both terms are same; here we consider at first (sub2) a given equation, multiplying which , by itself, we get an equation of higher order; and after that we go over to the equations[corresponding to binomials], which have the same unknown, but *different* second terms. Since the initial equations are so written that all the terms of them are on one side, and on the other side — 0, so here also , when equations of higher powers emerge, the issue is, as before, only about the multiplication of the binomials 2 by 2 , 3 by 3 etc.]]

By "multiplication of the binomials" here he means : representation of the product $(x-a_1)(x-a_2)\cdots(x-a_n)$, where, $a_1, a_2, \cdots, a_n$, are the roots of the unknown equation, in the form of a polynomial, expanded according to the powers of the unknown x.

The conspectus of chapter II (MacLaurin, part II, ch. II, §§ 12- 21,pp.139-147;sheets 73-76, Marx's pp. 53-56) contains three observations of Marx, of which the second is of special interest, since from it, it is clear, that Marx had some information about the works of Cauchy. Unfortunately, the source of this information could not be established. [This source could be : *Moigno F.*, Leçons de calcul différentiel et de calcul intégral, rédigées d'aprés les méthodes et les ouvrages publies ou inédits de Mr. L.A.Cauchy. 2 vol., Paris, 1840 et 1844. (see, PV, 66 and note [55]).—Tr.]

The first observation is related to §§12-13,which MacLaurin begins with the consideration of the "product of any number of simple equations", in order to thus proceed to the conclusion that : 1) the power of the equation thus obtained is equal to the number of its roots, 2) the number of terms of the equation is greater than its power by one and, 3) the coefficients of these terms are symmetric functions of the roots of the multiplied "simple" equations (sums of the products of these roots taken one, two etc. at time, with the signs + or - depending upon the evenness of the number of the factors of the products added) — then to extend it to any equation of higher power, without dwelling upon the question of legitimacy of such extention.

In this connection Marx writes (sheets 73-74,) Marx's pp. 53-54):

[[These two rules, as well as the formation of coefficients and the change of signs, in so far as all these are deduced from the multiplication of $(x-a)=0$, $(x-b)=0$, $(x-c)=0$, $(x-d)=0$ etc, i.e. from the *multiplication of the binomials* $(x-a)$, $(x-b)$, $(x-c)$, $(x-d)$ etc., are nothing new in comparison to the developments in the binomial theorem, and correspondingly, to that in the theory of combinations.]]

Heaving written, after this, the expressions for the coefficients of an equation through its roots, Marx made the second comment (sheet 74,Marx's p. 54) :

[[Thus emerge the symmetric equations. As many of these can be constructed, as one wishes. But from this, it still by no means follows, that every given equation must have a root, as MacLaurin, evidently, would have it. The proof later on proposed by Cauchy, seems doubtful.]]

Marx's third observation (sheet 76, Marx's p. 56) is related to the Cartesian rule of signs, connecting the number of changes of signs in the successive coefficients of an equation (with the independent terms other than zero) with the number of positive roots of the equation.

In §19(pp 144-145) MacLaurin formulates this rule as follows (without stopping to consider the "impossible" roots of an equation, "positive or negative", taking them as understood): "The number of positive roots of any equation is equal to the number of changes of signs of its term from + to – or from – to + ; the remaining roots are negative".

However, he proves the validity of this assertion only for the quadratic and the cubic equations, with the help of an examination of the possible particular instances (for example, when the equation has the roots $a, b, -c$ $(a, b, c > 0)$ and $a+b>c$ or $a+b<c$) and by operating upon the corresponding inequalities. Herein the equations are so considered, that as if all their roots could only be real. In spite of the absence of any kind of reference to the mode of generalising these arguments, MacLaurin concludes his arguments with the comment (p. 147) that, "the same mode of argument may be extended to the equations of higher degree, and the rule, mentioned in §19 may be applied to all types of equations" .

In his comments , placed not only within square brackets, but also within a box, Marx reflected upon the idea of a general proof of Descartes' rule (formulated without the assumption of "positive" or "negative" complex numbers) in the light of an example, which may be generalised without difficulty. He wrote:

Suppose $f(x)=0$, where $f(x)$ is a [complete] polynomial ; let its signs be

$$+ \quad - \quad + \quad + \quad - \quad - \quad - \quad + \quad + \quad - \quad + \qquad (1)$$

[A polynomial $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ is called "complete", if all its coefficients are different from zero. We shall be able to call the equation $f(x)=0$ "complete", if $f(x)$ is a complete polynomial.]

Let us introduce into the equation a new root m ; then it will be necessary to multiply $f(x)$ by $(x-m)$, and if we write down only the signs obtainable when multiplied [by $-m$], then we shall have (2), if m is positive ; hence $f(x)\cdot(x-m)$:

$$
\text{I)}\quad
\begin{array}{cccccccccccc}
+ & - & + & + & - & - & - & + & + & - & + & \\
 & - & + & - & - & + & + & + & - & - & + & - \\
\hline
+ & - & + & \pm & - & \pm & \pm & + & \pm & - & + & -
\end{array}
\quad
\begin{array}{c}(1)\\(2)\\ \\ \end{array}
$$

If m is negative i.e. $f(x)$ is to be multiplied by $(x+m)$, then (3)

$$
\text{II)}\quad
\begin{array}{cccccccccccc}
+ & - & + & + & - & - & - & + & + & - & + & \\
 & + & - & + & + & - & - & - & + & + & - & + \\
\hline
+ & \pm & \pm & + & \pm & - & - & \pm & + & \pm & \pm & +
\end{array}
\quad
\begin{array}{c}(1)\\(3)\\ \\ \end{array}
$$

Comparison of the results in I) and II) with (1) gives

$$
\begin{array}{cccccccccccc}
+ & - & + & + & - & - & - & + & + & - & + & \\
+ & - & + & \pm & - & \pm & \pm & + & \pm & - & + & -
\end{array}
\quad (\mathrm{I})
$$

Depending upon which of the two signs + or − will take the place of the equivocal sign ±, we shall get, if we put for ± here [everywhere] + :

$$
\begin{array}{cccccccccccc}
+ & - & + & + & - & - & - & + & + & - & + & \\
+ & - & + & + & - & + & + & + & + & - & + & -
\end{array}
\quad
\begin{array}{c}(1)\\(\mathrm{I})\end{array}
$$

If we put [everywhere] − :

$$
\begin{array}{cccccccccccc}
+ & - & + & - & - & - & - & + & - & - & + & -,
\end{array}
$$

in both the cases there will be *one change* of sign more sub I) and also in comparison sub II) [at least] *one more continuation of the same sign.*

a) Hence, every *additional positive root* entails, at least 1 *additional change of sign*, and every additional negative root — at least 1 *additional continuation of the same sign.*

b) *That is why, the number of positive roots of an equation can not be more than the number of changes of sign, and the number of negative roots not more than the number of continuations of sign.*

c) *If all the roots are real, then the number of positive roots= the number of changes in sign*, and the number of negative roots= the number of continuation of sign (constancy of sign).

Within a box, we read in Marx's hand, (sheet 76, bottom):

[[What has been put within [ ], in this note-book, beginning with p .49,under the title"From MacLaurin's Algebra", does not belong to MacLaurin.]]

This observation, placed inside a box, is clearly of cursory and provisional nature. In particular, not every thing in it has been stipulated in full. However, it is clear that, the smallest number of changes in sign as a result of the addition of a positive root (i.e., in instance I) will take place, when we shall substitute all "equivocal" ± signs everywhere by one and the same sign plus or minus*. That is why, even if under such substitutions the number of changes of sign is increased by one as a result, then it is increased at least by one, also in the other cases. Thus the

* In the cases, when in some of the places occupied by the signs ±, zeroes are found, the number of changes in sign will not be less, than in the cases when all ± signs are changed only by + or only by − . — Ed.

verification conducted by Marx, is quite sufficient. Since the number of negative roots of the equation $f(x) = 0$ is equal to the number of positive roots of the equation $f(-x) = 0$ and, if it is complete, then every constancy of sign in $f(x)$ mutually univocally corresponds to the changes of sign in $f(-x)$, and here Marx studies only the complete equations, so the observation within the box is fully valid, also in application to the negative roots, though, apparently, it has been substantiated directly (in a way analogous to what has been done for the positive roots.)

In the next page 57 (sheet 77) Marx naturally returns once more to the last point of his observation within the box, in connection with the notes of the rule for the transformation of an equation, whose roots differ only in signs, from the roots of a given equation (MacLaurin illustrates these roots in the light of the equation

$$x^4 - x^3 - 19x^2 + 49x - 30 = 0, \qquad (1)$$

transformable into

$$x^4 + x^3 - 19x^2 - 49x - 30 = 0; \qquad (2)$$

see, ch. III, §23, p.148). The text is marked on one side by the sign E . Here he writes:

[[It is very simply explained from (point c) placed within [ ] in the previous page. Roots of (1) were + 1, + 2, + 3, –5 ; the three positive roots correspond to three changes in sign in (1), from the first term to the 2nd, from the 3rd to the 4th, and from 4th to 5th. *To the one negative root –5* , corresponds one constancy of the sign, namely, in the 2nd and the 3rd terms :$(- x^3 - 19x^2)$.

Changing the signs into their opposites we get the *roots* –1, –2, –3, 5; that is why, 3 constancy of signs ; one : $(+ x^4 + x^3)$, and 2, namely : $(- 19x^2 - 49x - 30)$; meanwhile for one positive root 5 one change of sign, from the 2nd term to the 3rd : $(+ x^3 - 19x^{x^2})$.]]

Chapter III (§§23-32, pp.138-161) contains further rules : for transforming equations, permitting increase (and correspondingly decrease) in all the roots of an equation by one and the same number e ; "for freeing" an equation of the 2nd (generally of any intermediate) term, by multiplying all its roots by a given number, by changing it into an equivalent equation, with the coefficient of the old term, equal to one ; and "for freeing" an equation of fractional coefficients, by changing it into such an equation, whose roots are opposite of those of the given. Marx noted (sheets 78-83, Marx's pp. 58-63) all these in full. It does not contain any comment of Marx (which is not purely computational in character). The matter stands otherwise with chapter IV. Marx's interest in the algebraic origins of differential calculus prompted him to pay special attention to this chapter devoted to the algorithms for searching the multiple roots of an equation. The conspectus of this chapter contains a number of Marx's critical comments. Most of them are comments in the margins. Such, for instance, are the following :

I. MacLaurin reduced the question of searching the multiple roots of an equation, ***not equal to*** zero, into the question of searching the multiple roots of another equation , *equal to zero,* for this the given equation is transformed, by substituting in it, $y + e$ for x. Herein he explains that, in order to write the last (free) term of the transformed equation, it is enough to substitute e for x in the initial equation. Having noted this Marx adds (sheet 84, Marx's p.64) :

[[i.e., to make e a root of x, which MacLaurin does not say.]]

II. MacLaurin explains the rule for obtaining the other terms of the transformed equation from its free term, at first only in the light of some examples (without a general formulation) (§34, pp.163-164). After that he writes, that "its proof is easy to generalise" with the help of the binomial

theorem. Putting the quotation from MacLaurin (within quotation marks) a· · indicating the page : "p.164", Marx comments (sheet 84, Marx's p. 64) :

[[In reality here we don't have any kind of proof, only a fact is stated. It is observed when correct computational operations are performed.]]

III. From the following extract from his conspectus of (§36,p. 165 of MacLaurin), the character of this sort of comments of Marx becomes especially clear. Here we reproduce it in full (sheet 85, Marx's p. 68) :

4) Let us assume now, that 2 values of x are equal to each other and also to e. [[A very helpless expression for the fact, that x has 2 values e.]] then the 2 values of y will be $= 0$ in the transformed equation, since $y = x - e$ [[it should have been noticed already as the equation in e turned out to be identical with the initial one in x. From the very beginning it is said that, since $x = y + e$ (or what is the same, $y = x - e$), then x may be equal to e, only if $y = 0$. That is why, if the initial * equation has 2 values e for x, then 2 times $y = 0$.]]

Two longer observations of Marx are of a different character. They come directly, one after the other, on sheet 87 (and end up on sheet 88, Marx's 68, at the topmost line; in the photocopy of sheet 87, the numbering in Marx's hand is not visible). Here first of all Marx expresses his bewilderment at MacLaurin's failure to notice the connection between the method of seeking the multiple roots of an equation and differentiation. The text is marked by E sign on one side. He writes :

[[Most astonishing is the fact that MacLaurin, who discoverd *this method of searching the equal roots of an equation* and applied precisely the same method in differential calculus for expanding *in series* the function $f(x)$, given in a general form, never even mentioned , that here the *rule for* successive *deduction* of derived equations *has been algebraically proved*, so that if the original equation is ;

I) $x^4 - px^3 + qx^2 - rx + s = 0 = f(x)$,

[then]

II) $4x^3 - 3px^2 + 2qx - r = 0 = f'(x)$;

III) $6x^2 - 3px + q = 0 = f''(x)$;

actually $12x^2 - 6px + 2q$ directly divided by 2, gives $6x^2 - 3px + q$;

IV) $4x - p = 0 = f'''(x)$;

actually $12x - 3p$, divided by 3, gives $4x - p$;

V) $4(x^0) = 4$ [since $x^0 = 1$] $= f^{IV}(x)$.

This last one is no more an equation, but still it is the last product of that differentiation, thorough which these equations were algebraically deduced.]]

Marx's second comment contains a criticism of a suspected mistake of MacLaurin.

The issue is this : it is true that at first MacLaurin did not use general expressions for polynomials, and only in the light of some examples explained his idea, that if the equation

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_{n-2} y^2 + a_{n-1} y + a_n = 0$$

has two roots equal to zero (MacLaurin wrote "two values" of y, "equal to nothing"), then in its left hand side the polynomial must have as factors $(y - 0)$ and $(y - 0)$, i.e., must be of the form

* Here is a slip of pen in the manuscript : instead of "initial", here he wrote "transformed". — ED.

My^2 (where M is a polynomial), hence it follows that the coefficients a_{n-1} and a_n must be equal to zero.

However, in MacLaurin's "Algebra" the way in which this idea has been expressed is such, that it is not always clear to the reader, as to what, namely, the author wishes to say.

In application to the equation $x^3 - px^2 + qx - r = 0$, upon having the multiple root $x = e$ (i.e., when the substitution $x = y + e$ is made) and getting it transformed into an equation with the multiple root $y = 0.$, MacLaurin writes :

"In so far as we assume that $x = e$, the last term of the transformed equation, i.e., $e^3 - pe^2 + qe - r$, vanishes. And since the two values of y vanish, the penultimate term, i.e., $3e^2y - 2pey + qy$ will vanish at the same time, so that $3e^2 - 2pe + q = 0$ "(p. 166).

However, the words " two values of y vanish" have no exact meaning. If we impart upon the variable y the value zero *two* times, then it obtains *one* value (zero), and not two. Instead of saying that a (transformed) equation has two "roots", equal to zero (i.e., its left hand side is expanded into factors, amongst which $(y - 0)$ appears twice), MacLaurin speaks (here) of two "values" of y, equal to zero. It is not surprising that such a substitution of the word "root" by the word "value", would give Marx the (wrong) impression, that MacLaurin simply concludes about the equality of zero with the expression $3e^2 - 2pe + q$ from the equality of zero with the expression $3e^2y - 2pey + qy$ when $y = 0$, and that MacLaurin made a crude mistake.

Marx's second comment (sheet 87) shows that he really had such an impression. He placed it within square brackets and it is related to the immediately next point of MacLaurin's p.166 :

"If a biquadratic equation is proposed,namely $x^4 - px^3 + qx^2 - rx + s = 0$, and two of its roots are equal, then upon the assumption that $e = x$, two of the values of y must vanish and the equation in §34 is reduced to the following form :

[At issue here is the equation obtained upon the substitution of $x = y + e$, from the equation $x^4 - px^3 + qx^2 - rx + s = 0$.]

$$\left.\begin{aligned} y^4 - 4ey^3 + 6e^2y^2 \\ -py^3 - 3pey^2 \\ +qy^2 \end{aligned}\right\} = 0 .$$

So that $4e^3 - 3pe^2 + 2qe - r = 0$, or since $x = e$,

$4x^3 - 3px^2 + 2qx - r = 0$ " .

In his observation Marx wrote (the text is marked by a line on one side) :

[[That MacLaurin, in his demonstration, as we saw above, permitted a crude blunder, is expressed in sum by the fact that he says : If y takes the value 0 second time, then from there it follows, that its first coefficient [[since y does not figure in the *first derived* equation, $y^o = 1$]] is equal to zero. This is but playing with the word "value".

If in $(4e^3 - 3pe^2 + 2qe - r\,)\,y$, the value of $y = 0$, then it designates, according to all the rules of deduction, that we then have

$$(4e^3 - 3pe^2 + 2qe - r\,) \cdot 0\,,$$

where in place of y its value 0 has been put ; but thereby its coefficient vanishes, not because it is itself equal to zero, but because it has a factor $0 = y$. However, we should obtain $4e^3 - 3pe^2 + 2qe - r = 0$, and that is why the equation is to be divided by the factor $y - 0$

(since $y = 0$ gives $(y - 0)$ as a factor of the equation), i.e., by y [155], and then the co-efficient of y, as a term of the transformed expression, which $= 0$, also becomes $= 0$, *independently of* y [156]. *It is understandable, that the division by* y *in an equation* $= 0$, *where each term has as a factor some power of* y, *is accomplishable*. But if we divide first time by y, then y, y^2, y^3 will be the corresponding factors of the three remaining terms; if we divide once more by y, then y, y^2 will be the factors of the two remaining terms; and if we divide for the last time by y, then there remains only y with the coefficient 1.]]

By the words "as we saw above", Marx here indicated an analogous comment, which he made earlier, in connection with the question of the multiple roots of the cubic equation $x^3 - px^2 + qx - r = 0$ (MacLaurin, part II, §§33, 36). Transformation of his equation by substituting $y + e$ for x, where e is a root of the original equation, gives, since $e^3 - pe^2 + qe - r = 0$,

$$\left.\begin{matrix} +3e^2 \\ -2pe \\ +q \end{matrix}\right\} y + \left.\begin{matrix} 3e \\ -p \end{matrix}\right\} y^2 - y^3 = 0\,;$$

and in connection with the comment of MacLaurin, that if " two values of y vanish", then the co-efficient of y (in the transformed equation) "also must vanish" (p.166), Marx wrote the following (sheet, 85-86, Marx's pp. 65- 66):

Here $3e^2 - 2pe + q$ by no means becomes $= 0$, because y has a 2nd value $= 0$, for the equality with zero of the product of $3e^2 - 2pe + q$ and 0 [[this $0 = y$]] does not prove, that the coefficient of 0 is also 0, or else everything would be 0, since everything being multiplied by 0 becomes $= 0$. The fact that some second value of $y = 0$, is translated by MacLaurin, as designating that the *coefficient* of y must be equal to 0 !

In the same page, Marx wrote lower down :

If in an equation in x of the 4th power (*equation 1*) there are two equal roots and $e = x$, i.e., $x = y + e = y + x$, hence $y = 0$, then the two values of y (it should have been said : two coefficients of [y and] y) must vanish ...

Thus, here too, Marx raised an objection against MacLaurin's use of the word "value ". (Equality with zero of a coefficient of y is a consequence of the fact, that 0 is a multiple root of the equation, and not simply a second "value" of y.)

Taking notes from the next, fifth chapter of MacLaurin's "Algebra",Marx not only completed some calculations omitted by MacLaurin, but also illustrated his arguments by examples, which are not there in MacLaurin. Thus, Marx examines the proximation of the boundaries of the roots of an equation, in the light of the equation

$$x^3 - 2x^2 - 5 = 0,$$

which is not there in MacLaurin. That is why, it is natural to think, that having read MacLaurin, Marx turned to some other literature, containing further development of the questions considered by MacLaurin. Marx himself said about what had actually happened. Thus on p.64 (sheet 84), at the very beginning of his notes from chapter IV of MacLaurin's book, on the transformation of the equation.

$$x^3 - px^2 + qx - r = 0$$

by substituting $(y + e)$ for x, Marx wrote (inside a box):

Successors of MacLaurin inverted this, substituting instead of $(y+e)$ (for example) $(e+y)$ for x, so that x^3 turns into $(e+y)^3$ etc. Then in the above mentioned case the original equation is transformed into

$$\left.\begin{array}{ll} e^3 & +\,3e^2 \\ -\,pe^2 & -\,2pe \\ +\,qe & +\,q \\ -\,r & \end{array}\right\} y \left.\begin{array}{l} +\,3e \\ -\,p \end{array}\right\} y^2 + y^3 = 0.$$

Apparently, Hall belongs to the rank of such "successors of MacLaurin". We already had an encounter with an extract from his "Elements of Algebra", in connection with the question of further development of the algorithms of "involution" (raising the power, see pp.192-194).

Having noted the arguments, with the help of which MacLaurin proves, that the greatest absolute magnitude of the negative coefficients of an equation ("the greatest negative coefficient "), raised by one, is always the upper boundary of the roots of the equation, which he conducted in the light of the example of the cubic equation

$$x^3 - px^2 - qx - r = 0$$

separately for all the three cases, when successively p, q or r is this "greatest negative coefficient", Marx generalised this argument, clearly formulating its basic idea.

This note book comes to an end with the following observation of Marx (sheet 92; it is not there in MacLaurin):

[[In a *complete* equation, where not a single term is absent and the signs either remain unchanged, or successively take turns, [there] every root, taken by itself [i.e., as an absolute magnitude], must be less than the coefficient p ; for the latter is the *sum of the roots*, taken with the *opposite signs*; for *only the positive* (alteration of signs) or *only the negative* (constancy of signs) roots, it is, consequently, greater than each root.]]

OTHER MANUSCRIPTS ON ALGEBRA

S.U.N. 3934

It is a note book on algebra. It carries no special title and, it is a continuation of "Algebra II" (see above : manuscript 3933). Sheets 1-22 ; according to Marx's numbering 1-19, 23-25. The language is German ; in places it contains English words and expressions.

Sheet 1. The notebook begins with the following observation of Marx, which, however, has the character of a summary of the text that follows :

[[5) The *search for the real roots of an equation by finding the limits of the roots* has a sense only in that case, when the roots are *irrational*, i.e., are not whole numbers, and, hence, the values of the roots are represented only *approximately*. If the equation is *rational or commensurable*, i.e., if its roots are whole numbers, then they must be contained as factors in the last term of the equation = the product of all the roots. That is why, this last term is to be expanded into factors, with the help of the method — explained earlier — of deducing from them those, which constitute the roots.

But if the equation is incommensurable, then the following method, adduced by MacLaurin [is to be considered]. It consists of a transformation of the equation, assuming $x = y \pm k$, hence $y = x \mp k$; here k represents the approximate root greater or lesser than x ; MacLaurin begins with the latter. Upon substitution of $y + k$ for x, the transformed equation also becomes equal to zero, just like the initial one ; further, it is *of the same degree ;* depending on whether for x^n we write $(y + k)^n$ or $(k + y)^n$, the last or the first vertical series [157] of the transformed equation becomes an equation (where y is a factor only as y^0) (it is better to expand it the way Lacroix does — as $(k + y)^n$, since in the method of limits, every time you are required to begin with the term in k, namely : it is to be cast away as equal to zero), in which k enters in the same way, as does x in the initial equation, here $f(k)$ for $f(x)$. In fact, in this equation k figures as a root of the original equation, for $y + k = x$; but here $y = 0$, hence, $k = x$. That is why this equation cannot give any other limits, apart from the limits of the equation itself, already furnished by the original equation, namely, the limits of the positive terms (as well as of the negatives) and the limit 0[158].]]

Sheets 2-5. End of the notes from chapter V of MacLaurin's "Algebra"(for its beginning see the note book "Algebra II", manuscript 3933).

Sheets 5-8. Notes from chapter VI of MacLaurin's "Algebra". Marx gave it the title:"*F) Solution of equations with commensurable roots*".

Sheets 8-15.Conspectus of chapter VII of MacLaurin's "Algebra". It carries the heading given by Marx : "*G) Solution of equations by finding the equations of a lower order, which are its divisors*".

Sheets 15-19. Conspectus of the addition to chapter VII of MacLaurin's "Algebra". Marx gave it the heading : "*H) Reduction of equations by irrational divisors*".Marx began this note with the following comment :

1) *This chapter* was not written by MacLaurin, but by the publishers [[to whom MacLaurin gave the manuscript on completion]] :*Martin Folkes* (President of the Royal Society),*Andrew Mitchel* and *Rev. Hill,* Chaplain of the Archbishop of Canterbury.

It is an explanation of the "Rule", which *Newton* gave on p. 264 of his *Arithmetica Universalis.*

In this supplementary the issue is : the reduction of an equation of fourth degree $x^4 + px^3 + qx^2 + rx + s = 0$ to an equation of the form

$$(x^2 + \frac{1}{2}px + Q)^2 = n(kx + l)^2$$

by finding the suitable numerical values for k, l, n and an expression for Q through these numbers.

Marx did not take down the rule, according to which this reduction is carried out (pp. 213-214). He directly began his notes from its substantiation, contained in p. 218 of MacLaurin's book. Marx noted only the main part of this substantiation, ending it with the words (sheets 17-18) :

That is why, if the corresponding values of n, k, l, Q satisfy these conditions, i.e., if they correspond to each other, then it is proved, that they have been correctly surmised and that by adding $n(kx + l)^2$ to the given equation it is complemented to the square $(x^2 + \frac{1}{2}px + Q)^2$.

After this Marx noted example II from MacLaurin (pp. 215-216), then example I (pp. 214-215).He did not take notes from example III (pp. 216-217) and the entire text that followed the substantiation, which is devoted to the limits or particular instances of the rule. But Marx wrote: "*cf. further on pp. 216-222*". Apparently, he intended to return to this topic, once more. He left three pages blank in this note book and continued his notes from p. 23.

Sheets 20-22 (Marx's pp. 23-25). Notes taken from chapter VIII of MacLaurin's "Algebra". Marx gave it the heading : "I) *Solution of equations according to Cardan's rule and analogy*". From this chapter Marx noted the first part related to the solution of the cubic equations and the extraction of cubic roots. §§ 82 and 83 of chapter VIII, devoted to the solution of the equations of fourth degree remains unnoted : here the conspectus of MacLaurin's "Algebra" abruptly comes to an end. Marx never resumed it.

S.U.N. 3935

In this archival unit a few smaller manuscripts have been joined together. Contentwise they are close to the materials noted by Marx in his note books on algebra (see manuscripts 3932-3934).In all there are 16 such sheets.

1. Sheets 1-2. Two small sheets under the rubric "*VIII. Calculation of Logarithms*". Contentwise they are close to § 14 (pp. 15-19 in Marx's numbering) of the note book "Algebra II". The language is English. Its source is the book : J.Hind, "The Elements of Plane and Spherical Trigonometry", 3rd ed., Cambridge, 1837, ch. VII, §§ 161-165, pp. 154-159.

2. Sheets 3-6. Four sheets without a common title (mainly calculations). They contain calculation of the logarithms of numbers, calculation of the characteristics and mantissa; and further, repetition of the materials of the two previous sheets. Language — English. Its source is the same book by Hind, chapters : IV, §§ 89- 99, pp. 67-78 and VII, §§ 161-177, pp. 154-167 (with a few omissions).

3. Sheets 7-10. Four clearly written pages. Marx numbered them 1-4. Title : "*The Doctrine of Combinations*". It contains a few basic concepts and theorems of elementary combinatorics. It is sub-divided into the following parts : "*General*" — containing three points :" *1) The simplest starting point*", 2) here the concepts of form and complex, higher and lower forms, order and classes of forms have been introduced, 3) contains formulation of the problems of combinatorics. Then follows the parts:"*A) Permutation*" and "*B) Combination*". The notations are the same, as in the note-book "Algebra II"(pp. 27-38 in Marx's numbering). Language — German ; the source has not been established ; contentwise it is somewhat proximate to the corresponding paragraphs of : M.A.Stern, "Lehrbuch der algebraischen Analysis", 1860, and also to the aforementioned (see pp. 191-192) books by B.Thibaut and Fr.W. Spehr.

4. Sheets 11-13. Three sheets entitled : *"Method of Indeterminate Coefficients"*. The method of indeterminate coefficients is formulated and proved, and it is then applied to seeking the expansion of $\sqrt{1+x}$ into a powered series. Contentwise, it is related to point 3a), section III, "General binomial theorem" of the note-book "Algebra II" (see p. 196). The source is possibly the same. There is also a great amount of resemblance with the text of pp. 195-196 of the "Algebra" by Wood, in which, on p. 197 the same example of the expansion of $\sqrt{1+x}$ into a series, has been adduced. Language — German; the first phrase is in English.

Only in the first page, the text is in (natural) language; the rest are calculations.

5. Sheets 14-16. Three sheets, joined by the title : *"Powers and Roots"*. Apparently, it is an insertion to one of the notes on algebra, contemplated by Marx, since it ends with the words:*"End of the insertion"*. Here the operations with radicals, often represented in the form of fractional powers of arguments, have been examined. Language — German.

"SUCCESSIVE DIFFERENTIATION"

S.U.N. 3999

6 pages joined by the title : *"Successive Differentiation (according to G.W.Hemming, 1848, Cambridge)"*. This is a conspectus of some paragraphs of the book: "An Elementary Treatise on the Differential and Integral Calculus", by G.W.Hemming , Cambridge, 1848 (§§ 92 -116, pp. 60-77). The page numbers are according to the second edition of this book, dated 1852. Marx took notes from the following paragraphs only :

§ 92. Definition of an independent variable.

§ 94. Successive differentiation.

§§ 96-98. Relations between successive differentials and differential coefficients, when the independent variable is general.

§ 99. Form the above relations when the quantity (x), under the functional sign, is independent variable.

§ 107. To pass from an equation among differentials, with x as an independent variable, to one among differential coefficients with respect to x, and the converse.

§§ 108, 109. To pass from a general independent variable to x, and the converse.

In these paragraphs, the definitions of the independent variable and of the general independent variable are given, the relations among derivatives and differentials are considered, these are connected with invariance of the differentials of first order in respect of the choice of independent variable and the modes of transforming equations in derivatives into equations in differentials, and the converse. Herein a variable is called *independent* if its differential is taken as a constant. If the dependence among variables is given by an equation and no differential is taken as constant, then Hemming calls it an equation having *a general independent variable* or says that the independent variable is general. This conspectus does not contain any comment which is Marx's own. Apparently it was written at the time of writing section I, "Lagrangian deduction (somewhat modified) of Taylor's theorem, based upon an algebraic foundation" (of manuscript 4000),In the second half of the 70s of last century, when Marx took a special interest in the problem of meaning of successive differentiation (see : PV, 214).

The manuscript is written in English, with a touch of German.

THEOREMS OF TAYLOR AND MACLAURIN, FIRST SYSTEMATISATION OF THE MATERIAL

S.U.N. 4000

48 sheets of a note book (pp. 1-48 in Marx's numbering). Its content corresponds to that period of Marx's mathematical studies, when it still appeared to him that Lagrange's attempt to construct mathematical analysis as the theory of analytical functions is conclusive. Evidently, with the aim of investigating the Lagrangian theory and revealing its algebraic origins, Marx systematically arranged in this note book, the notes from all those sections of the courses of Boucharlat, Hind, Hall, Hemming, Lacroix, MacLaurin and of others at his disposal, in which the Lagrangian theory or questions related to it have been enunciated.

Four sections, of this note book, constitute its greater part (sheets 1-37). Marx gave them the titles:

"*I. Lagrangian deduction (somewhat modified) of Taylor's theorem, based upon an algebraic foundation*". Sheets 1-13.

"*II. Taylor's theorem, based upon a translation of the binomial theorem from the language of algebra into the differential mode of expression*". Sheets 14-20.

"*III. MacLaurin's theorem is also a simple translation of the binomial theorem, from the language of algebra to that of the differentials*". Sheets 20-28.

"*IV. More on Taylor's theorem*". Sheets 28-37.

Section I consists of several points, indicated by Latin capital letters from A) to F). In point A), Lagrange's attempt* to prove, that *in general*, leave a few exceptions, an arbitrary function $f(x+h)$ may be expanded into a series of integral positive ascending powers of h, has been enunciated according to Boucharlat (§§ 244,253, pp. 168-169, 173-175). It was an attempt to show, that in such a general case the equality

$$f(x+h) = f(x) + p\,h + Qh^2$$

must occur, where p is a function only of x, neither identically equal to zero, nor to infinity, and Q is a function of x and h, which is represented in its turn as $Q = q + Rh$ where q is a function of x, and R—a function of x and h, represented analogously as $R = r + Sh$ etc. Herein, the attempt to prove, that a multiplier of p can be neither a negative power of h, nor $\log h$, is also enunciated. The function p is defined as the derivative of $f(x)$ and for it the notation $f'(x)$ is introduced. Analogously, in point B), the derivative of derivatives (successive derivatives) are defined, and the notations $f''(x)$, $f'''(x)$, $f^{IV}(x)$ etc. are introduced (following Boucharlat § 248).

In point C) the interrelations among the coefficients of the expansion

$$f(x+h) = f(x) + ph + qh^2 + rh^3 + sh^4 \text{ etc.}$$

and the successive derivatives $f''(x)$, $f'''(x)$ etc. are determined by the method of indeterminate coefficients with the help of the equality

$$f(x + (h+i)) = f((x+i) + h)$$

(following Boucharlat, §§ 245-247, 249, 250). In the last of these paragraphs, the notations for the derivatives and the differential symbols have been introduced. Here the enunciation as per Boucharlat comes to an end. Further on, it has been shown according to Hind (§ 97, p. 127) that the Lagrangian derivatives are the limits of the ratios $\frac{\Delta y}{\Delta x}$ when $\Delta x \to 0$, i.e., they are "differential coefficients." Here Marx wrote (sheet 7) :

---

* Boucharlat wrote (p. 168) that he is modifying the method of Lagrange. — Ed.

In comparison with the results of differential calculus we find, that $f'(x)$ is the real equivalent for $\frac{dy}{dx}$, $f''(x)$ — the real equivalent for $\frac{d^2y}{dx^2}$ etc. or, conversely, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc., are differential expressions for the real differential coefficients, i.e., the derived functions of x in Lagrange.

After this came the following comment of Marx. It has the character of a summary. Contentwise it is related to §§ 251 and 252 of the text book by Boucharlat (save the first phrase, the source of which appears to be Hind's book ; see p.120).

Lagrange himself says, that dx is chosen instead of h, dx^2 instead of h^2 etc. only to establish the uniformity of notation.

Here the expression $\frac{dy}{dx}$ becomes the symbol of the operation, through which the coefficient of h in the expansion of $f(x+h)$ is obtained ; $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ etc. indicate, that the same process, being repeated, gives the coefficients of the other powers of h. Hence, it is necessary to establish, only as per *the rules of algebra,* what the $\frac{dy}{dx}$ etc. must be for each function. For example, what is the $\frac{dy}{dx}$ for x^m ? We are required to expand the function $(x+h)^m$ according to the binomial theorem, which gives $x^m + mx^{m-1}h +$ etc. ; since $\frac{dy}{dx}$ indicates the coefficient of the *first power* of h in this expansion, so $\frac{dy}{dx} = mx^{m-1}$ etc.

Hence, the entire problem is reduced to this : to expand the different types of functions, with the help of the analytical processes, which algebra can provide us. Hence the method presupposes an *algebraic expansion of all these functions* and thereby the differential expressions corresponding to them are obtained. Conversely, in the differential calculus, the differential forms, i.e., the operations indicated by them, serve the search for these functions, along their own short cut path.

[[Instead of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc. Lagrange also designates the derived functions by y', y'' etc.]]

[[Though in the method of Lagrange the *principles of differential calculus* are demonstrated independently, free of any reference to *limits, infinitesimals and evanescent quantities,* however, he is constantly required to have recourse to them, as soon as the issue concerns *applications,* for example, the determination of volumes, surfaces, length of curves, finding the expressions for subtangents etc. His method assumes an acquaintance with the analytical methods for expanding all types of functions of $x+h$ in integral ascending positive powers of h, which he has put forward; and often this expansion is quite difficult. That apart, the theorems of Taylor and MacLaurin, once established, provide, with great ease, the means to expand into series many functions, whose expansion through the methods of ordinary algebra, may be obtained along an extremely, tiresome and round about path.]]

Point C) comes to an end with the following comment of Marx. Here he obtained, from Newton's binomial theorem, the derivative for the function ax^m, utilising Taylor's theorem and, conversely, obtained Newton's binomial theorem, with the help of the theorem about the differential of a product.

[[We have just seen, that for obtaining the expansion for $f(x+h)$, for example, for ax^m, it is enough to expand $a(x+h)^m$ according to the binomial theorem:

$$m\,x^{m-1}ah\,,\quad m\,(m-1)\,x^{m-2}\cdot\frac{ah^2}{1\cdot 2}\,,\,\ldots$$

Then we know that the coefficient of $h = \dfrac{dy}{dx}$, the coefficient of $\dfrac{h^2}{1\cdot 2} = \dfrac{d^2y}{dx^2}$ etc.

Conversely, once Taylor's theorem is established, it is possible *to deduce the binomial theorem from it ;* it may be still easier to deduce it with the help of the most elementary differential operations.

For instance, since it has already been demonstrated, that

$$d\,(xy) = x\,dy + y\,dx\,,$$

so , in general

$$d\,(xyztu) = xyzt\,du + yztu\,dx + ztux\,dy + tuxy\,dz + xyzu\,dt\,.$$

Dividing both the sides by $xyztu$, we shall get

$$\frac{d\,(xyztu)}{xyztu} = \frac{xyzt\,du}{xyztu} + \frac{yztu\,dx}{yztux} + \frac{ztux\,dy}{ztuxy} + \frac{tuxy\,dz}{tuxyz} + \frac{xyzu\,dt}{xyzut}\,.$$

Hence,

$$\frac{d\,(xyztu)}{xyztu} = \frac{du}{u} + \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} + \frac{dt}{t}\,.$$

If now, x, y, z, t, u are made equal to each other, and consequently, it is assumed, for instance, that they are all equal to x, and that their number is equal to m, then

$$\frac{d\,(x^m)}{x^m} = \frac{m\,dx}{x}\,,$$

hence,

$$d\,(x^m) = \frac{mx^m\,dx}{x} = mx^{m-1}\,dx.$$

That is, $\dfrac{d\,(x^m)}{dx}$ or $\dfrac{dy}{dx} = mx^{m-1}$ and thus $\dfrac{d^2y}{dx^2} = m\,(m-1)\,x^{m-2}$ etc. and all [the derived] functions of x^m may be so obtained.]]

Marx gave point D) the title:"*D) Application of Lagrange's method to an elementary case*". It contains the note taken from the text book by Hind (pp. 126-127), of an example of application of Lagrange's method for finding out the derivative of the product of two functions y and z of the argument x by representing the augmented values of y and z in the form of $y + ph + \frac{1}{1\cdot 2}\,qh^2 +$ + etc. and $z + p_1\,h + \frac{1}{1\cdot 2}\,q_1\,h^2 +$ etc. and, for searching the coefficient of h (in the first degree) in their product with the help of formal operations with serieses.

Marx gave point E) the heading : "*E) More on the impossibility, that $p = 0$ or that h has [negative], fractional etc. indices in the general expansion for $f(x+h)$ where x is indeterminate, but is a variable, which can assume any value*".

In this point, at first notes are taken from § 253 (pp. 173-175) of the text book by Boucharlat, devoted to the proof, that the coefficient of h in the first degree may be equal to zero only in the particular instances, and then from § 259 (pp. 179-180), containing the Lagrangian proof, that in the expansion of $f(x+h)$, h can not enter into a fractional power, since in that case the number of different values of the expansion for $f(x+h)$ would be greater than the number of different values of the very expression $f(x+h)$. On this proof (Boucharlat mentioned that it belonged to Lagrange) Marx wrote (sheet 9) :

Earlier it was shown, that h in ph cannot have a fractional index. This could have been shown, in the same way, for all the other terms of the expansion. But Lagrange in addition gives the following interesting proof.

After this point, Marx wrote the following comment, as a summing up exercise, and without any title (sheet 11) :

Lagrange 1) *algebraically* proves what Taylor presupposes : that so long as x remains indeterminate, $f(x+h)$ may always be represented by an infinite series $= f(x) + ph + qh^2 +$ etc.; he provides an *algebraic basis* to the differential calculus; but it is to be used only as the starting point, since it is quite pointlessly tedious to develop algebraically, that which can be attained much more easily, through the proper methods of differential calculus.

2) From the very beginning he demonstrates, that the general series for the expansion of $f(x+h)$, where x is indeterminate, excludes all those particular instances, where Taylor's theorem is *inapplicable*.

3) By introducing the concept of the *derived functions of the variable,* he, in essence, gave differential calculus a new bearing, removing thereby a lot of useless difficulties.

If in Taylor's theorem it is inscribed that :

$$f(x+h) = f(x) + \frac{df(x)}{dx}\cdot\frac{h}{1} + \frac{d^2f(x)}{dx^2}\cdot\frac{h^2}{1\cdot2} + \frac{d^3f(x)}{dx^3}\cdot\frac{h^3}{1\cdot2\cdot3} + \cdots,$$

then this formula by no means contains the concept of derived functions $f'(x)$, $f''(x)$ etc., and merely says, that the *successive differential operations* took place in respect of *one and the same initial* $f(x)$; in it the successive differential coefficients are expressed as the *successively derived functions of x*.

On the other hand, when Lagrange writes :

$$f(x+h) = f(x) + ph + q\cdot\frac{h^2}{1\cdot2} + r\cdot\frac{h^3}{1\cdot2\cdot3} + \cdots,$$

then there it is assumed that in

$$f(x+h) = f(x) + ph + qh^2 + \cdots,$$

the coefficients p, q etc., were already reduced to their values equal to $f'(x)$, $f''(x)$ etc. [159].

Having explained in a few more words, that since Lagrange's arguments had a general character (were related to any function $f(x)$), they may be applied also to the derived functions, on which some calculations in point B) were based — Marx went over to point F), wherein section I of this manuscript was completed. In it he took notes from §§ 8, 9 (pp. 4-6) of "A Treatise on the

Differential and Integral Calculus, and the Calculus of Variations" by Thomas G. Hall. (We have its 5th edition, published in 1852.) This conspectus begins with the sentence :

F) Lagrange's demonstration, that when x is indeterminate, then $f(x+h)$ may be represented in the form of $f(x) + ph + qh^2 +$ etc., has been placed at the very beginning of some of the manuals on differential calculus, and is dealt with as follows.

Then follows the demonstration, not only by the method of indeterminate coefficients, but also of indeterminate indices of power, that if $f(x+h) = u + Ah^{\alpha} + Bh^{\beta} + Ch^{\gamma} +$, where $\alpha < \beta < \gamma \cdots$, then $u = f(x)$, $\alpha = 1$, $\beta = 2$, $\gamma = 3, \cdots$,

$$A = f'(x)\,,\ B = \frac{f''(x)}{1 \cdot 2}\,,\ C = \frac{f'''(x)}{1 \cdot 2 \cdot 3}\ .$$

The term Ah is called the *differential, and the coefficient A* — the *differential coefficient ;* for them the usual notations were introduced. Marx adduced this demonstration from Hall's book, pp. 4-7. Namely, that is what is had in view here.

As has already been noted, section II of the manuscript was written at a time, when to Marx appeared sufficiently well grounded the "proof", given by Lagrange, that in the *general case* the expansion

$$f(x+h) = f(x) + ph + qh^2 + rh^3 + \cdots,$$

where $p, q, r, \ldots$ are functions of x, must be valid.This fully corresponds to its title (see p.214). Here Marx states the proposition that : Taylor could proceed to his theorem *heuristically* by generalising Newton's binomial theorem; he adduces a number of considerations in support of this proposition and observes at the same time, that yet Taylor could not prove the legitimacy of such generalisation.In all the manuals at Marx's disposal, Taylor's theorem has been proved in the main identically. Apparently, in this connection Marx thought, that this proof belonged to Taylor himself. (For the details of these proofs see, Appendix, p. 333). Those parts of this section of the manuscript which are not notes from other sources, are being reproduced below, in full. Immediately following the heading of section II, Marx wrote on sheets 14-17:

A) *Boucharlat* remarked in the second note (Appendix) to his "*Traités du calcul différentiel et du calcul intégral* " : "With the exception of the differentials of circular functions, which are easily deduced from the trigonometric formulae, all other *monomial* differentials, like, for instance, the differentials of x^m, a^x, $\log x$ etc. were obtained with the help of the *binomial theorem. MacLaurin's* theorem was applied *for finding out the constant A in the formulae for exponential functions*, but could be managed without it [[on this afterwards]]. Hence it follows, that *all the principles of differentiation are based only upon the binomial theorem*"*.

But, on the other hand, *Taylor* established his *theorem*(which along with *MacLaurin's* theorem — the latter in its turn may be represented as a particular instance of Taylor's theorem — happens to be of utmost importance for the operations of differential calculus) at a time, when not only the binomial theorem was already known, but also the expansion of the *functions of x* furnished by it, through the methods of differential calculus itself, as well as, the so called elements of differential calculus, which were in general already developed.

---

*This Appendix is not there, in the 5th edition of the text book by Boucharlat entitled : "Elémens de calcul différentiel et de calcul intégral" (Paris, 1838), at our disposal. — Ed.

The function $f(x+h)$, in the second side [R.H.S.], in the side of the developed series, is always represented, in accordance with the binomial theorem, [by the terms] with factors $h^0 (=1)$, h, $\frac{h^2}{1\cdot 2}$, $\frac{h^3}{1\cdot 2\cdot 3}$ etc. (*with ascending integral and positive powers of h*, steering clear of the negative, fractional and logarithmic indices, on which we shall not be able to dwell here, after covering Lagrange's method) ; the *indeterminate coefficients* of h in its successive powers, i.e., the *different successively derived functions of x*, or the differential coefficients, naturally have different forms, depending upon, what sort of initial function $f(x)$ is to be expanded — for instance, depending upon, whether this function is x^m, or a^x, or $\log x$ or $\sin x$ or more complex[160] etc. But, evidently, Taylor's theorem is based upon the simplest application of the binomial theorem, i.e., $f(x) = x^m$.

That is why we shall expand $f(x+h) = (x+h)^m$ according to the binomial theorem. Then, for instance,

$$(x+h)^m = x^m + mx^{m-1}h + \frac{m(m-1)}{1\cdot 2}x^{m-2}h^2 + \cdots =$$

$$= x^m + mx^{m-1}h + \frac{1}{2}m(m-1)x^{m-2}h^2 + \cdots$$

In the third term, i.e., in $\frac{1}{2}m(m-1)x^{m-2}h^2$ (the same as what Lagrange wrote in the above mentioned expansion as $\frac{1}{2}f''(x)h^2$), there is a derivative of x, directly deduced from mx^{m-1}, namely, $m(m-1)x^{m-2}$; in order to have not half of this function, but the entire function as a whole, it is necessary to write here and afterwards $m(m-1)x^{m-2}\frac{h^2}{1\cdot 2}$ i.e., put the numerical divisor under h^2, h^3 etc.

Then we shall have

	according to the binomial theorem	considering x as a variable	
x^m	$= x^m$	$= f(x)$	
$m\,x^{m-1}h$	$= m\,x^{m-1}h$	$= f'(x)h$	$= \frac{dy}{dx}h$
$m(m-1)x^{m-2}\frac{h^2}{2}$	$= \frac{m(m-1)}{2}x^{m-2}h^2$	$= f''(x)\frac{h^2}{2}$	$= \frac{d^2y}{dx^2}\cdot\frac{h^2}{2}$
$m(m-1)(m-2)x^{m-3}\frac{h^3}{2\cdot 3}$	$= \frac{m(m-1)(m-2)}{2\cdot 3}x^{m-3}h^3$	$= f'''(x)\frac{h^3}{2\cdot 3}$	$= \frac{d^3y}{dx^3}\cdot\frac{h^3}{2\cdot 3}$
$m(m-1)(m-2)(m-3)x^{m-4}\frac{h^4}{2\cdot 3\cdot 4}$	$= \frac{m(m-1)(m-2)(m-3)}{2\cdot 3\cdot 4}x^{m-4}h^4$	$= f^{IV}(x)\frac{h^4}{2\cdot 3\cdot 4}$	$= \frac{d^4y}{dx^4}\cdot\frac{h^4}{2\cdot 3\cdot 4}$
etc.	etc.	etc.	etc.

Thus, Taylor already knew, how to find $d(x^m) = mx^{m-1}\,dx$ along the path of differential calculus, hence $\frac{d(x^m)}{dx}$ or $\frac{dy}{dx} = mx^{m-1}$ and, also further

$$\frac{d^2y}{dx^2} = m(m-1)x^{m-2} \quad \text{etc.};$$

in other words, he knew, that the derived functions of x deduced with the help of the binomial theorem, are identical to those which appear as successive differential coefficients; he also knew that while finding out these functions through differential calculus, h as well as its numerical coefficients $\frac{1}{1\cdot 2}, \frac{1}{1\cdot 2\cdot 3}$ etc. disappear; we get the functions $m\,x^{m-1}$, $m(m-1)x^{m-2}$ etc. as the results of differential operations, expressed by $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ etc.

On the other hand, the binomial theorem shows, that

$$f(x+h) \text{ here } (x+h)^m = f(x) + f'(x)h + f''(x)\frac{h^2}{1\cdot 2} + f'''(x)\frac{h^3}{1\cdot 2\cdot 3} + \cdots$$

Hence, in order to obtain the proper expansion of $f(x+h)$, the second term is to be multiplied by $\frac{h}{1}$, the third by $\frac{h^2}{1\cdot 2}$etc., in other words, h , h^2 etc. with their numerical multipliers — extinct in the process of differentiation — are to be *restored.*

Thus, for instance, when $x^m = x^3$,

$$f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3\,.$$

Thus, the derived functions of x are obtained through the binomial theorem, and they turn out to be those functions, which are obtained through differentiation :

$$3x^2 = 3x^2\,,\ 3\cdot 2x = 6x\,,\ 6x^0 = 6\,.$$

If we restore h , $\frac{h^2}{1\cdot 2}, \frac{h^3}{1\cdot 2\cdot 3}$ etc., extinct in differentiation, then we shall get :

$m\,x^{m-1}\ldots$ for $3x^2\ldots$ $3x^2h$,

$m(m-1)x^{m-2}\ldots$ for $6x\ldots$ $6x\frac{h^2}{2} = 3xh^2$,

for $6x^0$ [[i.e., $m(m-1)(m-2)x^{m-3}$ [or] the third [derived] function of $(x+h)^3 =$
$= 3(3-1)(3-2)\,x^{3-3} = 3\cdot 2\cdot 1\,x^0 = 6 \cdots \frac{6h^3}{1\cdot 2\cdot 3} = h^3$]].

Hence,

$$f(x+h) \text{ or } (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3\,,$$

in other words, the result obtained, is already known from the binomial theorem.

Further, as mentioned above, Taylor knew, that the series obtained through the binomial theorem, starting from the original function upto its derived functions, is represented in the differential [calculus] as follows : x^m as $f(x)$ or y, the original function ; the second function

in the binomial theorem $m\,x^{m-1}$ as the real value of the first differential coefficient $\frac{dy}{dx}$; the third function as the real value of the second differential coefficient $\frac{d^2y}{dx^2}$ etc. If now I substitute for these functions in

$$(x+h)^m = x^m + m\,x^{m-1}h + m\,(m-1)\,x^{m-2}\frac{h^2}{1\cdot 2} + m\,(m-1)\,(m-2)\,x^{m-3}\frac{h^3}{1\cdot 2\cdot 3} + \cdots$$

their differential expressions, and in place of $(x+h)^m$ put the indeterminate $f(x+h)$, then I shall get

$$f(x+h) = f(x) \ \text{(or } y\text{)} \ + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\cdot 2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\cdot 2\cdot 3} + \cdots$$

which is the theorem of Taylor.

[[We should note further, that if as the 4th derivative of $(x+h)^m$ we get

$m\,(m-1)\,(m-2)\,(m-3)\,x^{m-4}\frac{h^4}{1\cdot 2\cdot 3\cdot 4}$, whose differential expression is $\frac{d^4y}{dx^4}$, then for $(x+h)^3$, [it]

$$= 3\,(3-1)\,(3-2)\,(3-3)\,x^{3-4} = 3\cdot 2\cdot 1\cdot 0\,x^{-1} = 0\,,$$

being multiplied by $\frac{h^4}{1\cdot 2\cdot 3\cdot 4}$, it is again $= 0$, so that, in this case, $\frac{d^4y}{dx^4} = 0$. Thus, here too, the *binomial theorem* again shows, that as soon as x — the variable in differential calculus — is excluded from the derivative and, hence, the latter becomes a *constant*, the $\frac{dy}{dx}$ corresponding to it becomes $= 0$; i.e., the deduction of the new functions of x, and that is the new differentiation, comes to an end.]]

It is true, that here Taylor's formula was obtained for $f(x+h)$, only from the most elementary application of the binomial theorem, namely by substituting $x+h$ for x in x^m and by the subsequent expansion of $(x+h)^m$. But this changes absolutely nothing in the *generality of the result*, because 1) the factors h in their ascending integral and positive powers [[starting, if you wish, from $h^0 = 1$ for the first term of the series of expansion for $f(x+h)$]] will remain the same, whatever $f(x)$ be ; 2) the coefficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc., which are in reality the symbols of differential operations to be carried out, understandably, give different results, depending upon the specific character of the original function $f(x)$. It is one thing, for instance if $f(x) = a^x$, another — if ax^n etc. In all the cases they give $f'(x)$, $f''(x)$ and the subsequent derived functions of $f(x)$, all of which can, in addition, also be obtained algebraically, and again, essentially, on the basis of the binomial theorem, as Lagrange showed it in practice.

It is true, that $f(x+h)$ is undetermined, it does not have a determinate power and that is why, it is expanded into an infinite series. But $f(x+h)$ [$= (x+h)^m$] remains entirely indeterminate and expansiable only into an infinite series, till m acquires a determinate value ;

that is why, translated into the language of differential calculus, it also gives an infinite series, as is required by the given instance.

An authentic generalisation of this proof was given only by Lagrange. As we shall see now, in Taylor, this generalisation has the character of only a *hypothetical* assumption, and, besides, understandably, he did not investigate those conditions, which this *hypothesis* includes within itself.

[[A cursory glance is enough to see, that if

$$f(x+h) = f(x) + ph + q\frac{h^2}{2} + r\frac{h^3}{2\cdot 3} + \cdots =$$

$$= f(x) + f'(x)h + f''(x)\frac{h^2}{1\cdot 2} + f'''(x)\frac{h^3}{1\cdot 2\cdot 3} + \cdots =$$

$$= f(x) + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1\cdot 2} + \frac{d^3y}{dx^3}\frac{h^3}{1\cdot 2\cdot 3} + \cdots,$$

[then] the difference $f(x+h) - f(x)$ or $y_1 - y$ is equal to the infinite sum of the derived functions of x or of the differential coefficients. All the terms in the second side [R.H.S.] — in the side of the developed infinite series of the general expression $f(x+h)$ standing in the first side [L.H.S.] — which are connected with $f(x)$ by the + sign, taken together form the *difference* between $f(x)$ and $f(x+h)$. What concerns the *infinite series* of derived functions or differential coefficients, is that an (infinitely) overwhelming majority of these functions may in fact be represented only through an infinite series.

Owing to their very nature, the *exponential, logarithmic and trigonometric functions* can not be represented by *algebraic expressions with a finite number of terms* [161].

Again, from among the *algebraic* functions proper, an overwhelming majority, [for example], like, $\frac{a}{a-x}$ etc., can be represented only by infinite serieses. Only of the determinate algebraic functions, like, for example, $(x+h)^4$, there are determinate number [other than 0] of derivatives; ultimately the function becomes a *constant* (x is eliminated), that is, [the derivative] is also = 0, as it stands to reason also for the identical equations [with identically equal sides]. In the rest $f'(x) = 0$ (where f' signifies all the subsequent f'', f''' etc.) does not designate, that x became equal to 0, i.e., is eliminated, but only designates, that $f'(x) = 0$ *is an equation* [of determinate form] *of a determinate degree* : because every equation, if both of its sides are written on one side, gives zero on the other, and $f'(x) = 0$ just serves the search for x, through the fact, that the differential expression in one side becomes its real value, according to the other * ; in this connection [the equation] $f'(x) = 0$ plays quite a significant role in the theory of maxima and minima.]]

B) Instead of

$$f(x+h) = f(x) \text{ (or } y) + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\cdot 2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\cdot 2\cdot 3} + \cdots$$

* Evidently, here the intention is to state, that in the equation $f'(x) = 0$, the symbolic expression $f'(x)$ $\left(\text{or } \frac{dy}{dx}\right)$ is presupposed by its already substituted real value. — Ed.

the theorem is written also as :

$$f(x+h)=f(x)+\frac{df'(x)}{dx}\cdot\frac{h}{1}+\frac{d^2f(x)}{dx^2}\cdot\frac{h^2}{1\cdot 2}+\frac{d^3f(x)}{dx^3}\cdot\frac{h^3}{1\cdot 2\cdot 3}+\cdots$$

In full, according to Lagrange :

$$f(x\pm h)=f(x)\pm ph+\frac{q\,h^2}{1\cdot 2}\pm\frac{r\,h^3}{1\cdot 2\cdot 3}+\cdots$$

or

$$f(x\pm h)=f(x)\pm f'(x)h+f''(x)\frac{h^2}{1\cdot 2}\pm f'''(x)\frac{h^3}{1\cdot 2\cdot 3}+\cdots$$

C) Instead of presenting his theorem as the binomial theorem translated into the language of differential calculus, Taylor communicated it through a hypothesis, outwardly obtained with the help of a *general* proof.

1) *Let us assume* that the function $f(x+h)$ is expanded in ascending, positive and integral powers of h. Then

$$y_1 \text{ or } f(x+h)=y \text{ (or } f(x)\text{) }+Ah+Bh^2+Ch^3+\cdots, \qquad \text{(I)}$$

where A, B, C etc. are indeterminate coefficients, unknown functions of x.

In fact, since the equation (p. 15, last line)*

$$f(x+h)=f(x)+\frac{dy}{dx}h+\cdots,$$

was found through the binomial theorem and the already known results of differential calculus, it was not difficult to again substitute the indeterminate coefficients instead of $\frac{dy}{dx}$ etc., i.e., instead of the differential coefficients or derivatives; and the method of indeterminate coefficients is often applied in algebra, for example, in the expansion of logarithms, like A, B etc., in order to then *conversely* deduce *from them*, with the help of the differential calculus itself, the differential coefficients, and thereby provide them with a *general* derivation. What Taylor introduces here *independently* of the differential calculus, consists of just the fact, that $f(x+h)$ may be expanded into the series $f(x)+Ah+$ etc.; but for him it is only a hypothesis; it was proved for the first time by Lagrange.

If it is assumed, that he found his theory privatim, in the form in which we enunciated it sub II A), then the subsequent substitution of A, B, C etc. in place of the derivatives of x or their differential expressions — as the starting point for the differential operations proper to be carried out — was not, to be sure, wittingly such an intricate affair.

2) Evidently, there is only one way[to ensure] that the coefficients A, B, C etc. are determined from the equation

$$y_1=y+Ah+Bh^2+Ch^3+\cdots$$

: to make two out of this one equation, whose first sides [L.H.S.], i.e., the unexpanded expressions of the functions, are *one and the same*, and the second sides [R.H.S.], i.e., the

* See, PV, 222 — Ed.

terms of the function expanded into a series, assume different forms. Since the two first sides [L.H.S.] are identical, so the second sides [R.H.S.] must also be identical, hence, the terms with the factors containing h in the same powers (y has the factor $h^0 = 1$) may be equalised.

If differentiation is carried out with the assumption that, x is a constant and h — a variable, then y disappears, since y is a function of x, not containing h, and we get A without h (with $h^0 = 1$), the remaining terms give the numerical coefficients, because h^1, h^2 etc. have been furnished with the numerical indices of power. If differentiation is to be carried out, starting from the other assumption — that h is a constant, and x is a variable — then we shall get $\frac{dy_1}{dx}$ in ascending line as $\frac{dy}{dx} + \frac{dA}{dx} h +$ etc. The trick of this method is revealed *in the differentiation of the first terms*

$$\frac{dy_1}{dh} = A + \cdots, \quad \frac{dy_1}{dx} = \frac{dy}{dx} + \cdots$$

As soon as

$$A = \frac{dy}{dx}$$

is already found, the remaining coefficients, $\frac{d^2y}{dx^2}$ etc., are somehow obtained all by themselves.

After this the conspectus (sheets 18-19) is related to Taylor's theorem. These notes are taken from §§ 55-57 (pp. 34-37) of the text book by Boucharlat (on the contents of these paragraphs, see : Appendix, p. 338).

Section II comes to a close with point D) (sheets 19-20). It carries the title:

"*D) The mode of proving (on the basis of differential calculus) Taylor's theorem, wherein the indices of the powers of h are also considered as indeterminate, and are sought for at the same time with the indeterminate coefficients A, B etc. or P,Q, etc. of h*".

It is a note taken from § 74 (pp. 83-84) of the text book by Hind.

Section III of this manuscript is devoted (as is evident from the title mentioned above) to MacLaurin's theorem : it proposes that, this theorem could have been obtained *heuristically*, from Newton's binomial theorem. However, here, to begin with, Marx stresses not only the resemblance, but also the difference of MacLaurin's theorem from that of Taylor. The difference, first of all, consists of this, that while in Taylor's theorem the expansion of the function in series takes place in the neighbourhood of, though fixed, but nevertheless indeterminate (and in that sense *a variable*) point x ; in MacLaurin's theorem it takes place in the neighbourhood of a determinate point O, so that all the coefficients of the expansion are *constants* (values of the function and its derivatives at the point O). Marx writes :

A) *Taylor's theorem* gave the formula permitting the representation of every function in x (under the above mentioned conditions), when x increases by a positive or negative increment h, i.e., when $f(x)$ turns into $f(x \pm h)$, in the form of a series, whose first term is $f(x)$, and the following terms $\frac{dy}{dx}$ etc., having as factors h in ascending powers, are the *differential*

coefficients or, to be more precise, the symbols indicating, how along the path of differentiation, the successive functions of x are derived, the sum of which, taken together with their factors h, h^2 etc. = the difference between $f(x+h)$ and $f(x)$.

MacLaurin's theorem must give an expansion in series of the *very function* in x, as for example,

$$y = \frac{1}{a+x},\ y = (a^2+bx)^{\frac{1}{2}},\ y = (a+x)^m, \cdots$$

(besides *in ascending powers of* x). The *function* of $x = (a+x)^{-1}$ or $(a^2+bx)^{\frac{1}{2}}$, or $(a+x)^m$ etc. Since $f(x)$ must be expanded in ascending powers of x, so here x plays the same role, as the increment h in Taylor's theorem. It is the second term of the binomial and that is why here it appears only as the factor in ascending powers ; as there we had h^0, h^1, h^2 here we have x^0, x^1, x^2 etc. For example, what is actually developed in Taylor's theorem, is the first term : the derived functions of the *variable* x, meanwhile h, the increment, the second term, figures only as the factor in ascending powers, beginning with $h^0 = 1$. Conversely, likewise figures the *variable* x in MacLaurin's theorem ; consequently, through this theorem one should obtain the development of the first term, which, here, is a *constant magnitude* ; the trick consists, namely, of this : in order to obtain the algebraic deduction of the constant coefficients, contained in $f(x)$, with the help of differential calculus ...

Then for obtaining the expansion in powers of x, for $(c+x)^n$ (corresponding to what is obtained according to MacLaurin's theorem) with the help of Newton's binomial theorem, Marx at first writes the expansion according to the latter theorem :

$$(c+x)^n = c^n + n\,c^{n-1}x + \text{ etc.}, \qquad (1)$$

paying attention to the fact, that for $(x+c)^n$ it would look otherwise :

$$(x+c)^n = x^n + n\,x^{n-1}c + \text{ etc.} \qquad (2)$$

Then he dwells upon an explanation of the sense, in which in the first of these expansions with the *constant* coefficients c^n, $n\,c^{n-1}$, $\cdots$, the latter may be considered as functions. He writes (sheet 21) :

But we shall call these derivatives, functions of c, in the sense, in which, if I divide a^4 by a in the order : $\frac{a^4}{a} = a^3$, $\frac{a^3}{a} = a^2$ etc., [then] a^3, a^2, a, may be called functions, derived from a^4.

After this Marx considered the function $(c+x)^n$ as a function of x and differentiated it in respect of x, thus obtaining successively :

$$(c+x)^n = y = f(x),\ n\,(c+x)^{n-1} = \frac{dy}{dx} = f'(x),\quad n\,(n-1)\,(c+x)^{n-2} = \frac{d^2y}{dx^2} = f''(x) \quad \text{etc.}$$

Then assuming $x = 0$ in all these equalities, he observed that in consequence the coefficients c^n, $n\,c^{n-1}$, $n\,(n-1)\,c^{n-2}$, etc. of the expansion (1), are obtained (as $f(0)$, $f'(0)$, $f''(0)$, $\cdots$), and concluded (sheets 21-22) :

And thereby, proceeding from the binomial theorem, we obtain the whole of such a secret, as MacLaurin's theorem.

We note, nevertheless, that *in the binomial* $(c+x)^n$ *and in the derivatives of it,* as soon as we proceed to assume that $x=0$ and to thus obtain the constant c^n with its derivatives, it is a matter of complete indifference to us, whether we write $(c+x)^n$ or $(x+c)^n$, since in both the cases $0+c$ and $c+0$ is always $=c$. But initially it is not a matter of indifference, in respect of the fact, that x *figures* as a factor *with the ascending powers of* c^n etc., quite like h in Taylor's theorem.

After this Marx went over to the proof of MacLaurin's theorem, about which he took notes from Boucharlat's text book. The corresponding point of the manuscript carries the heading : "*D) MacLaurin's account*". It starts (sheet 22) with the words :

1) Here the *equation* is independent of the differential calculus, *from which follows*.

$$f(x) \text{ or } y=A+Bx+Cx^2+Dx^2+Ex^4+\cdots Ux^n, \qquad (1)$$

(In fact everything is borrowed from algebra, even *the successive differentiation* etc. ; see the note book "Algebra I", p. 73d, sq.)*

Later on Marx noted § 31 (p. 20-21) of Boucharlat's book in full(on its content, see : Appendix, pp. 336-337). Marx finished it (on the last line of sheet 22) with the words :

There is absolutely nothing here, that was not borrowed from algebra — [this includes] the initial equation and the deduction of functions.

In point "*2) Examples of application of MacLaurin's Theorem*", Marx noted from Boucharlat, at first § 34, p. 23, which contains an application of MacLaurin's theorem for the deduction of the binomial theorem of Newton. He completed this conspectus (sheet 23) with the words :

Thus the binomial theorem

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{1{\cdot}2}a^{m-2}x^2 + \cdots,$$

is in its turn deduced, from the theorem of MacLaurin, deduced from it.

Then follows the notes from §§ 32, 33 (pp. 21, 22) of Boucharlat's book. It contains an application of MacLaurin's theorem to the expansion in a series in powers of x, of the function : $y=\frac{1}{a+x}$ and $y=\sqrt{a^2+bx}$.

Marx took notes of his point"*3) Failures of MacLaurin's Theorem*", from Hind's book, from which he noted, in order of points 1), 2), the general considerations about the instances in which MacLaurin's theorem may turn out to be inapplicable (p. 75, § 70) ; α) the example of the function $u=\sqrt{2x-1}$, where according to Hind, the "impossibility of carrying out the expansion in the form, required by the theorem [of MacLaurin], is indicated by the symbol $\sqrt{-1}$, which appears in every term [of the] series" (Hind, p.74); β) the considerations related to the impossibility of expanding $\log x$ in a series of powers of x (§ 70, pp. 74-75); γ) the example of the function $u=ax^{3/2}+bx^{1/2}+cx^{-1}+\cdots$, the inapplicability of MacLaurin's theorem to it is conditioned by the presence of fractional and negative powers of x in its expression (§ 70, p. 75); δ) the example

* Here p. 179 and afterwards. — Ed.

of the function $u = \sqrt{x - x^2}$, to which MacLaurin's theorem is not directly applicable, but for which the expansion in the series

$$u = x^{1/2} - \frac{1}{2}x^{3/2} - \frac{1}{8}x^{5/2} - \frac{1}{16}x^{7/2} - \cdots$$

is obtained by representing u in the form of $\sqrt{x} \cdot \sqrt{1-x}$ and by expanding $v = \sqrt{1-x}$ according to MacLaurin's theorem (§ 71, pp. 75- 76 ; in Marx's manuscript pp. 25-26).

In connection with the example $u = \sqrt{2x-1}$, considered sub α), Marx writes there (in sheet 24 of the manuscript) :

Since all the terms in the series of expansion have the factor $\sqrt{-1}$, so with it nothing can be done; this function may be expanded in "*possible expressions*"[162], but not *in ascending powers of* x, i.e., not through an application of MacLaurin's formula. The "distress" is in this, that we cannot get rid of $\sqrt{-1}$, and this [arises] again from the nature of the "constant" element in $f(x)$; namely, if in $f(x) = \sqrt{2x-1}$ we put $x = 0$, then we shall get $\sqrt{-1}$, and hence, the second side [R.H.S.] must also be reducible to $\sqrt{-1}$, as it happens in $\sqrt{-1}\,(1 - x - \frac{x^2}{2} - \frac{x^3}{3} - \text{etc.})$; if we put here $x = 0$, then what remains is $\sqrt{-1} \cdot (1) = \sqrt{-1}$.

Marx took notes of point "*E) Expansion of the function $f(x)$ in decreasing, and not increasing powers of the independent variable x*" (sheet 27), from § 72, pp. 81-82 of Hind's book.

Here, the expansion of the function in decreasing powers of x is obtained from the expansion in increasing powers of z, of the function, obtained from the given substitution $x = \frac{1}{z}$.

In point "*F) MacLaurin's theorem as a special instance of Taylor's theorem*"(sheets 27-28) Marx adduced a deduction of MacLaurin's theorem from Taylor's theorem, following Boucharlat's (§ 62, pp. 39-40) and Hemming's (§ 150, pp. 107-108) books.

Section IV of the manuscript (sheets 28-48) carries the title : "*IV. (More on Taylor's theorem)*".

This section begins with point "*A) Just as Taylor's theorem is deduced from the binomial theorem, likewise, conversely, the binomial theorem may be deduced from Taylor's theorem*". Here, the example 1 from § 74 of Hind's book is noted, on which the author writes : "This example is a general proof of the binomial theorem,though it is not unusual to assume the binomial theorem, which can be established proceeding from the algebraic principles, for the proof of Taylor's theorem" (p. 85). At first Marx noted sub a) the deduction of Taylor's theorem from Newton's binomial theorem, which followed the words quoted above, and then, under the title : "b) *Conversely, the deduction of binomial theorem from Taylor's theorem*", he noted the first part of this example.

Marx gave the next point the title : "*B) Taylor's and MacLaurin's theorems* " (sheet 29). Here Marx at first adduces the initial equations

$$1)\, f(x_1) \text{ or } y_1 = y + Ah + Bh^2 + Ch^3 + Dh^4 + \text{etc.};$$

$$2)\, f(x) \text{ or } y = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}$$

of the theorems of Taylor and MacLaurin, respectively, and then comments :

Regarding the *initial equation* itself: it is borrowed from ordinary algebra.

Marx again returns to his notes from the chapter on multiple roots, from Lacroix's "Elements of Algebra" and MacLaurin's "Treatise of algebra", paying special attention to the fact that : the algorithms for finding out the multiple roots, are in fact based upon successive differentiation. Thus on p. 34 of this manuscript Marx writes :

The only thing that interests us here, is the lowering of the power of the general equation by one; carried out successively, it is the algebraic method of executing the *successive differentiation.*

Immediately before this we read in the manuscript (sheets 33-34) :

I borrowed the algebraic expansion from the inventor of MacLaurin's theorem himself, i.e., from Colin MacLaurin, 6th edition ("A Treatise of Algebra in three parts etc.", London, 1796). The publisher —*Anne MacLaurin*, MacLaurin's widow. It is a posthumous production. By the way, it is said in the preface that : "*Sir Isaac Newton's Rules,* in his "*Arithmetica Universalis*", concerning the resolution of the higher equations, and the affections of their roots, being, *for the most part, delivered without any demonstration*, Mr. MacLaurin had designed, that this Treatise should serve as a Commentary on that work. For we find all those difficult passages in Sir Isaac's book, which have so long perplexed the students of algebra, clearly explained and demonstrated".

Regarding the art of deducing the equations of lower degrees from those of the higher degrees, it is well known that Newton discovered it with the help of differential calculus or, conversely, being the author of the binomial theorem, he discovered his entire theory of differentiation with its help. *MacLaurin (Colin)*, born in 1698, in Scotland, died in 1746. In 1720 (at the age of 22) he published [his] treatise on curves, which startled even Newton.

Taylor (J.Brook) born in 1685, in Edmonton (Middlesex), died in 1731(at the age of 46). He published [his] "*Methodus incrementorum directa et inversa* ", London, 1715-17, which, so to say, is the résumé of his theorem. Apart from this, he also published a number of mathematical and, some metaphysical works.

The direct source of this part of section IV of this manuscript (sheets 29-37) are §§ 204, 206, 207 (pp. 280-281, 283-285) of Lacroix's "Elements of Algebra", from which Marx took notes earlier, in the note book "Algebra I". Having written down the equation, subsequently obtained by Lacroix from the general equation

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} + \cdots + Tx + U = 0$$

by substituting $x = y + a$, where a is a multiple root of the general equation, Marx concludes (sheet 37):

This is also exactly the result, that would have been obtained by successive differentiation.

The last 11 pages of the note book (sheets 38-48 of the manuscript 4000) contain a few observations related to the binomial theorem, Taylor's series, as well as notes taken from the paragraphs on total differential from Boucharlat's, Hall's and, Sauri's books.

This part of the manuscript does not contain any sub-title : Marx only numbered his points.

Point 1 begins with the words (sheet 38) :

1) The *binomial theorem* (which can be extended to the *polynomials*) is the greatest discovery of *algebra proper.* Not only did the solution of the *equations* of determinate power, as it happened earlier, become possible with the help of this theorem,but also *the general theory of equations.*

After this Marx adduced MacLaurin's account of the method of obtaining the equations of higher orders, by multiplying the equations of first degree.

Point 2 (sheets 39 and 40) begins with the words:

2) The binomial theorem not only permitted development of the general theory of equations (including those with some unknowns), it also served the development of combinatorics, theory of trigonometric, exponential etc. functions ; it is the general *basis* of *differential calculus* ; and that is why, the question that naturally arises is : having discovered the binomial theorem, as well as the differential calculus, did not Newton derive it, or his disciples *Taylor and Maclaurin* (the former chronologically preceeds the latter) — though their generalising formulae made the technical application of the differential calculus unusually easy — did not they draw their results, even if *on the quiet*, from an application of binomial theorem? *

On sheets 41-46 Marx at first took notes from §§ 63, 64 (pp. 40-43) of Boucharlat's book, devoted to the differentiation of equations with two variables, and then from §§ 26-28 (pp. 15-18) of the same text book, related to the differentiation of composite functions. After that Marx returned to the differentiation of an equation with different variables and took notes from §§ 66-68 (pp. 44-45) of the same book, devoted to the concepts of total differential and partial differentials. Here Marx also took notes from the concluding § 70 (pp. 46-47) of this section of Boucharlat's book, related to the differentiation of inverse function, as well as from the Appendix 5 (p. 500) on the same theme.

Later on under the title : "*Another method for the general equation of different variables* " (sheet 46) Marx took notes from § 96 (pp. 87) of chapter VIII, "Functions of two or more variables", of Hall's book.

In conclusion he listed the formulae for the first, second and third differentials of the product of two functions, from Sauri's book (volume III, p. 3).

This manuscript came to an end with three observations of Marx (on sheets 47, 48) under the general title : "*In respect of Taylor's theorem and Lagrange's expansion (p.1, 29)*". The first of them is devoted to an application of Taylor's theorem to the approximate calculation of the increment of a function, according to Hind (§§ 81, 82, pp. 96-97).

The second is about the advantages of the Lagrangian notations for the derived functions.

The third gives the general characteristics of the instances of inapplicability ("the exceptions") of Taylor's theorem. Marx writes (sheet 48) :

III. The failures (so called) of Taylor's series occur when it can not give the development of the function $(x + h)$; it happens, when the *particular value* of the function is *inexpressible in integral and positive powers of h, in combination with finite coefficients.*

---

* On this question see the footnotes on pp. 90, 232 and 333. — Tr.

TAYLOR'S THEOREM, MACLAURIN'S THEOREM AND THE LAGRANGIAN THEORY OF DERIVED FUNCTIONS

S.U.N. 4001

It is a manuscript of 27 sheets, under the general heading :*"Taylor's theorem, MacLaurin's theorem and Lagrangian theory of derived functions"*. (In Marx's numeration pp 1-8, 5-6,4-5, an unnumbered page, 7-20). Apparently this manuscript is a rough draft of an intended general account of the entire material on this theme, which Marx noted earlier. Evidently, it is related to the 2nd half of the 70s of the last century, i.e., to that period, when Marx still defended the "algebraic" point of view of Lagrange, on the nature of differential calculus.

This manuscript has three sections, of which the first two have as their titles only the Roman numerals I and II, and the third carries the title : " *III. The Lagrangian theory of functions*".

Section I (sheets 1-2) has been published in full in the present volume (pp.88-89). Section II (sheet 2) begins as under :

II

1) Let us take the simplest expression of the binomial, for instance $(x+c)^m$.Since $x+c=c+x$, the numerical value of the binomial does not change at all, whether we write $(x+c)^m$ or $(c+x)^m$. Nevertheless, the serieses by which these two identical expressions are represented, have different forms. In the first case, derivatives of x are unfolded, whereas the second term c figures only as a factor in ascending powers, like h in Taylor's series. On the contrary, in $(c+x)^m$, where c is the first term, and x is the second, the derivatives of c are unfolded; whereas the second term x figures only as a factor in ascending powers, like the variable x in MacLaurin's theorem.

In connection with the fact, that the expressions in x, c, and correspondingly in x, h (in transition to $x+h$ from x) appear here, Marx latter on (on sheet 5) made the following insertion regarding this* :

We note here that: in the statement of (see p. 3) the binomial expansion we call $f(x)$, $f'(x)$ etc. the derived functions of x^m ; and this is permissible, since the concept of function at first emerged from [those] indeterminate equations, where there are more unknowns than equations, and that is why, the value of x changes, when, for instance, changes the value of y. Later on the concept of function was transferred to the *unknowns* in an equation, without taking stock of the known magnitudes, as the latter appear as constants. Finally, in the calculus of variables, it is spoken of, for example, about $f(a)$, when x takes a particular value a. That is why, we can, without being confused, speak about the functions of x in respect of the binomial $(x+h)^m$, as well as in respect of the binomial $(c+x)^m$ — about $f(c)$ and the derivatives $f'(c)$, $f''(c)$ etc.

With the aim of showing how, proceeding from the binomial theorem, it is possible to go over to such a generalisation of it, which must lead to the Taylor's theorem — Marx begins here (sheets 3-4) with a translation of Newton's binomial formula into the language of differential calculus; this has been done exactly in the way it was done in his manuscript 4000 (see p. 219).

Marx puts this intuitive transition, this conjecture, leading to the functions of a more general form, from the powered functions (let us recall, that for Marx, as well as for Lagrange, the issue was, still,

* On p.2 Marx wrote : +++ p. 5 ***. On p. 5 he made the insertion cited here, after writing : "ad ***". It begins with the words : "We note here that ". — Ed.

in essence, only the analytical functions of the real variables, i.e., the functions, which may be expanded into powered series) as follows (sheet 26) :

If now the equation has to become *general* then instead of $f(x) = x^m$ we must put $f(x) = y$, where the variable x has no determinate power, but is capable of any power, so that the function $(x+h)^m$ takes the general form $f(x+h)$. But what happened in the left hand side of the equation, must be repeated also in the right hand side, i.e., we must strike out the latter terms, provided by the power m of the binomial $(x+h)^m$, and substitute them by + etc. etc., in order to indicate the possibility of the infinite appearance of the new derived functions of the general $f(x+h)$.

Then we shall obtain

$$y_1 \text{ or } f(x+h) = f(x) \text{ or } yh^0 + Ah + Bh^2 + Ch^3 + Dh^4 + \cdots$$

and this is the basic equation, from which Taylor proceeds for the exposition of his theorem.

Thus,

$$y_1 \text{ or } f(x+h) = f(x) + \frac{dy}{dx}h + \frac{d^2y}{dx^2} \cdot \frac{h^2}{1 \cdot 2} + \cdots$$

Hence, for every given function x,in which x changes, i.e., turns into $x+h$, we should only compute the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$, i.e., the successive derived functions of x, and then restore again the factors $h, \frac{h^2}{1 \cdot 2}$ etc., extinct in the process of differentiation, in order to obtain the expansion for $f(x+h)$.

Thus, Taylor's theorem appears as a simple translation of the binomial theorem, from the algebraic language into the language of differential calculus.

Then Marx prefaced the proof of Taylor's theorem in the general instance, with a discussion once more of the question, as to whether Newton himself also discovered Taylor's theorem, proceeding from his binomial theorem.To this and to an analogous question regarding Taylor Marx's answer is as under (sheet 5) :

The question arises : did not Newton, having discovered, the binomial theorem, through a secret application of the latter, also discover, for his personal use, the theorem of Taylor, which is such an unusually simplified application of the differential calculus ? To this question, one ought to unconditionally answer in the negative*. In that case he would have made a brilliant display of this finding. Had he noticed this simple connection, then neither Taylor, nor MacLaurin, nor even Lagrange would have had anything more to discover, and the differential calculus would have, in essence, been completed by him.For, though Taylor's (and correspondingly MacLaurin's) theorem is directly related only to the functions of *one and only one independent variable,* it remains the basis also for the expansion of the functions of many variables, given in an explicit, as well as in an implicit form.

---

* We have indicated earlier, that the later developments in Newton-studies have provided an affirmative answer to this question. See, PV, p 90, 230 and 333. —Tr.

The same is more doubtful in respect of Taylor, who had at his disposal, on the one hand the Newtonian algebra ("Arithmetica Universalis"), and the differential calculus of Newton and Leibnitz on the other.

It is true, that one could have said, that in algebra the issue is always only the binomials of a determinate power m, as in $(x+h)^m$, whereas $f(x+h)$ includes every determinate power only as a moment, but absolutely excludes it as a boundary. However, this retort could have been turned against Taylor, for the binomial theorem has much greater generality, than his theorem.

The first permits h^{-1}, $h^{1/m}$, $h^{n/m}$ etc., more precisely, h in any possible power as factors of the functions of x, whereas Taylor's theorem is applicable (i.e., does not fail) only if the functions of x are such that, though they are indeterminate and capable of any change, $f(x)$, $f'(x)$ etc. are all finite expressions (which do not at all lose their variability) and, besides, in the expansion, their factors are ascending, positive and integral powers of h. But Taylor did not even attempt to demonstrate, that the indeterminate $f(x+h)$, which admits of any expansion, can be represented after the pattern of the binomial expansion. He, in fact, arouses suspicion, because, for example, in $(x+h)^m$, where owing to the indeterminateness of m, the series may be made infinite, he limited himself to writing for $(x+h)^m$ the series $f(x)+f'(x)h+\cdots+$ etc., forgetting, therein, that inspite of the endlessness of $(x+h)^m$, in so far as we left m indeterminate, its end is known to us, for the penultimate term can contain only x, and the last term can only be $x^0 h^m = h^m$ [163].

All the same it appears to me to be beyond any dispute, that Taylor did not have the slightest idea about this simple connection between his theorem and the binomial theorem. He acted entirely on the grounds of differential calculus, without returning to its sources.

Having noted, that Taylor proceeded from the equation

(A) $f(x+h)$ or $y_1 = y\,(\text{or } f(x)) + Ah + Bh^2 + Ch^3 + Dh^4 + \cdots$ etc.,

and dwelling once more upon the mode of emergence of this type of polynomial in algebra (in the general theory of equations), Marx concludes (sheet 6) :

Taylor changes nothing in the initial equation obtained with the help of the binomial theorem, apart from making $f(x+h)$ bereft of powers, that is, capable of any expansion, owning to which he also makes the right hand side inaccessible for completion with the help of + etc. He uses the binomial theorem (or, what is the same, the general form of an equation with one unknown, provided), only in so far as it gives him his initial equation, without the proof, which is applied here. The polynomial itself is considered by him from the stand point of differential calculus.

After this (sheets 6-7) Marx stated the proof of Taylor's theorem according to Boucharlat (§ 57, pp. 36-37), which he apparently ascribed to Taylor himself.

Further (sheet 8), he adduced an analogous, but more general proof of Taylor's theorem, according to Hind (§ 74, pp 83-84), where the initial expansion,

$y_1 = y + P\,h^\alpha + Q\,h^\beta + R\,h^\gamma + S\,h^\delta +$ etc.

contains not only the indeterminate coefficients $P, Q, R, S, \ldots\ldots$, but also the indeterminate indices of power $\alpha, \beta, \gamma, \delta\ldots.$ With this point 1) comes to an end.

Point 2) of section II (sheet 9) carries the title : "2) *MacLaurin's theorem*". With the aim of obtaining the induction of MacLaurin's theorem, starting from a generalisation of the binomial theorem of Newton, here Marx begins by applying the latter to the expansion of $(c+x)^m$, and then substitutes the coefficients of the powers of x in this expansion by indeterminate coefficients. Thus having obtained the equation

$f(x) = (c+x)^n = A\,x^0 + B\,x + Cx^2 + D\,x^3 + E\,x^4 + \cdots + ncx^{n-1} + x^n,$

he concludes further (sheets 26, the insertion "Zu MacLaurin, p.5"):

As earlier, we can, upon obtaining Taylor's [initial equation], generalise this equation into $f(x) = A + B\,x + C\,x^2 + \cdots$ +etc. etc. (without coming to an end), and this is the initial equation of MacLaurin.

In this connection Marx specially dwells upon the fact, that such an order of arrangement of the terms of a series is inverse, in respect of their arrangement in the polynomial, representing the left hand side of the general equation of n-th power; and that (sheet 27) thus :

. . .. we shall get rid of the term x^n, appropriate only for the equations of a determinate power, which, till the conversion of the series, formed its first term, and hence, which must now be its last term. It is substituted by + etc. With this the polynomial acquires that general form, which is essential, when we substitute $(c+x)^n$ or any other determinate function of x by the general expression $f(x)$, where $f(x)$ does not have any power, but includes all the powers in its expansion.Then we shall get the general :

$f(x)$ or $y = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$

— the basic equation, from which MacLaurin begins the exposition of his theorem.

Here the insertion comes to an end. The subsequent text (sheet 9) reads:

Thus, the starting point of MacLaurin's exposition

y or $f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$

is already an algebraic expression (with indeterminate coefficients, for the binomial $(c+x)^n$, in which the known c is the first term, and x— the second.Hence, in order to expand any arbitrary function of x, i.e., to represent it in the form [of a sum] of the products of its constant functions and the ascending integral powers of x, it is necessary only to translate this algebraic expression into the language of differential calculus, i.e., to find out the differential symbols for the coefficients A, B, C, D etc.

Here, as in manuscript 4000, Marx specially deals with the fact, that while in Taylor's series the coefficients are the derived *functions*, in MacLaurin's series the coefficients are constants. Then Marx describes as follows, the circumstance, where these constants are values of the derived functions of x, when $x = 0$, and that is why they may be found with the help of differential calculus:

But here, from the very beginning, there arises a difficulty, which is alien to Taylor's theorem. With the help of differential calculus, only the functions of the variables may be obtained *directly*, while what is at issue here is, conversely, the expansion connected with the variables of constant functions. On the other hand, differentiation is possible, only when x turns into x_1 or into $x+h$, as in Taylor's theorem. The issue here is not about the functions obtained as a result of the change in x thanks to the positive or negative increment, but about the representation of the general expression $f(x)$ in an expanded form, with the factors of x in

ascending power, as, for example in the ordinary algebra, which represents $f(x) = \frac{a}{a-x}$ with the help of successive division, in the form of the series

$$\frac{a}{a-x} = 1 + \frac{1}{a}x + \frac{1}{a^2}x^2 + \frac{1}{a^3}x^3 + \cdots$$

If now in $(c+x)^n$, we take x as a variable and expand this $f(x)$ with the help of the differential calculus (hence, turning $f(x)$ into $f(x_1)$ or into $f(x+h)$ etc.), then we shall get :

$$\begin{aligned} &(c+x)^n && = y && = f(x) \\ &n\,(c+x)^{n-1} && = \frac{dy}{dx} && = f'(x) \\ &\frac{n\,(n-1)}{1\cdot 2}(c+x)^{n-2} && = \frac{1}{2}\frac{d^2y}{dx^2} && = \frac{1}{2}f''(x) \\ &\frac{n\,(n-1)\,(n-2)}{1\cdot 2\cdot 3}(c+x)^{n-3} && = \frac{1}{6}\frac{d^3y}{dx^3} && = \frac{1}{6}f'''(x) \end{aligned}$$

etc.

Thus all the while we get new binomials, and not an independent expansion of the constant function in c. However, we attain the ultimate goal by assuming in all the binomials $x = 0$, i.e., after we have already used the variable for obtaining an expansion, we remove it.

The latter part of point 2) is again, as in manuscript 4000, devoted to the multiple roots of an algebraic equation. Marx begins it (sheet 11) with the words:

In order to indicate once more the algebraic element of the differential calculus, we shall further refer to MacLaurin's deduction proper.

Marx completes this deduction, consisting of the search for the indeterminate coefficients of the expansion $f(x)$ or $y = A + Bx + Cx^2 + Dx^3 +$ etc., with the help of successive differentiation and then with the supposition of $x = 0$, thus (sheet 11) :

Concerning the process of successive differentiation, specially applied here, [it may be said], that MacLaurin, in his "Algebra"— which, in his own words, is, in essence, a commentary on Newton's "Arithmetica Universalis" — developed this process purely algebraically, namely, in that part, where the successive lowering of the power of an equation by one, with the aim of finding out its multiple roots, as well as the detection of the unknown [roots], have been discussed.

In MacLaurin's "Algebra" the method of seeking the multiple roots has been enunciated only in the light of the examples of the equations of third and fourth power (see the description of manuscript 3933, pp. 206-208). That is why, here, as in manuscript 4000, Marx enunciates this method according to Lacroix's "Elements of Algebra", where the algorithm has a more general character(where it has been constructed in application to an equation of any power m). Underlining the circumstance, where the result turns out to be coinciding with what is obtained by successive differentiation of the initial equation, Marx explains(sheets 13-14) that coincidence as follows :

Regarding this algebraic form of successive differentiation, it should be noted that: we get the equation as a result of the assumption that $x = y + a$, i.e., $x - a = y$. If a itself is a root of the equation, then $x = a$, i.e., $x - a = 0$, *hence* $y = 0$. In $x - a = y$ the difference between x

and a has been posited, and besides, this difference $= y$. Then, if $x = a$, i.e., [if] the difference posited at first, is again taken away, then owing to this we get a two-fold result : on the one hand a as one of the roots of the equation or *a particular value of* x; on the other hand, $y = 0$. Thus, the setting up and removal of the difference appears, as in the differential calculus, as the supposition, on the one hand, of something positive, and on the other, as the supposition of 0.

The last part of section II (sheets 14-16), is devoted to MacLaurin's theorem as a particular instance of Taylor's theorem.It has been reproduced in the present volume (see, pp.89-90).

Point 1) of section III of the manuscript carries the title : "*III. The Lagrangian theory of functions*" (sheets 16-17). It has also been reproduced in the present volume (see, pp. 90-92).

After this, the manuscript contains a detailed account (according to Boucharlat, §§ 244-260, pp. 168-180) of the Lagrangian proof, that in the *general instance* $f(x+h)$ can be expanded into a series of integral ascending powers of h, and that such a series must be that of Taylor.

Going over to the last example, of this part of the manuscript, of an application of Lagrange's method for finding out the derivative of the product of two functions, Marx writes on sheet 24 :

Before making, in conclusion, a somewhat long comment on the method of Lagrange, let us consider at first, the simplest example of its application. Suppose, for instance, that $f(x) = uz$ *is to be differentiated.*

Marx's concluding remarks (point H) (sheets 25-26) have been reproduced in the present volume (see, pp. 91-92)

OTHER MANUSCRIPTS ON THE DIFFERENTIAL CALCULUS

S.U.N. 4002

Some separate sheets(four in all), with Marx's numbering 1,2 and 1,2, carrying respectively the titles : "*Taylor's Theorem*" and "1) *Taylor's Theorem*". The contents of both the pairs of sheets coincide. Most likely, one pair of sheets served as the draft for the other. Contentwise it corresponds to those parts of the manuscripts 4000 and 4001, where Taylor's theorem has been obtained by induction, from Newton's binomial theorem. However, in this connection here it has been said, that :

To all appearance, Taylor did not in fact arrive at his discovery so simply.

The manuscript stops suddenly at the heading : "*MacLaurin's Theorem*".

S.U.N. 4003

A few sheets, in the main containing calculations (in part also notes), which Marx jotted down, evidently, while reading the text-books by Hind, Boucharlat and others ; in all 26 sheets (Marx's numeration :1-7, III, further 6, a page without number, 3-4, again a page without number).

Sheets 1-7, entitled : "*Lagrange (Derived Functions)*", numbered 1-7 by Marx, contain: calculations related to §§ 95-97 (pp. 120-127) of Hind's book, in which Lagrange's method has been enunciated ; a quotation (with Marx's reference to Hind) from § 99, wherein it is stressed that in practice, the expansion into a series is carried out with the help of the usual methods of differential calculus, and not following Lagrange ; and finally, calculations, related to § 74, to which Marx turned in connection with the reference, found by him in § 96, to an analogous method (of indeterminate indices of power), applied earlier.

Sheets 8-9, carry the title:"*Lagrange's Method*". It is the beginning of the conspectus of § 244 (pp. 168-169) of Boucharlat's book. The language is English. This note abruptly comes to an end with the calculations related to an example borrowed from Hall (p. 3).

Sheets 10-18 are related to the instances of applicability of Taylor's theorem. The first two instances

$$y = x^2 + \sqrt{x-a} \quad \text{and} \quad y = \frac{a}{(b-x)^3}$$

are from § 77 (pp. 92-93) of Hind's book;the second two :

$$y = b + \sqrt{x-a} \quad \text{and} \quad y = \frac{1}{(x-a)}$$

are from § 69 (pp. 52-53) of Hall's book.Sheets 19-22 contain rough calculations related in the main to the deduction of Newton's binomial theorem with the help of Taylor's series.

Sheet 24 contains extracts and diagrams from § 88 (pp. 111-112) of Hind's book. In this paragraph Hind attempted to substantiate Newton's method of fluxious with the help of the method of limits.

Sheets 25 and 26, contain the differentiation of xy according to Leibnitz and Poisson; the notes are taken from Sauri, volume III, p. 3 and Hall, p. 4. It is being reproduced here in full :

THE DIFFERENTIATION OF xy

1) *Leibnitz's method.*

$f(x, y) = xy$

$df(x, y) = (x + dx)(y + dy) - xy,$

$df(x, y) = xy + x\,dy + y\,dx + dx\,dy - xy,$

$\therefore\ df(x, y) = x\,dy + y\,dx + dx\,dy,$

We neglect $dxdy$ as an infinitesimal of 2nd order. (As to this neglecting, the same in Newton, only [the] notation [is] different.)

$\therefore\ df(x, y) = x\,dy + y\,dx\,.$

2) *According to Poisson.*

If we have yz, then according to the general proposition:

$y_1 = y + Ah + Bh^2 + \cdots,$

$z_1 = z + A_1 h + B_1 h^2 + \cdots$

We know from the same proposition:

$\dfrac{dy}{dx} = A$ and $\dfrac{dz}{dx} = A_1\,.$

Multiplying the 2 equations:

$z_1 y_1 = zy + Azh + Bzh^2 + \cdots + A_1yh + AA_1h^2 + \cdots + B_1yh^2 + \cdots$

Hence :

$$\frac{z_1y_1 - zy}{h} = Az + A_1 y + (Bz + AA_1 + B_1y)\,h + \cdots$$

making $h = 0$:

$$\frac{z_1y_1 - zy}{0} = Az + A_1 y\,, \text{ [and] } \frac{d\,(zy)}{dx} = Az + A_1 y\,.$$

Putting in the values of A and A_1 :

$$d\,(zy) = z\,dy + y\,dz$$ [164].

THE NOTE BOOK "A. I."
A NEW SYSTEMATISATION OF THE MATERIAL ACCORDING TO THE COURSES OF HIND AND BOUCHARLAT

S.U.N. 4036

This a note book carrying the title "*A. I.*". (It seems that the large Roman numeral I was written later on, in pencil).In all there are 42 pages; in Marx's numeration first comes 1a, then the pages 1-35, then pages 37-41, and the last page is without number.

It is a systematic conspectus of the first three chapters of Hind's book, with a few insertions from Boucharlat's book. In the note book"B(continuation of A). II" Marx already took notes from the fourth chapter of Hind's book, the beginning of which is devoted to the differentiation of trigonometric functions (see, manuscript 4038). But in note book "A.I." Marx anticipates this section in his section IV, containing the notes from chapters I and II of : Hind, "The Elements of Plane and Spherical Trigonometry", 3rd. ed., Cambridge, 1837, pp. 7-46. He makes an analogous insertion, devoted to logarithms, in connection with the differentiation of logarithmic and exponential functions.

In section I (sheets 1-3), corresponding to chapter I of Hind's book, devoted to the "definitions and preliminary observations", Marx noted only the examples 1-9 of Hind (pp. 20-24), numbering them with the first small letters of the Greek alphabet ($\alpha - \iota$). In all these examples the derivative is sought, proceeding directly from its definition as the limit or the "last value" of the ratio $\frac{\Delta u}{\Delta x}$.

In the examples α) and β), the functions $u = ax$ and $u = ax^3 - bx^2 + cx - e$, respectively, are differentiated. Marx wrote a note on the margins of the example β). This example (as it was written by Marx) (sheet 2) and the corresponding note (comment) of Marx is being reproduced here in full.

β) $u = ax^3 - bx^2 + cx - e$; $u_1 = ax_1^3 - bx_1^2 + cx_1 - e$;

$u_1 - u = a(x_1^3 - x^3) - b(x_1^2 - x^2) + c(x_1 - x)$;

$u_1 - u = a(x_1 - x)(x_1^2 + x_1x + x^2) - b(x_1 - x)(x_1 + x) + c(x_1 - x)$;

hence :

$$\frac{u_1 - u}{x_1 - x} \text{ or } \frac{\Delta u}{\Delta x} = a(x_1^2 + x_1 x + x^2) - b(x_1 + x) + c .$$

/If the increment of x be diminished sine limite x_1 becomes $= x$.

$$\therefore \quad \frac{du}{dx} = 3ax^2 - 2bx + c \ ;$$

$$\therefore \quad du = 3ax^2\,dx - 2bx\,dx + c\,dx \ .$$

β) It should be noted that here is a difference from α). There*we had $\frac{\Delta u}{\Delta x} = a$, and that is why when it turns into $\frac{du}{dx}$ nothing changes, apart from the form of the left hand side. In β)

* A slip of pen in the manuscript :"here"has been written, where as, it should be "there". —Ed.

both the sides : with the decrease of the increment of x sine limite x_1 becomes $=x$, and this gives us at the same time, in the right hand side the first *derived function* of x and in the left hand side the conversion of $\frac{\Delta u}{\Delta x}$ into $\frac{du}{dx}$.

But when $x_1 = x$, then in reality we get

$$\frac{\Delta u}{\Delta x} = \frac{u_1 - u}{x_1 - x} = \frac{0}{0},$$

so that it is only "seemingly"managed with that, which does not linger herein ; but the advantage of elementary deduction consists of this, that in the right hand side 0 is no more met with as a factor, exterminating the determinate terms; *rather, conversely,* it is *immediately* detected, that the conversion of x_1 into x gives a new function of x in the right hand side, meanwhile in the left hand side it is shown, through the conversion of $\frac{\Delta u}{\Delta x}$ into $\frac{du}{dx}$, that this new function is the limiting and ultimate value of $\frac{\Delta u}{\Delta x}$.

In the example γ), where the function

$$u = \frac{a^2 + x^2}{a - x}$$

is differentiated directly from the definition of the derivative, after the words:

"If we do the same with the help of *differential calculus*, then"

Marx adds the deduction of $\frac{du}{dx}$ according to the rule for differentiating the quotient.

In section II (sheets 4-13), which is a short conspectus of chapter II ("On the differentiation of algebraic functions of one independent variable") of Hind's book, Marx took down, for the most part, only the formulae and examples contained in this chapter, without adducing any proof, not even for the differential of a product. However, Marx completed the calculations pertaining to these examples, in full, sometimes even in greater detail, than did Hind. Bulk of these pages of the manuscript contains only calculations.

Contentwise section III (sheets 14-26) is related to chapter III ("On the differentiation of exponential and logarithmic functions of one independent variable", §§ 34-46) of Hind's book. Having noted the whole of § 34, in which the derivative of an exponential function is sought with the help of a reference to the fact, that "it was demonstrated in algebra", that

$$\log a = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \cdots,$$

Marx makes an insertion (sheets 14-17) from Hind, ch. VII, pp. 154- 159, under the title:"*Insertion from trigonometrical algebra*". Its content in fact coincides — right upto the mistake in the sixth decimal place of the expansion for e — with §§ 13 and 14 of section VII of the manuscript 3933 ("Algebra II") described in pp. 186-187.

At the end of the insertion on sheet 17 Marx wrote : "*End of the insertion*", and then under the heading : "*continuation of p. 13(prior to the insertion)*" he returned to § 34 of Hind's book, this time taking notes from it very briefly. After this he took notes from §§ 35-37, and also from example I of § 45 of chapter III of Hind's book.

In the insertion the serieses for a^x and $\log x$ were obtained by applying the binomial theorem. While deducing the derivative of a^x; proceeding directly from the definition of the derivative, Hind used the series for $\log a$. In Boucharlat's book the derivative of a^x was sought with the help of the expansion of the expression a^{x+h} into a series in powers of h and by looking for the coefficient of the first power of h. The latter was derived by using MacLaurin's theorem and the method of indeterminate coefficients.

In this connection Marx once again turned his attention to the connection between the methods of differential calculus and those used in the courses of algebra. He wrote on sheet 19 :

If we again compare with this, the development of the same problem, resting upon the differential calculus itself, then we shall again observe, how in Taylor's and MacLaurin's theorems, a *simple translation* from one mode of expression into another, in which 1) the common basis, which is also the binomial theorem, is used by MacLaurin's theorem, for *further development,* which gives us $f(x)$ [[in the given case $y = a^x$]] expanded into a series of integral, positive and ascending powers of x, where, then it is assumed that $x = 0$, in order to find out that which is in fact required to be found out, i.e., the coefficients A etc., which are [functions] derived from the *constant* a. Let us examine this deduction.

After this Marx took detailed notes from §§ 36-40 (pp. 24-28) of Boucharlat's book devoted to the differentiation of the exponential function, and then again returned to Hind, and again took detailed notes from §§ 37-46.

Section IV carries the title : *"Preliminary recapitulation of the trigonometric developments"*. It is a detailed conspectus from the text book by Hind, mentioned above and, from : Hall Th. G., "A Treatise on Plane Trigonometry", London, 1833, ch. I. Here the deduction of ordinary trigonometric formulae, required for the differentiation of trigonometric functions, has been discussed. Pages 1a and 41 of this note book contain the usual proof of the theorem about the differential of product.

The content of the last, unnumbered page is related to the next note book entitled:"B (continuation of A). II". It is the beginning of the first draft of the work on differential. It comes to an end with the words :*"See, further note book II, p.9"*. This page has been reproduced in the present volume (see, pp. 40-42).

"II. NOTE BOOK I"

CONTINUATION OF THE SAME MATERIALS

S.U.N. 4037

It is a note book carrying the title :*"II. Note book I"*. 18 completed pages. Conspectus of the text books by Boucharlat and Hind on differential calculus.

Sheets 1-3 begin with the heading :*"I.A) Maxima et Minima of functions of one variable"*, then follows the Roman numeral "I". Conspectus of §§ 91-101 (pp. 64-70), related to the section under the same title, of Boucharlat's text book (French edition, 1838).

Sheets 4-18 begin with the title :*"II. Additionally on Maxima and Minima"*. Conspectus of chapter VII (§§109-120 and 123-124; pp. 147-171, 173-177) of the text book by Hind. From the last §124 Marx notes only example 1; the last three pages (177-179) of chapter VII remain unnoted.

THE NOTE BOOK "B (CONTINUATION OF A). II"

FIRST DRAFTS OF MARX'S OWN POINT OF VIEW ON THE NATURE OF DIFFERENTIAL CALCULUS AND DRAFTS OF THE HISTORICAL ESSAY

S.U.N. 4038

39 completed pages ; it carries the title : "*B (Continuation of A).II*" (the large Roman numeral II, is written in pencil, like the numeral I in the note book with the title "*A.I*"; see, the manuscript 4036).The first page is unnumbered, then follows Marx's numerations pp. 1-37, the last page of the note book is again unnumbered.

Sheet 1 (an unnumbered page). It contains a name index, years of birth and death [of the authors] and the titles of the classics of differential calculus. It has been reproduced in the present volume under the title : "*A page of the note book entitled "B (continuation of A). II*"(see, pp. 65-66).

Sheets 2-8 (Marx's 1-7), carry the title : "*IV(continuation of IV A). Differentiation of trigonometric functions*". Marx's conspectus of the paragraphs devoted to the differentiation of trigonometric functions, successively from the courses of : Sauri (vol. III, § 27, pp. 36-37), Hall (ch.II, §§ 29-31, pp. 18-19), Hind (?) (ch. I, §18, pp. 22-23), Hemming (?)(ch. III, § 30, p. 23), Hind (ch. IV, §§ 47-48, pp. 46-47) and, Boucharlat (§§ 43-51,pp. 30- 33). (The notes of interrogation indicate, that the source has not been established with complete certainty.)

Marx began his notes with § 27 of Sauri's book (Sauri still adhered to Leibnitz's method, and that is why, for him sin of an infinitesimally small arc was simply equal to the arc itself). Marx enunciated this point under the title (p.1):

"a) *One of the simplest forms of the account based on the differential calculus itself* ".

In Hall's book(which proceeds from the method of Lagrange) the augmented value of the function u is expressed in the form:

$$u + \frac{du}{dx} + U h^2 ;$$

on this Marx observes within brackets (sheet 3, point b)):

(where U is the representative of all further coefficients with higher powers of h).

The following method, noted by Marx in point c), is based on the equality

$$\frac{u_1 - u}{x_1 - x} = \cos\frac{1}{2}(x_1 + x)\,\frac{\sin\dfrac{x_1 - x}{2}}{\dfrac{x_1 - x}{2}},$$

where $u = \sin x$, and in transition to the limit when $x_1 \to x$. Hind applied such a method in § 18, in the example of the differentiation of the function $u = \sin 2x$, illustrating the definition of the derivative. Marx applied the same method for finding out the derivative of the functions $u = \sin x$ and $u = \cos x$. Was this paragraph of Hind's book Marx's source here or did he have some other source at his disposal? This we could not ascertain. Further, going over to the method enunciated by him in point d), Marx wrote (sheet 4) :

d) The previous method (c) presupposes the basis of differential calculus only in the *final translation into its language,* of the result, which is obtained *without the help of this* calculus.

This happens in yet fuller from, in the following method, which proceeds directly from the *finite differences*, turning into differentials in the end.

Here, the method, utilising the formula

$$\sin(x+\Delta x)-\sin x=2\cos(x+\frac{\Delta x}{2})\sin\frac{\Delta x}{2},$$

has been discussed. Having written down this formula, Marx continues (sheet 4) :

From the stand point of finite differences it is concluded further that:

if the arc Δx is *very small*, then $\cos(x+\Delta x)=\cos x$, *approximately*, and $\sin\frac{\Delta x}{2}$ is nearly $=\frac{\Delta x}{2}$.

Hence,

$\sin(x+\Delta x)-\sin x$ or $\Delta\sin x=2\cos x\cdot\frac{\Delta x}{2}=\cos x\cdot\Delta x$, *nearly*.

Translated into differential form, it gives $d\sin x=\cos x\,dx$, where Δ is simply substituted by d, and the sign for *nearly* = by the = sign.

It could not be established, from where Marx borrowed this (entirely non-strict) deduction. In Hemming's book, which also begins with the above mentioned equality, the deduction of the formula $dy=\cos x\,dx$, where $y=\sin x$, is obtained by dividing Δy (i.e., $\Delta\sin x$) by Δx and through a transition to the limit.

Going over to the next point Marx wrote (sheet 5) :

e) This (differential) method differs from the previous ones, emanating from differential calculus (i.e., from a) and b)), in this that, the increment is not instantly considered as the differential.

Here Hind's method (§§ 47, 48, pp.46, 47) has been discussed. In this book the function $u=\sin p$, where p is, in its turn, a function of x—is differentiated, proceeding from the equality

$$\frac{u_1-u}{h}=2\cos(p+\frac{1}{2}i)\frac{\sin\frac{1}{2}i}{h}=\left[\cos(p+\frac{1}{2}i)\frac{\sin\frac{1}{2}i}{\frac{1}{2}i}\right]\frac{i}{h},$$

through a transition to limit when $h\to 0$. Here the increments i and h of the variables p and x are in fact not identified directly with the differentials dp and dx.

In that last point f) of this section from §§ 43-51 (pp. 30-33) of Boucharlat's book, notes are taken under the title : "*f) (Boucharlat)*" (here, elsewhere Marx did not indicate his sources). Here the starting point is the equality

$$\frac{\sin(x+h)-\sin x}{h}=\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}$$

which is transformed into the equality

$$\frac{\sin(x+h)-\sin x}{h}=\frac{\sin x(\cos h-1)}{h}+\frac{\sin h\cos x}{h}.$$

When $h = 0$, both the terms of this equality turn into $\frac{0}{0}$, and Boucharlat, who is in fact required to compute $\lim\limits_{h \to 0} \frac{\cos h - 1}{h}$, says in this connection (we reproduce here Marx's note on sheet 6) :

"Hence, we must give this term $\left[\frac{\cos h - 1}{h}\right]$ *another form*".

After this Boucharlat transformed the expression $\cos h - 1$ into $\frac{\sin^2 h}{\cos h + 1}$ and then the transition to limit is effected by the simple assumption of $h = 0$ and $\frac{\sin 0}{0} = 1$.

Sheets 9-16 (Marx's pp.8-15). The text begins with a reference provided by Marx:"*(see, the beginning of this in note book I, repeated on p.10 of this note book)*". Here Marx has in view the last(unnumbered) page of manuscript 4036.

This text is the first draft of Marx's work on the differential. It has been published in the present volume (pp. 41-52). It occupies the following sheets : the last unnumbered page of the note book "A.I" (the beginning) (sheet 42), (sheet 11, p. 10 of the present note book under the title:"*Here is the beginning of page 8*", sheet 12 (p. 11) upto the words :"*further on p.8 top and p. 9*", sheet 9 (p. 8), after this the remaining part of the text continues on sheet 12 (p. 11), sheet 10 (p.9) and sheets 13-16 (pp. 12-15).

Sheets 16-35 (Marx's pp. 15-34). Drafts of the essay on differential calculus. These have been published in the present volume (pp. 64-86) under the title : "On the History of Differential Calculus".

Sheets 36-38 (Marx's pp. 35-37). These pages contain the beginning of a note, which Marx did not continue. He did not even formulate clearly, what, namely, he wished to say. The beginning of this note reads (sheet 36) :

from it the secret of differential calculus can not be extracted directly; it can be found only from the opposite method, where the augmented value of x appears only as x_1, i.e., the difference retains its difference-form $x_1 - x$, also in the subsequent deduction. Hence, the secret which remains hidden in the method used by us, is required to be revealed.

The meaning of the subsequent part of the text is not sufficiently clear. We may only surmise, what Marx intended to explain, namely : why, for obtaining the derivative, at first the difference $x_1 - x$ different from 0, is required to be formed, and then removed. Actually, here considering the function x^2, Marx at first formed the difference $x^2 - x^2$, which is represented in the form of $(x + x)(x - x)$, whence dividing both the parts by $x - x$, he obtained further $x + x = 2x$. After that he wrote (sheet 37) :

But though till now we deduced correctly : from x^2 we went over to the difference $x^2 - x^2$, then obtained $(x - x)(x + x)$ from this difference, and from the latter — the divisor $x - x$, i.e., all the time we have applied only the expressions derived from the initial x^2 ; however, during this time we forgot that $x - x = 0$ in the expression $(x - x)(x + x)$, that is also in $0 \cdot (x + x) = 0$, before any further division by $x - x$ may occur. Nevertheless, this development showed us :

1) Persistently drawing upon the treatment of x^2 and x^2, whence also of x and x,as different magnitudes, that is also of $x^2 - x^2$ and $x - x$ as *actual differences*, we could find out the derivative of x^2, namely $2x$. But how?

2) After violating all the rules of algebra, we still, in the end, treated $x - x$ in the expression $(x - x)(x + x)$ as an actual difference, i.e., acted as if the first x is a magnitude different from the second x, and that is why permitted ourselves to divide $(x - x)(x + x)$ by $(x - x)$, [but] as soon as we obtained the positive expression $x + x$ without any other factor,apart from 1, we suddenly remembered that, x and x are not at all different, but are identical magnitudes and that is why $x + x = 2x$.

Here the note abruptly comes to an end. Later, on sheet 38, under the title: "*3) ad to p.35*" the deduction of the derivative $2x$ from x^2, by forming the differences $x^2 - x^2$ and $x - x$, written on sheet 36, is repeated (with some other expressions).

The last, unnumbered page of the note book (sheet 39), contains only some calculations related to the determination of the subtangent to parabola.

SOME SEPARATE SHEETS CONTAINING MATHEMATICAL CALCULATIONS

S.U.N. 4040

A double-page writing paper (pp. 1-3), containing a summary of the formulae of differential calculus. Source : Hind's book, chapters I ,II.

S.U.N. 4048

Some separate sheets (10 in all), containing calculations related to various questions, referred to by Marx in his notes. These sheets do not contain anything new in comparison to the notes. On one sheet the general equations for the curves of second and third order have been written.

NOTES ILLUSTRATING D'ALEMBERT'S METHOD AS EXEMPLIFIED BY THE DIFFERENTIATION OF A COMPOSITE FUNCTION

S.U.N. 4143

A manuscript of 14 pages; Marx numbered them with the small Latin Letters from a to n.

Sheets 1-7 (Marx's pages a to g). Marx's notes illustrating d'Alembert's method as exemplified by the differentiation of a composite function. These have been reproduced in the present volume, under the title: "*Analysis of d'Alembert's method in the light of yet another example*" (pp. 102-106).

Sheets 8-9(Marx's page h, i) carry the heading : "*Lagrange*". These sheets contain calculations related to the differentiation of the product of two functions u and z, according to Lagrange's method, i.e., by representing the product of the augmented values of the functions in the form of the product of two serieses

$$u + ph + \frac{1}{2}qh^2 + \cdots \text{ and } z + p_1h + \frac{1}{2}q_1h^2 + \cdots$$

and a search for the coefficients of the first power of h. Source : Hind, § 96 (pp. 126-127).

Sheets 10-14 (Marx's (turned over) k to n), devoted to the differentials of second order. Here the question specially discussed is : how to differentiate expressions of the type $f'(x)dx$. Thus on sheets 12 and 13 we read :

I

$$x^3, \quad 3x^2 = f'(x), \quad 6x = f''(x).$$

I) $3x^2$ or $f'(x) = \dfrac{dy}{dx}$.

If we proceed from $f'(x)$ as $\varphi(x)$, i.e., assume that $\varphi(x) = 3x^2$, then moving along the aforementioned path, we shall get :

II) $6x = \varphi'(x) = \dfrac{d\varphi}{dx}$. Here φ is written only to make it different from the first $f'(x)$.

$$d(3x^2) = d f'(x) = d\left(\frac{dy}{dx}\right),$$

$$dy = f'(x)\,dx, \quad d(dy) \text{ or } d^2y = d(f'(x)\,dx).$$

If dx is a constant, then [it] $= d(f'(x))\,dx$.

$$\therefore \quad \frac{d^2y}{dx} = d(f'(x)).$$

But $d(f'(x)) = f''(x)\,dx$,

$\therefore \quad \dfrac{d^2y}{dx} = f''(x)\,dx \quad \text{or} \quad \dfrac{d^2y}{dx^2} = f''(x)$.

[[If we take $3x^2$ as the first derivative, then it is equal to $f'(x)$ and $\dfrac{dy}{dx} = f'(x)$.

That is why :]]

a) $3x^2 = f'(x) = \dfrac{dy}{dx}$. Thus, $dy = f'(x)\,dx$.

That is why :

$d^2y = d\,(f'(x)\,dx) = f''(x)\,dx^2$.

b) $6x = f''(x)$.

If dx is taken to be a constant, then

$$d^2y = d\,(f'(x)\,dx),$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d\,(f'(x))}{dx}, \text{ but } \; d\,(f'(x)) = f''(x)\,dx.$$

Hence

$$\frac{d^2y}{dx^2} = f''(x)\,.$$

In this method the formula $\dfrac{d^2y}{dx^2} = f''(x)$ is obtained, only owing to the hypothesis, that

$$d\,(f'(x)\,dx) = d\,(f'(x))\,dx\,,$$

i.e., that dx is taken as a constant. This account may serve as a proof only if there is no arbitrariness in the hypothesis that dx was a constant, in the result $dy = f'(x)\,dx$, where now dx is not considered as $dx = d\,(f(x))$, but as the differential of the magnitude x. If it turns out to be a constant in the first differentiation, then it so happens also in the second.

ON THE NON-UNIVOCALITY OF THE TERMS "LIMIT" AND "LIMITING VALUE"

A COMPARISON OF D'ALEMBERT'S METHOD WITH THE ALGEBRAIC METHOD

S.U.N. 4144

8 sheets of notes. Marx numbered them with Latin letters from A to H. It has two parts (A-D and E-H). These have been published in the present volume, under the titles: *On the non-univocality of the terms "limit" and "limiting value"* (pp. 96-98) and *Comparison of d'Alembert's method with the algebraic method"* (pp. 99-101) respectively.

ROUGH NOTES ON THE DIFFERENCES BETWEEN THE METHODS OF MARX AND D'ALEMBERT

S.U.N. 4145

Two sheets of rough notes devoted to explaining that the difference between the methods of Marx and d'Alembert, is not to be simply reduced to the fact that, instead of h, $x_1 - x$ is written. With this aim Marx differentiates the function $f(x) = x^2$, representing at first, its augmented value, in the form of $f(x + h)$, and then in the form of $f(x + (x_1 - x))$. Applying the binomial theorem in both the cases, he obtains the derivative as the coefficient of h in the first power, corresponding to $(x_1 - x)$. After this Marx finds out the derivative of the same function, representing at first, the difference $x_1^2 - x^2$ in the form of the product $(x_1 - x)(x_1 + x)$, and then by dividing it by $x_1 - x$ and assuming finally, that $x_1 = x$. In this connection regarding the first method Marx observes :

Here we have the desired function $2x$ at once in the ready-made form, in the first derivative, as the coefficient of $(x_1 - x)$; it is obtained directly with the help of the binomial theorem.

On the second method Marx wrote an incomplete sentence in the margin, after which dots follow:

Hence, when not only $x_1 - x$ is to be removed, but when the assumption $x_1 = x$, or $x_1 - x = 0$, gives $x_1 + x = 2x$, i.e., the function $2x$ is deduced through the reduction of $(x_1 - x) \times (x_1 + x) \cdots$

DRAFT MANUSCRIPTS ON THE CONCEPT OF DERIVED FUNCTION

S.U.N. 4146

A manuscript of 9 sheets. It contains: a) draft of the manuscript "On the concept of the derived function" (see PV, 19-25 and the description of manuscript 4147, given below), sheets 1-8 (Marx's numbering : 1-6, 5 and two pages without number) — the different readings found in different places of the draft and in the fair copy of the published manuscript " On the concept of derived function", have been indicated there in foot-notes; b) the observations contained on sheet 9 — these are being reproduced below, in full.

ON SUBSTITUTING THE SYMBOL $\frac{0}{0}$ BY THE SYMBOL $\frac{dy}{dx}$

I

It has been shown, for instance, that

1) When

$$y = x^m = f(x), \quad y_1 = x_1^m,$$

we get

$$\frac{dy}{dx} \quad \text{or} \quad \frac{0}{0} = mx^{m-1}.$$

It was shown, that the derived function $f'(x)$ or mx^{m-1} is obtained from the initial

$$f(x) = x^m,$$

by assuming $x_1 = x$, i.e., $x_1 - x = 0$.

But this assumption of $x_1 - x = 0$, or of $x_1 = x$, turns $\frac{y_1 - y}{x_1 - x}$ into $\frac{0}{0}$, and instead of the latter we write $\frac{dy}{dx}$, in order to indicate the origin of this $\frac{0}{0}$, i.e., [to indicate] what sort of a ratio of actual differences, in the aforementioned case of $\frac{y_1 - y}{x_1 - x}$, in the end turns into $\frac{0}{0}$.

This is all the more validated, owing to the fact that as result we get

$$\frac{0}{0} = mx^{m-1} = f'(x),$$

and in the left hand side of the equation, this result $\frac{0}{0}$ was obtained, thanks to the movement which ensued from the variable x standing on the right hand side.

$\frac{0}{0}$ may be = any magnitude X, as

$$0 = X \cdot 0 = 0.$$

Since here $\frac{0}{0}$ is not equal to an arbitrary X, but $= mx^{m-1}$, so through $\frac{dy}{dx}$ or $\frac{df(x)}{dx}$ we may indicate : in consequence of what sort of movements of the independent variable x, in a determinate function $f(x)$, did the symbol $\frac{0}{0}$ emerge.

2) However, since the meaning of $\frac{dy}{dx}$, whose special value naturally changes depending upon the determinate form of $f(x)$ itself, has been fixated once and for all, as soon as we operated on the grounds of differential calculus; the task is [thereby] inverted. Namely, the special value of $\frac{dy}{dx}$, like, for example, mx^{m-1} above, i.e., the derivative to which it corresponds, is required to be found out through differentiation.

ON THE CONCEPT OF THE DERIVED FUNCTION

S.U.N. 4147

The first among Marx's works of 1881, on the nature and history differential calculus. In it Marx introduced the concept of algebraic differentiation and the corresponding notations for the process of finding out the derivative, for a certain class of functions. It was written in the form of a letter to Engels, on 8 sheets of writing pad, with an envelope attached to it carrying the heading "*For General*" (sheet 9). It has been reproduced the present volume, under the title "*On the concept of the derived function*" (pp. 19-25). On this manuscript also see the Preface and note[1].

PRELIMINARY DRAFTS AND VARIANTS OF THE MANUSCRIPT ON THE DIFFERENTIAL

S.U.N. 4148

Three groups of sheets, photocopies, described below in points a), b) and c), have been united in this archival unit.

a) Photocopies of 6 sheets of double page writing papers, sheets 1-14 (Marx's numbering : 5-15 (14 twice), and photocopies of two separate pages 16 and 17. These have been published in the present volume : sheets 1-10 (pp. 5 - (first) 14) under the title : "Second draft" (see pp. 53-59 and note[41]); sheets 11-14 (pp.(second)14-17) under the title : "Third draft "(see, pp. 60-62). Photocopies of the pages 1-4 of the second draft are missing (see, note [41]).

b) Sheets 15-27 (in Marx's numbering : pp.ad 3, once more ad 3, 4-9, unnumbered, 11-13) — draft of the manuscript "On the differential" (see below manuscript 4150). Variants of parts of this manuscript have been indicated in the foot notes to it (see, pp. 26-39). The paragraph reproduced below is from the draft. It is absent in manuscript 4150. With it point 4) of the draft, corresponding to point 4) of section I of the fair copy (sheet 20), comes to an end.

The initial course consisted of this : forming the difference $f(x_1) - f(x)$, to obtain at first

$$\Delta f(x) = \Delta y ,$$

and then $\frac{\Delta f(x)}{\Delta x}$ or $\frac{\Delta y}{\Delta x}$, with the aim, of finally deriving from there,the derivative $f'(x) = \frac{dy}{dx}$, by assuming $y_1 - y = 0$, to obtain in conclusion $dy = f'(x)\, dx$. dy is the *last* symbolic result of the difference-form. Conversely, dy becomes the starting point, as Δy earlier. With the help of operations indicated by it $f'(x)\, dx$ is obtained, and in conclusion we obtain $\frac{dy}{dx} = f'(x)$ from $dy = f'(x)\, dx$, meanwhile, as in the initial deduction, $dy = f'(x)\, dx$, i.e., the differential of y, is obtained as the last result from $\frac{dy}{dx} = f'(x)$.

This paragraph was written on a separate page,which Marx numbered with the numeral 6. After it, there is a sentence on this page, with which the next point 5) begins.

5) Let us now examine the differential $dy = f'(x)\, dx$.

This sentence is not there in the fair copy, where this point 5) corresponds to point 1) of section II. In this connection it may be mentioned that the remaining part of p.6 of the draft is blank ; in the fair copy this page is absent.

c) Sheets 28-30, are, content-wise, related to the differentiation of the product of two functions and to the search for the second differential and the second derivative. The sheets not numbered by Marx have the character of incomplete fragments. They do not contain anything new, in comparison to the material [herein] published.

FOUR VARIANTS OF THE DRAFTS OF ADDITIONS TO THE MANUSCRIPT ON THE DIFFERENTIAL

S.U.N. 4149

In the first draft, the most complete initial plan of this supplement to the manuscript "On the differential" (see, the description of the manuscript 4150) has found expression. Apparently it plays the role of an outline. This draft contains ten pages and consists of four sections, with the following titles provided by Marx :

Sheets 1-4. "*A) Additionally on the differentiation of* xy", p.1 — beginning of p.4.

Sheets 4-5. "*B) The equation* $y^2 = ax$ ", pp.4-5.

Sheets 6-9. "*C) Differentiation of* $\frac{u}{z}$ ", pp. 6-9.

Sheets 9-10. "*D) Implicit form*", end of p.9-p.10.

In the second draft, sheets 11-16 (Marx's pp. 1-5 and one page without number), the first two sections (sheets 11-13 and 13-15) have been fair copied. First of all, here Marx corrected a slip of pen in the title of section A) : struck out xy and wrote uz. However, here too, we have many cancellations at the end of every section. It shows that Marx was not satisfied even with this new version.

In this connection it is of interest to note that in the following two drafts (third and fourth) these first two sections do not exist at all. Apparently, later on Marx dropped the intention of including them in the supplementary. Drafts third and fourth, each contain only two sections :

"*A) Differentiation of* $\frac{u}{z}$ ", pp. 1-3.

"*B) Differentiation of an implicit function* " *, end of p.3 - p.5.

The third and the fourth drafts contain nearly no cancellation, but a lot of rough calculation regarding the expansion of the fraction $\frac{y}{y-x}$ into a series has been appended to the third one.

Here all the four drafts will be described at the same time. It will be based on the four sections projected by Marx in the first draft, but the contents of each of them will be elucidated, in the main, in accordance with the variant, more carefully edited by Marx. (Herein all the existing variants will be adduced in the foot notes or otherwise.)

We shall begin with section A) of the first two drafts.

Point 1) of this section has been published in the present volume (see, p.62 and note [47]), according to the second draft, where Marx made a fair copy of it .

Apparently, in the following points 2) and 3) Marx wanted to bring to light the following : should we not get rid of the assumption, that the variables u and z are both functions of one and the same independent variable x ? Should we not proceed only from the presupposition that u and z are interdependent ? In fact Marx wrote on this (we quote from the second draft) :

* In the third draft : "B) Differentiation of implicit functions". — Ed.

2) So far as the origin of the symbolic differential coefficient inside the "derivative" $f'(x)$ is concerned, $d\,(uz)$ could also have been expanded, *without the equation*

$$y = uz\,,$$

namely, as a function uz, considered in isolation.

Let us assume at first, that *u and z depend upon x*; then

a) uz;

when x becomes x_1:

b) $u_1 z_1$.

That is why, subtracting a) from b):

$$u_1 z_1 - uz\,;$$

expanded in factors *

$$z_1\,(u_1 - u) + u\,(z_1 - z)\,;$$

since both depend upon x:

c) $\dfrac{z_1\,(u_1 - u)}{x_1 - x} + \dfrac{u\,(z_1 - z)}{x_1 - x}$; assuming $x_1 = x$, i.e., $x_1 - x = 0$:

d) $z\dfrac{du}{dx} + u\dfrac{dz}{dx}$.

3) We could, finally, develop also *without x* and assume that *u and z are mutually dependent upon each other.*

However, here Marx could not put across the problem with exactitude, and it is possible, that namely that is why, later on he refrained from including the sections A) and B) of the first two drafts in his supplement to the article "On the differential".

Section B) is being reproduced below according to the first draft, where it has been enunciated in greater detail. From this draft it is also evident, that at that stage of his work — to which the manuscripts "On the concept of the derived function" and "On the differential" and then the additions to the latter are related — Marx was first of all interested in what he himself called, the "purely algebraic" aspect of the differential calculus; nevertheless, he was also preoccupied with the question of geometrical application of the latter (namely, as *geometrical).* To all appearence, initially Marx wished to investigate this question in the first drafts of the supplement to the manuscript "On the differential"; but afterwards he decided to postpone it till the completion of one of the latter "issues", evidently, he considered the question to be serious enough, so as not to talk about it in passing, but to devote to it a special entry. Unfortunately, this intention of Marx could not be actualised.

The text of this section is being reproduced below, in accordance with the first draft (sheets 4-9):

---

* Marx's expression "expanded in factors", is related to the transformation of the differences $u_1 z_1 - uz_1$ and $uz_1 - uz$ into the products $z_1\,(u_1 - u)$ and $u\,(z_1 - z)$.— Ed.

B) THE EQUATION $y^2 = ax$

1) If this equation is treated as the equation of the parabola*, then this object itself dictates the path **, demanding that it be turned over. If $y^2 = ax$, then inevitably also : $ax = y^2$, where x is the dependent, and y — the independent variable. This is the proper path, since the *general formula for sub-tangent to curves* $= y\dfrac{dy}{dx}$, hence, in the given *particular* equation of parabola dx must finally be expressed in y and then substituted in the general formula.

When divided by a, $ax = y^2$ gives : $x = \dfrac{y^2}{a}$, an equation with one dependent variable in the first power, and besides one of the most elementary functions of the independent variable y. However, we shall keep it for the subsequent geometric application, since I personally wish to make a few more prefatory general remarks on the method used by me***. Now only

2) $y^2 = ax$ will be considered, purely analytically.

$y_1{}^2 = ax_1$, when x turns into x_1. Hence,

$y_1{}^2 - y^2 = a\,(x_1 - x)$, $(y_1 - y)\,(y_1 + y) = a\,(x_1 - x)$.

If we divide, in the right hand side , $x_1 - x$ by itself, then we shall get $a \cdot 1 = a$; in the left hand side : $\left(\dfrac{y_1 - y}{x_1 - x}\right)(y_1 + y)$; hence

$$\left(\frac{y_1 - y}{x_1 - x}\right)(y_1 + y) = a\,. \qquad [(1)]$$

This first result of ours should not be let out of view, since it vividly shows, that, exactly as in our very first example, where we had $y = ax$ and obtained

$$\frac{y_1 - y}{x_1 - x} = a\,,$$

[here too] the *entire differential operation onesidedly* occurs in the symbolic left hand side****.

If now in (1), in the left hand side we assumed that $x_1 = x$, hence $x_1 - x = x - x = 0 = dx$, then thereby $y_1 = y$, hence, $y_1 + y$ will turn into $y + y$, into $2y$, $y_1 - y$ into $y - y$, into 0, into dy, and we get :

3) $\dfrac{dy}{dx} \cdot 2y = a$.

* In the second draft, here is an addition : "which has not occured yet in this entry". — Ed.

** For the solution of the problem of subtangent to the parabola (see, the text that follows). — Ed.

*** This sentence is absent in the second draft. In its stead there we read : " Here we shall not be concerned with it any more". — Ed.

**** In its stead the second draft contains only : "The subsequent *differential operation occurs onesidedly* in the symbolic, i.e., in the left hand side". — Ed.

The form $2y\frac{dy}{dx}$ in no way differs from the forms $z\frac{du}{dx}$ or $u\frac{dz}{dx}$ developed sub A) 2), and that is why, nothing more is required to be said about its deduction, in this connection.

The difference consists of this, that $\frac{du}{dx}$, $\frac{dz}{dx}$ are developments upon uz*, as symbolic differential coefficients ; they emerge as multipliers of the dependent variables u and z, in the right hand side, [while] operating with y^2, we see the reverse : the one and only variable y emerges in the left hand side, as the multiplier of the differential coefficient $\frac{dy}{dx}$; this is explained simply by this (see, *above* B) 2a)), that the difference $y_1 - y$ from the very beginning has as its multiplier $(y_1 + y)$, owing to which the assumption that $y_1 = y$ must equally give the positive result $2y$, i.e., twice the dependent variable y, as well as the negative result $y_1 - y = y - y = dy$.

Section C) of the first draft is devoted to differentiation of the quotient $\frac{u}{z}$, which Marx carries out first (in point a)), using the ready-made formula for the differential of the product. In points b) and c) Marx sought the differential of the quotient $\frac{u}{z}$, without using this formula. Here he proceeds directly from the definition of the derivative. Herein, for *"brevity of procedure"* he takes z as the independent variable, and comments (p.9) :

But it would be a mistake to conclude from this, that the expression for the general ratio of dependence, of y upon the independent variable x, standing in the left hand side, which we obtained earlier for the symbolic differential coefficient, in its finite form $\frac{y_1 - y}{x_1 - x}$, has in its differential form $\frac{dy}{dx}$, now been supplanted by some other form of the ratio of dependence.

Is it not clear from this, that, to all appearence, Marx did not at all intend to replace the usual mathematical modes of expressing the dependence between variables, by some other (generalised) mode of expressing their "interdependence"?

In point b), having formed the difference $\frac{u_1}{z_1} - \frac{u}{z}$ and having transformed it into $\frac{z(u_1 - u) - u(z_1 - z)}{zz_1}$, Marx then assumes $z_1 = z$ (owing to which he has $u_1 = u$) and gets at once

$$d\frac{u}{z} = \frac{z\,du - u\,dz}{z^2}. \qquad [(1)]$$

In point c), this differential is obtained, starting from the division of the difference $\frac{u_1}{z_1} - \frac{u}{z}$ by $z_1 - z$, i.e., through an initial search for the derivative of the quotient. This is in full agreement with

* In the manuscript, owing to a slip of pen, here we read "xy". — Ed.

the method usually used by Marx. Nevertheless, having obtained, in point b), the right hand part of the formula (1), without an initial division of the difference $\frac{u_1}{z_1} - \frac{u}{z}$ by $z_1 - z$, Marx observes (sheet 8):

The method is absolutely the same, as earlier, only in the result there is something new: the difference of the independent variable $z_1 - z$ stands in the numerator, whereas the positive expression of equality $z_1 = z$ stands in the denominator in the form of z^2.

If Marx made a fair copy of the manuscript "On the differential", before sending it to Engels, then, apparently, the additions to it, were read by somebody (most probably by Engels or Moore) while these were still in the rough copy-stage (which is the first draft of the supplement). Evidently, the aforementioned place called forth some critical comments of the reader, in answer to which Marx wrote the last two variants of the supplement. Wherein the sections C) and D) of the first draft served as the basis, as has already been noted, for the third and fourth drafts, where they were turned into sections A) and B).

Section A) of the third draft , is devoted to the differentiation of the quotient $\frac{u}{z}$. It was written directly in the form of an answer to a critical observation. In his answer, Marx explained, that the expression $\frac{z\,du - u\,dz}{z^2}$ obtained by him, was not the derivative of $\frac{u}{z}$, but the differential $d\frac{u}{z}$.

(Let us remember that for Marx — just as it was for Euler, see, Appendix, "On Leonhard Euler's Calculus of Zeros" — quantitatively speaking, a differential was generally equal to zero.)

In the fourth draft , which is the edited, final, fair-copy-stage of the work on the supplements, section A) lost the character of an answer to some critical comments. That is why to depict the entire course of Marx's work, we shall adduce not only the text of this section according to the fourth draft,but also (in foot notes) all the existing differences, from the text of the third draft.

Section A) consists of 3 points : 1), 2), 3). The first two of these are being reproduced here in full. Point 3) is to be found on p.63 of the present volume, in accordance with the third draft (which is fuller on this point)

A) DIFFERENTIATION OF $\frac{u}{z}$

1) Let us assume that in $\frac{u}{z}$, the independent variable is z, and u is the dependent variable.

Just for a change, this time we shall consider the function, given in an algebraic form, independently of the form of the equation [in the third draft, instead we read : "For the sake of a change this time we shall consider $\frac{u}{z}$ as a function of u and z, not connecting it with a third variable , dependent on $\frac{u}{z}$, in the form of an equation" — Ed .], whatever that might be — this can always happen (among the functions given in an algebraic form there may be expressions containing sines, cosines etc., logarithms and exponential expressions like a^x).

a) $\frac{u}{z}$, if z grows into z_1, then* :

b) $\frac{u_1}{z_1}$; subtracting a) from b) :

c) $\frac{u_1}{z_1} - \frac{u}{z}$; arriving at a common denominator : $\frac{zu_1 - uz_1}{z_1 z}$; expanding the numerator [into a difference of products] :

d) $\frac{z(u_1 - u) - u(z_1 - z)}{z_1 z}$. If now z_1 becomes $= z$, hence, $z_1 - z = 0$, then **:

e) $\frac{z\,du - u\,dz}{z^2}$ [" hence, $d\frac{u}{z} = \frac{z\,du - u\,dz}{z^2}$ " (3rd draft) —Ed.] :

This expression seems to be strange. In fact it was obtained at the cost of a *total change in method*, for — see d) — $z_1 - z$ instead of being in the denominator, found itself in the numerator, and d) was turned into its differential expression e), only owing to the fact that we reduced $z_1 - z$, situated *in the numerator*, into $z_1 - z = 0$***.

This apart, though it was assumed, that in $\frac{u}{z}$, u is the dependent variable, and z— independent, we would have obtained the same result, had we assumed, that in the numerator of the expression d) $z(u_1 - u) - u(z_1 - z)$ (where the role of $z_1 - z$ does not at all differ from the role of $u_1 - u$), $u_1 = u$, $u_1 - u = 0$, and that z depends upon u ****.

It is true that we can so interpret the whole process. In d) the denominator $z_1 z$ turns into zz, i.e., it is assumed that $z_1 = z$, that is, $z_1 - z = 0$, wherefrom, on the one hand, in the numerator $z_1 - z$ turns into dz, on the other, $u_1 - u$ into du (since u depends upon z, i.e. when in the denominator $z_1 = z$, then $u_1 = u$, i.e., $u_1 - u = u - u = du$).

Thus the method could have been saved in respect of the numerator, however, only to give it up fully. Namely, its general result, which was [as follows] : the ratio of dependence of a variable upon another must be represented as $\frac{y_1 - y}{x_1 - x}$, if y is presupposed to be the dependent, and x —the independent variable.

---

* " if z grows into z_1, then u into u_1, hence" (3rd draft). — Ed.

** "If z_1 becomes $= z$, then $z_1 z$ turns into zz or z^2, $z - z = dz$, $u_1 - u = du$, hence :" (3rd draft). — Ed.

*** In the third draft in place of this paragraph, we read : "What has struck you, is the appearence of this result. I suppose this, because otherwise you would not have thought that the differentiation of $\frac{u}{z}$ presented a peculiar case, in the development of which the method was undergoing some modification". —Ed.

**** This paragraph is absent in the third draft. —Ed.

[Instead of the last two paras, in the third draft : "Actually , dz (e) (in the final form of $z_1 - z$ (d)), the differential particle of the independent variable z, stands in the numerator as the multiplier of u, while z itself, in the positive form of z^2 (in the finite form $z_1 z$ (d)), is situated in the denominator. Thus, it seems,that we proceed from the finite ratio sub (d) to its differential expression sub (e), assuming sub (d):in the *numerator* $z(u_1 - u) - u(z_1 - z)$, $z_1 = z$, hence $z_1 - z = 0$, and this looks like a change in the method, more so, as herein the *denominator*, instead of being made to represent the removed difference $z_1 - z = dz$, is rather turned from $z_1 z$ into z^2 " . —Ed.]

2) Let us again use the equation form*.

a) $y = \frac{u}{z}$;

b) $y_1 = \frac{u_1}{z_1}$; $y_1 - y = \frac{u_1}{z_1} - \frac{u}{z}$;

c) $$\frac{y_1 - y}{z_1 - z} = \frac{\frac{u_1}{z_1} - \frac{u}{z}}{z_1 - z} = \frac{\frac{zu_1 - uz_1}{z_1 z}}{z_1 - z} = \frac{(zu_1 - uz_1) \cdot \frac{1}{z_1 z}}{z_1 - z};$$

d) $$\frac{y_1 - y}{z_1 - z} = \frac{(z(u_1 - u) - u(z_1 - z)) \cdot \frac{1}{z_1 z}}{z_1 - z}.$$

If now we put in the right hand side $z_1 = z$, hence, $z_1 - z = 0$ etc. etc., then :

e) $$\frac{dy}{dz} = \frac{(z\,du - u\,dz) \cdot \frac{1}{z^2}}{dz}$$

and,

f) dy or $d\frac{u}{z} = (z\,du - u\,dz) \cdot \frac{1}{z^2}$;

hence :

$$dy = \frac{z\,du - u\,dz}{z^2}.$$

Hence, the difficulty arose only from this, that the *differential* occupied the place of the *differential coefficient* **.

Let us compare (see, the previous manuscript) with what was obtained while differentiating uz :

* In the third draft : "The secret is explained as soon as we again use the inital form of the equation". — Ed.

** In the third draft : "The riddle was completely solved. The *differential coefficient* represented sub e), was obtained also in 1) e), and here sub f), the result is the *differential*". — Ed.

A) $\frac{dy}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$ and B) $dy = z\,du + u\,dz$.

The difference between :

a) $d(uz) = z\,du + u\,dz$ and b) $d\frac{u}{z} = (z\,du - u\,dz)\cdot\frac{1}{z^2}$, arises only from the difference between the functions being differentiated.

Sections D) of the first draft and B)of the third and fourth drafts are devoted to the differentiation of an implicit function. In the first draft Marx examined the example of finding the derivative of the function $y(x)$, in terms of x, given by the implicit equation of second degree $y^2 - 2yx + \frac{b}{c} = 0$, borrowed from Hind's book (p.23, example 8 ; in Hind the equation is : $u^2 - 2ux + a^2 = 0$).

In the third and fourth drafts Marx examined the function $y(x)$, given by an implicit equation of the same type

$$y^2 - 2yx = 0 .$$

Marx then expands (in the third and fourth drafts) the obtained result

$$\frac{dy}{dx} = \frac{y}{y-x}$$

into a series by division at an angle :

$$\frac{y}{y-x} = 1 + \frac{x}{y} + \frac{x^2}{y^2} + \frac{x^3}{y^3} + \cdots,$$

and explains, that in the given case he thus obtained,

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots,$$

since it follows from the equation $y^2 - 2yx = 0$ (when $y \neq 0$) that $\frac{x}{y} = \frac{1}{2}$.

ON THE DIFFERENTIAL

S.U.N. 4150

Marx's second work, of the year 1881, on the nature and history of differential calculus. In it Marx's own method of differentiation is actualised, using the theorem of the derivative of product.It is a manuscript of 13 sheets of writing paper. An envelope (sheet 14) with the heading :"*For Fred*", was attached to it. It has been published in full in the first part of the present volume (pp.26-39),under the little :"On the differential". On this manuscript see also the preface and note [13] .

---

* COMPUTATIONS RELATED TO THE METHOD OF LAGRANGE

S.U.N. 4300

8 Sheets of detailed calculations, related to the section:" On the method of Lagrange, to prove the principles of differential calculus without having recourse to limits, infinitesimals or any other evanescent quantities ", of Boucharlat's book.(In the 5th edition of this book at our disposal,it is §§ 244-254,pp. 168-176; on its content, see : Appendix," Theorems of Taylor and MacLaurin and Lagrange's theory of analytical functions, in the sources consulted by Marx").

Marx systematised the material according to his own plan. He divided it into six parts, numbered them with the Roman numerals I-VI, with additional indexes inside each part.

It contains no important addition to Boucharlat's account or to Marx's manuscripts 4000 and 4001 (see,the descriptions in pp.214 and 231). That is why, here only the following observation of Marx,is being reproduced. It is of interest from the point of view of his understanding of the "infinity" of a series.

Having written in part II the formula

$$f(x+h) = f(x) + ph \text{ (or } f'(x)\,h) + qh^2 + rh^3 + sh^4 + th^5 + \cdots$$

and having raised the problem of explaining the rule of successive formation "of the derived" functions p, q, r, s, t etc., starting from the initial function $f(x)$, Marx summed up in the following words, what he has already done (sheet 3):

The first result

1) gives at all events, that

$f(x+h) = f(x) + ph$ (or $f'(x)h$) + the terms with h^2, h^3, $\cdots$

2) *that the series has no power* , i.e. owing to its own nature, it can always be continued.

TAYLOR'S THEOREM ACCORDING TO HALL AND BOUCHARLAT

S.U.N. 4301

In the manuals at Marx's disposal, Taylor's Theorem was proved in two different ways :

1) with the help of the ready-made apparatus of differential calculus, but proceeding from the assumption, that "in the general instance", for any x and h ,

$$f(x+h) = f(x) + A\,h + B\,h^2 + C\,h^3 + \cdots, \tag{1}$$

where $A, B, C,$ .... are unknown functions of x,to be determined (this was done in the text books by Boucharlat and Hind, with the help of the method of indeterminate coefficients);

2) by the method of Lagrange, i.e. without the apparatus of differential calculus, and conversely, by defining the *derivative* as the coefficient of the first power of h in expansion (1), and Lagrange wanted to substantiate this method "purely algebraically " (in Hall's book this attempt of Lagrange has been enunciated in a form, which was given to it by Poisson, who attempted to complete it).

Criticising the proof of Taylor's theorem, presented in the books of Hind and Boucharlat, Marx observed (see, PV, 88-92,231-236 and also pp.93-94, 264-301) that their initial assumption is unfounded. In the present manuscript he gave an account of the proof of Taylor's theorem, according to Boucharlat, but wanted to utilise Poisson's method for substantiating the validity of the initial assumptions of this proof. There are two copies of this manuscript (both in English) : a rough, and a fair copy, identical in content. The fair copy (pp. 1-4 in Marx's numeration) consists of four points, designated by the Roman numerals I-IV, in the rough copy (pp. 1-4 and one more unnumbered page) the same material has been split into five points (I -V). The content of the manuscript and the quotations from it, are being reproduced below, in accordance with the fair copy. As has already been indicated, Marx thought, that not only the formulation of Taylor's theorem, but also its proof adduced in the books of Boucharlat and Hind, belonged to Taylor himself. In none of these books there was any bibliographical reference. Nevertheless, the fact that Marx looked for the title of the corresponding work of Taylor and, speaking of Taylor adduced the [corresponding] bibliographical information, induces us to assume that he wanted to verify this hypothesis from the primary source, which, unfortunately, he could not manage to do.

Accordingly, Marx formulates in point I,the initial assumption of Boucharlat, as follows :

Taylor's theorem may be considered to be the resume of his "*Methodus incrementorum etc.*" (London, 1715-1717). He proceeds from the following *supposition* :

If $y = f(x)$ and $y_1 = f(x+h)$, then this latter function may be developed as a series according to the ascending powers of h, thus:

$$y_1 = y \text{ (or } yh^0) + A\,h + B\,h^2 + C\,h^3 + \cdots,$$

where A, B, C etc. are unknown functions of x.

That is why two problems emerge : we have to 1) prove that validity of "Taylor's supposition", and 2) find out the unknown coefficients $A, B, C, \ldots$..

It is well known that the class of functions investigated by the mathematicians of the 18th and the first half of 19th century were such, that it was natural to assume that a function may be represented by a powered series only " in exceptional cases". Referring to Lagrange, who tried to prove it, Lacroix wrote in his big "Treatise" (p.160): "With this aim we shall use a very elegant analysis, carried out by Lagrange in 1772 and enlarged thereupon by the highly sparkling comments of Poisson". (In this connection Lacroix mentioned in the list of literature, a paper of Poisson, published in the third volume of the journal "La Correspondance sur L' École Polytechnique".)

Hall began his book following Poisson. Marx used this book in his point I, devoted to substantiating "Taylor's supposition". After the aforementioned place Marx writes :

But Poisson and other French mathematicians have afterwards *proved* : if $y = f(x)$ and $y_1 = f(x+h)$ then : $y_1 - y$ or $f(x+h) - f(x)$ is expressible by a series of the form $A\,h + B\,h^2 + C\,h^3 +$ etc., hence :

$$y_1 = y + A\,h + B\,h^2 + C\,h^3 + \cdots, \tag{1}$$

the chief object of the differential calculus being to find the values of the coefficients A, B, C etc. I shall not give here the general demonstration, but an example.

Then Marx adduces from Hall (§ 7, pp 3-4) an example (No. 3) of the expansion of the value of the function.

$$y = A\,x^m + B\,x^n + C\,x^p + \text{ etc.},$$

into a series of ascending powers of h, when x is substituted by $x + h$.

Point I comes to an end with the following observation of Marx (sheet 2), related to the adduced "general demonstration" by Hall :

The demonstration of Poisson etc. offers, moreover, the great advantage that it cannot be given without stating already the cases, in which the serial development with ascending integral powers of h and unknown or indeterminate functions of x leads to irrational results — thus predetermining the limits of the applicability of Taylor's theorem, its so-called "*failures*".

In points II and III Marx gives an account of the second problem, from Boucharlat's book : the unknown functions A, B, C etc. in expansion (1) are sought through the method of indeterminate coefficients. In point II the following *lemma* is noted: "If in a function y of x, the variable x is changed into $x + h$, then we shall get one and the same differential coefficient, irrespective of the fact, whether x will be a variable, and h a constant or h — a variable, and x a constant " (Boucharlat, § 55, p.34).

In point III the equations are written with the help of differentiation first in respect of h and then in respect of x, and application of the lemma (quoted above) ; and from these equations the coefficients A, B, C etc. are determined (see, Appendix, p.338). Taylors theorem, thus obtained, is then applied to the search for the expansion in Taylor's series of the function $\log(x+h)$ (in the draft this example occupies a special point IV).

Point III ends with the following words of Marx (sheet 4) :

Since all transcendental functions of x —exponential, logarithmic, trigonometrical— (in fact *all* functions of x save those possessing a common algebraic form)— refuse by their nature, their expansion in a finite number of algebraic terms, it is self-evident that *the differential coefficients of such functions of* x can only be expressed by an infinite number of terms, whence it follows that the *corresponding functions* of $x + h$— or Taylor's series— can also in general be but expressible by a series of terms indefinitely continued.

Taylor's theorem may also be written * [as]

$$f(x+h) = fx + \frac{d\,(fx)}{dx}\frac{h}{1} + \frac{d^2\,(fx)}{dx^2}\frac{h^2}{1\cdot 2} + \frac{d^3\,(fx)}{dx^3}\frac{h^3}{1\cdot 2\cdot 3} + \cdots + \frac{d^n\,(fx)}{dx^n}\frac{h^n}{1\cdot 2\cdot 3} + \cdots$$

This is without any doubt the formula which has led Lagrange to his *theory of functions*. Other fellows denote the successive differential coefficients by p, q, r etc., and write then :

$$f(x \pm h) = f(x) \pm p\frac{h}{1} + q\frac{h^2}{1\cdot 2} \pm r\frac{h^3}{1\cdot 2\cdot 3} + \cdots$$

It is understandable, that if $f(x+h)$ is expressible in Taylor's series, then $f(x)$ has derivatives of any order. Lagrange attempted to prove, that "in the general instance" the affair is like this : $f(x+h)$ is expanded into Taylor's series, but his successors—authors of the text books used by Marx —did not at all doubt, that every function is differentiable, and besides an infinite number of times. From their point of view only a polynomial had a finite number of derivatives. It is not surprising, that in this connection Marx made a distinction between two instances : functions expressible by a finite series, and functions expressible by an infinite series.

In the last, fourth point, Marx gave an account of the theorem on the method of indeterminate coefficients, following Boucharlat (in the fifth edition of his book, it is the sixth appendix). The following words of Marx precede the demonstration of this theorem:

As the *method of indeterminate coefficients* is of frequent use in the differential calculus, I shall add a very simple demonstration by *Boucharlat* (Frenchman).

The time of writing this manuscript could not be established. However, there is some ground to assume, that it was written after the manuscripts 4000 and 4001, but before manuscript 4302 (see, below). In fact the ideas developed in manuscript 4301, are yet to be found in the manuscripts 4000 and 4001, in which the first results of the work on the theorems of Taylor and MacLaurin and Lagrange's theory of analytical functions have been summed up. The attempt to use the method of Lagrange-Poisson, not as something prior to the differential calculus, but within this very calculus, so as to remove with its help, the inadequacies in the demonstration of Taylor's theorem, completing the construction of differential calculus, happens to be a new one. At the same time, it is evident from Marx's last incomplete manuscript, wherein he criticises the "demonstrations", adduced in the books of Hall and Boucharlat, that this attempt did not satisfy Marx.

---

* In the notation adopted by Marx. —Ed .

AN INCOMPLETE MANUSCRIPT ENTITLED "TAYLOR'S THEOREM"

S.U.N. 4302

The task of dating this manuscript presents no difficulties, since in it Marx refers to the manuscripts sent to Engels as " in the first two" (this refers to the manuscripts "On the concept of the derived function" and "On the differential"). Hence it was written, not earlier than 1881, to all appearence, in 1882.

This manuscript, in 45 sheets, remains only in the form of a rough draft, and that too at a very early stage of it. Marx returned many times to the beginning and to the other places of this manuscript and that is why it does not contain his continuous numbering of the pages. It would be natural to divide it into the following parts, for description.

I. Sheets 1-4 (in Marx's numeration pp. 1-3 and 4 lines of p. 4). Marx gave this part the title :"*Taylor's Theorem* ".

II. Sheets 4-7 (pp. 4-7) (upto the diagram on p. 7). Title : "*Ad p.1 Additionally*".

III. Sheet 7 (p.7 under the diagram), begins with the words : "*All this better be so begun*" and ends with the indication within brackets : "*Continuation on p.* 11^a".

IV. Sheets 8-9 (pp. 9, 8). Title : "*Preliminaries of successive differentiation (prefatory to p. 1)*". Page 9 ends with the following words within brackets : "*Contin. on previous p. 8*"

V. Sheet 10 (p. 10). Title : "*Ad p. 5*".

VI. Sheets 1-18 (pp. 11-17, and once more 17). Page 11 begins with the words : "*Contin. of p. 8 and 9*"

VII. Sheets 19-23 (pp. 11^a, 11^b, 11^7*, 11^8, 11^9 upper half). Continuation of the text, begun on p. 7. Page 11^a begins with the heading (in the left upper corner after a cancellation) : "*Ad. p.* 11^a.

VIII. Sheets 23-25 (p. 11^9 beginning with the title : "*Prelude to p. 1 ; Taylor's Theorem*", p.1 and an unnumbered page). Two sheets, on which Marx wrote at the same time, continuing from certain places of p.1 into the unnumbered page, which he called "*the opposite page*". It begins with the Roman numeral I, under which he wrote "a)". To all appearence, it is the number of the formula for the expansion of $(x+h)^5$.

IX. Sheets 26-38 (pp. *a, b, c, d, e, f, g, h, i,* again *h*, again *i, j, k*). Page "*a*" begins with the words: "*Ad p.1; Taylor's Theorem*". On p. "*c*" (fourth line from top Marx begins a new section under the title : "*Taylor's Theorem*". Page "*k*" carries the heading: "*MacLaurin's Theorem*".

X. Sheets 39-40 (pp. *o, p*). Title : "*Successive Differentiation*".

XI. Sheet 41 (p. *s*). Title : "*Ad p.1 (Taylor's Theorem)*".

XII. Sheets 42-45. Four separate pages. The first three are numbered 1-3 in pencil, the last one is unnumbered. Page 1 carries the title : "*Towards Taylor's Theorem, p. 1*". Page 2 has the heading: "*Ad. Taylor's Theorem (p.1)*. Page 3 carries the title: "*Ad Taylor's Th. p.1, eq, III*", written in pencil.

It is evident from the above list, that Marx returned to the beginning of this manuscript at least 8 times; that is to say, this beginning did not satisfy him. So many variants, as well as the fact, that

---

* 11^7, probably because, by mistake Marx took the previous page 11^b to be 11^6. —Ed.

all of them has the character of rough, unchecked drafts (in places containing explicitly wrong calculations and conclusions), makes the task of ascertaining Marx's intended plan of the manuscript, a difficult one. The text of this manuscript is being reproduced below almost in full. However, a closer acquaintance with the manuscript, provides sufficient ground, to draw the following conclusions about Marx's ideas and intentions.

1) The unsatisfactoriness of all the modes of substantiating Taylor's theorem, found in the manuals of Boucharlat, Hind, Hall and others, was clear to Marx. The attempt, contemplated in manuscript 4301, to correct the deficiencies of Boucharlat's proof, by substantiating his initial assumption, no more satisfies Marx, as he sees, that Boucharlat's initial assumption is not at all tenable : not every function $f(x+h)$ may be expanded into a series according to the ascending integral powers of h.

2) That is why Marx thought that Taylor's theorem was obtained through a generalisation of Newton's binomial theorem, which permits the expansion of $(x+h)^m$ into the series indicated. Such a generalisation had to isolate a class of functions $f(x+h)$, for which this expansion is possible, even if, in its turn, in a certain generalised sense. But certain difficulties are connected with this sort of generalisation, and Marx highlighted them. First of all, here a transition (even a "leap", as Marx calls it) from a finite polynomial to an infinite (indefinitely continued) series, is essential. Further, in the binomial theorem, $x+h$ is a simple sum of arbitrary x and h ; in differential calculus $x+h$ is a mode of expressing (what we would now call) the local change of the variable x , which Marx expressed in his algebraic method of differentiation, with the help of the indeterminate difference $x_1 - x$ (see, the manuscript "On the differential", pp.26-39). In this connection Marx had to specially ponder upon the difference between the indeterminate differences $x_1 - x$, corresponding to $y_1 - y$, to be "removed" in the process of differentiation (in the transition to limit), and the fixated difference $f(x+h) - f(x)$, by no means in need of such a "removal " (which also is designated by $y_1 - y$), and which is to be computed (approximately) through an expansion into the Taylor's series.

3) It is natural that Marx wishes to investigate the confusion generally connected with the concepts of constant and variable, with the concepts of a function as an analytical expression (functions "in x ") and a function as a correspondence (functions "of x"), with successive differentiation. Of special interest in this connection is a note, which Marx placed in that section of the manuscript, which is devoted to successive differentiation, and which we marked out under the heading : *On the word " function"* (see, below p. 268).

The draft and fragmentary character of this manuscript makes its reading and description very difficult. The difficulty is there right from the arrangement of the different parts of the manuscript, especially as Marx many times returns to its beginning and to the other parts. All corresponding indications provided by Marx have been taken into account, as far as possible. In all the other cases, the texts have been joined together according to the questions considered (but always mentioning the part number given in the list on p.264, so that the reader may retrieve the content of each of these parts, if she or he so wishes). Some parts of the manuscript, containing only calculations (which are, by the way, easier to do all by oneself, than to follow the way another person does it) have been omitted. Certain explicitly mistaken places have also been omitted (but the mistakes have been indicated).

In consonance with Marx's directions we shall start with part IV (sheet 8), and then proceed to part VI of the aforementioned list.

PRELIMINARIES OF SUCCESSIVE DIFFERENTIATION

(PREFATORY TO PAGE 1)

We know that if we have : $y = f(x)$, then $d f(x) = f'(x)\, dx$.

Hence

1) $\dfrac{d f(x)}{dx} = f'(x)$; being differentiated in its turn $f'(x)$ gives $d f'(x) = f''(x) dx$;

hence :

2) $\dfrac{d f'(x)}{dx} = f''(x)$; further : $d f''(x) = f'''(x) dx$;

hence :

3) $\dfrac{d f''(x)}{dx} = f'''(x)$; further : $d f'''(x) = f^{IV}(x) dx$;

hence :

4) $\dfrac{d f'''(x)}{dx} = f^{IV}(x)$ etc.

A. The "derived" functions, in their turn, may be considered independently of the entire chain of functions, connecting them with the original function ; on the other hand, each of them may be presented also as the "derivative" of the previous original function. In this case, those very functions appeared as :

1) y or $f(x)$ with the derivative $f'(x)$, hence, as before $\dfrac{d f(x)}{dx} = f'(x)$; this function in itself $= \varphi(x)$;

2) y or $\varphi(x)$ and the one deduced from it : $d\varphi(x) = \varphi'(x) dx$ and $\dfrac{d\varphi(x)}{dx} = \varphi'(x)$; this function, in its turn, is an independent function, say $F(x)$;

3) y or $F(x)$ and the one deduced from it : $d F(x) = F'(x) dx$ and $\dfrac{d F(x)}{dx} = F'(x)$.

Thus, we have :

1) $f'(x) = \dfrac{d f(x)}{dx}$,

2) $df''(x) = \dfrac{d f'(x)}{dx}$; if here we put the value of $f'(x)$ then we shall have

$$\frac{d\left(\dfrac{d f(x)}{dx}\right)}{dx} = \frac{d\,(df(x))}{dx^2} ;$$

hence,

$$f''(x) = \frac{d\,(d f(x))}{dx^2} ,$$

an expression, wherein we leave the meaning of the numerator, for the present, indeterminate.

(Contin. on previous page 8.)

Marx continues this procedure on p.8 (sheet 9), till he obtains :

$$f^{IV}(x) = \frac{d\left(\frac{d \cdot d \cdot d f(x)}{dx^3}\right)}{dx} = \frac{d\,(d \cdot d \cdot d f(x))}{dx^4} .$$

Hence ,

$$f^{IV}(x) = \frac{d \cdot d \cdot d \cdot df(x)}{dx^4} .$$

With this, part IV of the manuscript (according to our list) comes to an end.

Part VI on p. 11, is preceded by the words : "*Contin. of p. 9 and 8*"; it begins with a repetition of the entire aforementioned procedure. Then Marx writes (sheet 11) :

We have seen, how these various formulae were obtained in the form of *symbolic expressions* of *derived functions*, hence, as the *symbols of operations already carried out ;* and from the earlier account it is understandable, that they become the *symbolic operational formulae*, formulae, indicating only those *operations which are yet to be carried out* for finding out the real equivalents corresponding to them, or the derived functions. But these formulae themselves are still to be analysed in detail.

1) Let us consider, at first, the numerators of the symbolic differential coefficients for $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{IV}(x)$, namely: $df(x)$, $d\,(df(x))$, $d\,(d\,(df(x)))$, $d\,(d\,(d\,(df(x))))$ etc.

These expressions emerge only because, the various functions appear in the chain of their deduction from the original function and that of one from the other, i.e. as *successively deduced functions*, each of which always springs from the one immediately preceding it. But they appear in the other equations too, in their turn, themselves as the original functions, i.e. irrespective of the chain connecting one with the other and with the initial, original function, which, in its turn, may carry the birth-mark of some other orginal function, of which, initially, it is the derivative .

Beyond the mutual connections, these functions appear as follows : for example y or $f(x) = 5x^4$ (at the first glance, it is possible to indentify this original function as the *first derivative of* x^5).

Hence,

$$f'(x) = \frac{df(x)}{dx} \text{ or } \frac{dy}{dx} = 5 \cdot 4 \cdot x^3 .$$

Let us call this independently appearing derivative $\varphi(x)$.

But still preliminarily the following is to be noted :

$$f'(x) = \frac{df(x)}{dx} \text{ or } \frac{dy}{dx} = \frac{y_1 - y}{x_1 - x} ,$$

since the latter will be reduced to $\frac{dy}{dx}$ thanks to the equalisation $x_1 = x$, or $x_1 - x = 0$.

To fixate this, we shall designate the [expression] $\frac{y_1 - y}{x_1 - x}$, which has undergone this metamorphosis, through $\left(\frac{y_1 - y}{x_1 - x}\right)$, where, consequently, the brackets indicate that $\frac{y_1 - y}{x_1 - x}$ has turned into $\frac{dy}{dx}$.

Thus, we obtained

$$f'(x) = \frac{d f(x)}{dx} \text{ or } \frac{dy}{dx} = \left(\frac{y_1 - y}{x_1 - x}\right).$$

But here a few more remarks should be made about the word *function* .

Since the following insertion is of interest, independently of the question of successive differentiation, it is being reproduced below in the form of a separate note (sheets 12 - 13).

ON THE WORD "FUNCTION"

Initially the word "function" was introduced in algebra, while investigating the so-called indeterminate equations, whose number was less than the number of unknowns entering into them. Here, for example, the value of y changes, when in place of x its numerical values, for example, 3, 4, 5 etc., are substituted. Here y is called a function of x, because it must submit to the latter's command, just as every functionary does, even the great Wilhelm I is himself dependent on somebody.

In consonance with this, in the differentioal calculus the word "function" was transported with this sense, upon the dependent variable, for example, upon y.

Consequently y or $f(x)$, i.e., y or $5x^4$ in our example, signified a function of x, and besides it is that function of x, which is given by the determinate expression $5x^4$, because the value of y changes along with the changes in the values, which x produces through its own variation, in its own specific expression $5x^4$.

However, when Lagrange introduced the definition of the "derived" functions, and along with that also the definition of the original function, from which they have been deduced, then there arose the confusion which continues till date. Lagrange's function entered into all the modern treatises of calculus, where, however, the word "function" is used at the same time, also in the previous sense : thus, for example, if $y = 5x^4$, then we have : y or $f(x)$, or still more specifically, *y or* $f(x)$, *or* the function $5x^4 = f(x)$, *or* $5x^4$.

The confusion may be removed only by reading :

y as the function *of* x, i.e. as dependent upon x in each particular case or as the *function of x in its determinate expression* $5x^4$, which = the orginal function *in* x : $5x^4$; exactly in the same way in respect of the derivatives : y is always the function *of* x, they [the derivatives] are functions *in* x. In the last sense, the word "function" designates for the *original function*, that algebraic combination, in which x appears initially, for example, as $5x^4$ and for the *derived function* [it designates] those new values , which appear instead of $5x^4$, as a result of the variations of x and of the differentiations corresponding to them [165].

In Lagrange when the expression $f(x)$ stands to the left of the algebraic expression in x, [then it] has the meaning of only a *general* and that is why *indeterminate expression,* standing

opposite the particular [166] ; and $f(x+h)$ has the meaning of a general unexpanded expression, standing opposite its developed expression, a serial expansion, as, for example, in algebra $(x+a)^m$ is the general unexpanded expression, while on the right hand side, on the side of serial expansion, stands x^m + etc.

This is quite enough and adequate for determinate purposes ; nevertheless, we should not fail to distinguish the functions *of* x from the functions *in* x, since only from this distinction does it follow, that the functions *of* x may have concrete existence, as distinct from functions *in* x, like, for instance, the existence of the ordinate, when x is the abscissa [[and the orginal function in x is simply an expression in x]] etc.

After this Marx went over to explaining the meaning of the expressions : $df(x)$, $d(df(x))$, $d(d(df(x)))$ etc., placed by him in the numerators, i.e. of the differentials, now of any order. With this aim, first of all he successively differentiated the function $y=x^5$, using for the derivative $\frac{dy}{dx}$, the new notation $\left(\frac{y_1-y}{x_1-x}\right)$, introduced by him, which is also extended to the differential, designating the differential dy through the difference y_1-y, taken within brackets, i.e. through "(y_1-y)". If $y=f(x)$, then Marx writes there upon $(y_1-y)=(f(x_1)-f(x))$ or — as he says, " in another method" — $(y_1-y)=(f(x+h)-f(x))$, thus subsitituting x_1 by $x+h$. Here we read (sheets 13-14) :

If we shall at once consider each of the various *derived functions*, in its turn, as the original function, i.e. independently of the entire chain of deduction, then : if we have

1) $y=f(x)=x^5$, then $y_1=f(x_1)=x_1^5$.

Having designated y_1-y, when owing to the fact that $x_1-x=0$ it becomes dy, by (y_1-y), and exactly in the same way x_1-x by (x_1-x), that is also $\frac{f(x_1)-f(x)}{x_1-x}$ by $\frac{(f(x_1)-f(x))}{(x_1-x)}$, as soon as x_1-x becomes $=0$, we shall have :

$$\frac{(y_1-y)}{(x_1-x)} \text{ or } \frac{(f(x_1)-f(x))}{(x_1-x) \text{ or } h}=f'(x)=5x^4$$

and

$$dy \text{ or } (y_1-y) \text{ or } (f(x_1)-f(x))=f'(x)(x_1-x) \text{ or } f'(x)\,dx=5x^4(x_1-x) \text{ or } 5x^4dx.$$

And in another method :

$$\frac{(y_1-y)}{(x_1-x)} \text{ or } \frac{(f(x+h)-f(x))}{(x_1-x)}=f'(x)=5x^4,$$

and in the same way

$$dy \text{ or } (y_1-y) \text{ or } (f(x+h)-f(x))=f'(x)(x_1-x)=5x^4(x_1-x) \text{ or } 5x^4dx.$$

Further, having designatged the obtained derivative $5x^4$ by $\varphi(x)$, Marx treats it just as the function $f(x)$, i.e., just as x^5, was treated. He repeates it also for the function $F(x)$, obtained by differentiating $\varphi(x)$. After that he concludes (sheets 14-15) :

We see here, that

1) dy or $(y_1 - y) = (f(x_1) - f(x))$,

2) dy or $(y_1 - y) = (\varphi(x_1) - \varphi(x))$,

3) dy or $(y_1 - y) = (F(x_1) - F(x))$

etc.

or in another mode of expression :

1) dy or $(y_1 - y) = (f(x+h) - f(x)$,

2) dy or $(y_1 - y) = (\varphi(x+h) - \varphi(x))$,

3) dy or $(y_1 - y) = (F(x+h) - F(x))$

etc.,

that, hence, dy or $(y_1 - y)$ expresses the removed difference of the *different functions of* x, and besides that of the *different functions,* obtained in 1), 2), 3) etc., every time through the *differentiation of the respective original functions in* x,

$$f(x) = x^5,\ \varphi(x) = 5x^4,\ F(x) = 5{\cdot}4\,x^3.$$

Thus, dy or $(y_1 - y)$ in 1), 2), 3) has three entirely different values, which are represented by the *differentials* :

1) dy or $(y_1 - y) = f'(x)(x_1 - x)$,

2) dy or $(y_1 - y) = \varphi'(x)(x_1 - x)$,

3) dy or $(y_1 - y) = F'(x)(x_1 - x)$.

Conversely, in the expressions of the *symbolic differential coefficients*

$$1)\ \frac{(f(x_1) - f(x))}{(x_1 - x)},\quad 2)\ \frac{(\varphi(x_1) - \varphi(x))}{(x_1 - x)},\quad 3)\ \frac{(F(x_1) - F(x))}{(x_1 - x)},\ \ldots,$$

$x_1 - x$ is *the same*, as in the *differentials,* where $x_1 - x$ appears in all the three equations, as one and the same factor. But this is self-understood. If we take the functions $f'(x)$, $\varphi'(x)$, $F'(x)$ in the concrete expressions given for the examples, then :

1) $f'(x) = 5x^4$, 2) $\varphi'(x) = 5{\cdot}4x^3$, $F'(x) = 5{\cdot}4{\cdot}3x^2$.

Though all the three are different *functions* in x, they all have this in common, that they are *functions in one and the same variable* x. They were all obtained through *differentiation,* namely through the assumption that $x_1 = x$, that is $x_1 - x = 0$, $(x_1 - x) = dx$.

After this Marx went over to obtaining these very derivatives with the help of the operational formulae of differential calculus, i.e., he began not with the derivatives, but with the symbolic formulae of the differentials. Here Marx writes (sheet 15) :

If we now proceed from the operational method, where the actually symbolic *differential, emerging from the differential coefficients* serves as the *starting point,* in order to find out, conversely, these first ones, then, for example,

1) dy or $(y_1 - y) = f'(x)(x_1 - x)$;

that is why

$$\frac{dy}{dx} \text{ or } \frac{(y_1 - y)}{(x_1 - x)} = f'(x)\ .$$

Exactly in the same way :

2) dy or $(y_1 - y) = \varphi'(x)(x_1 - x)$;

hence

$$\frac{(y_1 - y)}{(x_1 - x)} = \varphi'(x), \ldots$$

In the differntial $(x_1 - x)$ or dx appears as a factor of the first derived function in x, in respect of $f'(x)$, $\varphi'(x)$, $F'(x)$ etc. Thus, the latter are obtained only by freeing them from their accompanying multipliers through division, in other words, by dividing $(y_1 - y) = dy$ by $(x_1 - x)$ or dx.

Further (sheets 17-18), Marx carries out all these successive differentiations once more for the function $f(x) = x^5$, right upto finding out the third derivative, starting every time with the formula for the differential, i.e., the formula of the form $d\varphi(x) = \varphi'(x)dx$ (here, in the calculations there are quite a few slips of pen). Summing up, Marx writes (sheet 17) :

If we now compare this with the *expressions obtained through substitution* (p .11*), then, having equalised the different expressions for the same $f'(x)$, $f''(x)$, $f'''(x)$ there and here, or as in our example [the expressions] $5x^4$, $5{\cdot}4x^3$, $5{\cdot}4{\cdot}3x^2$, we shall have :

1) $\dfrac{df(x)}{dx} = f'(x)$ (resp. $5x^4) = \dfrac{dy}{dx}$,

2) $\dfrac{d(df(x))}{dx^2} = f''(x)$ (resp. $5{\cdot}4x^3) = \dfrac{dy'}{dx}$,

3) $\dfrac{d(d(df(x)))}{dx^3} = f'''(x)$ (resp. $5{\cdot}4{\cdot}3x^2) = \dfrac{dy''}{dx}$.

Thus, the expression on the left hand side does not stand for any thing other than the fact that, sub 1) the original function is *differentiated for the first time*, sub 2) — for the second time and sub 3) — for the third time and since we designate this first differentiation by $df(x)$, so we can designate the second $d(df(x))$ by $d^2f(x)$, the third $d(d(df(x)))$ by $d^3f(x)$, where d, d^2, d^3 do not singify any thing apart from the fact that, $f(x)$ is differentiated for the first time, the result thus obtained is again differentiated etc. Thus we obtain :

1) $\dfrac{df(x)}{dx} = \dfrac{dy}{dx}$, 2) $\dfrac{d^2f(x)}{dx^2} = \dfrac{dy'}{dx}$, 3) $\dfrac{d^3f(x)}{dx^3} = \dfrac{dy''}{dx}$.

Since $f(x) = y$, we can, everywhere, write in the left hand side y instead of $f(x)$, and we get :

1) $\dfrac{dy}{dx} = f'(x) = \dfrac{dy}{dx}$, 2) $\dfrac{d^2y}{dx^2} = f''(x) = \dfrac{dy'}{dx}$, 3) $\dfrac{d^3y}{dx^3} = f'''(x) = \dfrac{dy''}{dx}$.

[In each of these equations] the difference between the two expressions is only apparent.

On the left hand side all the derivatives have been expressed as derivatives of the original function $f(x)$;

on the right hand side they have not only been expressed as derivatives of the original function, but also each as the derivative of the preceding one.

---

* see, PV, 267 — Ed.

1)But $dy = (y_1 - y) = (f(x_1) - f(x)) = df(x)$,

2) exactly in the same way

$$dy' = (y_1' - y') = (f'(x_1) - f'(x)).$$

Thus, if sub 1) *the sublated difference between the original* $f(x)$ and $f(x_1)$ is represented, then sub 2) [is represented] the *sublated difference between the first derivative* $f'(x)$ *and its function* $f'(x_1)$, *changed owing to the increment of* x *to* x_1.

But

$$d^2 f(x) = d\,(d\,(f(x)))$$

does not express anything else ; since $df(x)$ is the *differential* of the original function $f(x)$, it is equal to $f'(x)\,dx$, and that is why $d\,(df(x))$ is the differential of the first differential $f'(x)\,dx$ *.

$d\,(f(x))$ is the first differential of the original function $f(x)$.

$d\,(df(x))$ or $d^2 f(x)$ is the second differential in respect of the orginal function, but [it is the] first differential in respect of the first differential $f'(x)\,dx$.

The same for $d\,(d\,(df(x)))$ etc.

$d\,(y'\,dx)$ is d^2 in respect of y and in the same way $d\,(y''\,dx^2)$ is d^2 in respect of dy, hence, d^3 in respect of y.

Thus, the most suitable form is obtained for calculation : if $f(x) = y$, then

$$f'(x) = \frac{dy}{dx},\ f''(x) = \frac{d^2y}{dx^2},\ f'''(x) = \frac{d^3y}{dx^3},\ \cdots.$$

Expression of this confusion is a remnant of the Newtono-Leibnitzian methods amidst the modern mathematicians, who not only ...**

Here part VI (of our list) breaks off. One may think, that speaking of the confusion, which happens to be the remnant of the methods of Newton-Leibnitz, Marx had in view [the practice of] operating with the expressions of the form $\frac{d^ny}{dx^n}$, as ordinary fractions, when x is not an independent variable. For Marx the source of information about such confusion could, for example, be the text book by Hemming. In fact Hemming writes (see, the note on p. 65) about the notations $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}$: "These expressions are sometimes used instead of $f'(x)$, $f''(x)$ etc. also in the instances, when x is not an independent variable. But in this case it is clear, that the numerators, no more represent the successive differentials of y and that these expressions in reality cease to be fractions, but become simply symbols, equivalent to $f'(x), f''(x), \cdots, f^{(n)}(x)$ ". Let us note in this connection, that in the entire text cited, for Marx x is always an *independent* variable.

The account proper of Taylor's theorem begins with part I (in our list above sheets 1-4) . It is being reproduced below in full. While reading this part, we should remember, that Marx uses the term "ableiten" ("to deduce") in a sense wider than what is in current use; for Marx the derivative is "deduced" from the original function (i.e., obtained from it according to definite rules), Taylor's theorem is "deducd" from Newton's binomial theorem (i.e., it emerges as a generalisation of this theorem, herein the permissibility of which is still to be substantiated).

* Here is a slip of pen in the masunscript (and it is repeated below several times): instead of "differential" Marx wrote "derivative". This slip of pen has been corrected. — Ed.

** Here the sentence breaks off. — Ed.

TAYLOR'S THEOREM

Let us take

$$1)\ (x+h)^m = x^m + mx^{m-1}\frac{h}{1} + m(m-1)x^{m-2}\frac{h^2}{1\cdot 2} + m(m-1)(m-2)x^{m-3}\frac{h^3}{1\cdot 2\cdot 3} + \\ + m(m-1)(m-2)(m-3)x^{m-4}\frac{h^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

This is the binomial theorem.

If we now assume that in $(x+h)^m$, x is a variable, and h is its increment, then, when $h = 0$, $(x+h)^m = (x+0)^m = x^m$.

That is why, before its increase, the original function in x is x^m, or

a) $x^m = f(x) = y$,

b) $(x+h)^m = f(x+h) = y_1$.

That is why the above mentioned equation is transformed into

$$2)\ f(x+h) \text{ or } y_1 = x^m + mx^{m-1}\frac{h}{1} + m(m-1)x^{m-2}\frac{h^2}{1\cdot 2} + m(m-1)(m-2)x^{m-3}\frac{h^3}{1\cdot 2\cdot 3} + \\ + m(m-1)(m-2)(m-3)x^{m-4}\frac{h^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

If in the right hand side we assume that $h = 0$, then it will become euqal to 0 also in the left hand side, and we shall again get $y = x^m$ or $= f(x)$ (see a)). Thus, the first term of the serial expansion for y_1 or for $f(x+h)$, is inevitably $= f(x) = y$.

Hence, equation 2) is transformed into

$$3)\ f(x+h) \text{ or } y_1 = y + mx^{m-1}\frac{h}{1} + m(m-1)x^{m-2}\frac{h^2}{1\cdot 2} + m(m-1)(m-2)x^{m-3}\frac{h^3}{1\cdot 2\cdot 3} + \\ + m(m-1)(m-2)(m-3)x^{m-4}\frac{h^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

Regarding the coefficients of h we know that

the *derivative of* $x^m = mx^{m-1}$,

the *derivative of* $mx^{m-1} = m(m-1)x^{m-2}$,

the *derivative of* $m(m-1)x^{m-2} = m(m-1)(m-2)x^{m-3}$,

the *derivative of* $m(m-1)(m-2)x^{m-3} = m(m-1)(m-2)(m-3)x^{m-4}$ etc.

That is why equation 3) is transformed into

$$4)\ f(x+h) \text{ or } y_1 = y \text{ or } f(x) + f'(x)\frac{h}{1} + f''(x)\frac{h^2}{1\cdot 2} + f'''(x)\frac{h^3}{1\cdot 2\cdot 3} + f^{IV}(x)\frac{h^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

But since

$$f'(x)=\frac{dy}{dx},\ f''(x)=\frac{d^2y}{dx^2},\ f'''(x)=\frac{d^3y}{dx^3},\ f^{IV}(x)=\frac{d^4y}{dx^4},\ \ldots$$

and since we may substitute for the derived functions, their symbolic equivalents, so :

$$5)\ f(x+h) \text{ or } y_1=f(x) \text{ or } y+\frac{dy}{dx}\frac{h}{1}+\frac{d^2y}{dx^2}\frac{h^2}{1\cdot 2}+\frac{d^3y}{dx^3}\frac{h^3}{1\cdot 2\cdot 3}+\frac{d^4y}{dx^4}\frac{h^4}{1\cdot 2\cdot 3\cdot 4}+\cdots$$

and this is the *Taylor's theorem*, i.e., the general operational formula for differntiating *every* $f(x)$, [when x] increases by a positive or negative increment h [167]. Herein it is necessary only to represent y through the given functions in x and, as we shall unfold corresponding to them $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc., to substitute the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc. thus obtained, in the above mentioned formula, having modified therein, their numerical multipliers, through multiplication by h, $\frac{h^2}{1\cdot 2}$, $\frac{h^3}{1\cdot 2\cdot 3}$ etc.

From the point of view of the algebraic method applied by us, this theorem is, till we act in its characteristic way, inapplicable ; though, as has already been said, on the basis of the data obtained with the help of this method, it can be directly deduced from the binomial theorem. That is why, from its stand point, only the following remarks may be made about this most general and the most comprehensive of all the operational equations of differential calculus :

a) *To be at all applicable*, it requires (see, equation 4), that the original function in x be expandable, not only into a series of determinate, and in this sense finite, functions of x, but that, apart from it, it should be a series of functions of the indicated sort, with the factor h in ascending, integral and positive powers. Lower down, we shall again return to this.

b) From equation 5) it follows that

$$f(x+h)-f(x) \text{ or } y_1-y=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1\cdot 2}+\frac{d^3y}{dx^3}\frac{h^3}{1\cdot 2\cdot 3}+\frac{d^4y}{dx^4}\frac{h^4}{1\cdot 2\cdot 3\cdot 4}+\cdots$$

But y_1-y is a finite difference, $=\Delta y$, because it is only the difference between $f(x)$ in its original condition and the same $f(x)$ in its increased form. This difference is not reduced to dy. Hence it follows, that the finite difference

$$y_1-y \text{ or } f(x+h)-f(x)$$

is expressible by a *sum of differential coefficients* with the factors h in ascending (integral and positive) powers.

This sum is the increment attained by the dependent variable y, when x increases by h.

But from equation 4) it follows, that it does not signify anything other than the fact, that $f(x+h)-f(x)$ (or y_1-y) is expressible through the sum of functions with the increment h in ascending etc. powers, deduced from $f(x)$.

But we know that

$$\left.\begin{array}{l} f'(x) = \dfrac{dy}{dx} = \dfrac{y_1 - y}{x_1 - x} \\ \text{where it is assumed} \\ \text{that } x_1 = x,\ x_1 - x \\ \text{that is, also } y_1 - y = 0, \\ f''(x) = \dfrac{(y') - y'}{(x_1 - x)} = \dfrac{d^2y}{dx^2}, \\ \text{since } x_1 - x = 0; \\ f'''(x) = \dfrac{(y'') - y''}{(x_1 - x)} = \dfrac{d^3y}{dx^3} \\ \text{since } x_1 - x = 0, \text{ etc.} \end{array}\right\} (A)$$

If the various derivatives are viewed as functions, successively deduced from the original function $f(x)$, then the essence of the matter presents itself like this.

But if we consider each of the derivatives only in respect of *its own immediate original function*, i.e., in respect of that $f(x)$, from which it springs directly, then we shall get only a series of functions $f(x)$ and $f'(x)$, not having any [further] connection [among themselves], but that is only the differential expression $\frac{dy}{dx}$ for each of these differential coefficients.

Thus for example at first [we have] :

$$\left.\begin{array}{l} 1)\ f(x) = x^3, \\ \quad f'x \text{ or } 3x^2 = \dfrac{y_1 - y}{x_1 - x}, \quad \text{where } x_1 - x = 0, \quad \text{that is} = \dfrac{dy}{dx}; \\ 2)\ f(x) = 3x^2, \\ \quad f'(x) = 6x = \dfrac{y_1 - y}{x_1 - x}, \quad \text{where } x_1 - x = 0, \quad \text{that is} = \dfrac{dy}{dx}; \\ 3)\ f(x) = 6x, \\ \quad f'(x) = 6 = \dfrac{y_1 - y}{x_1 - x}, \quad \text{where } x_1 - x = 0, \quad \text{that is} = \dfrac{dy}{dx}. \end{array}\right\} (B)$$

If we adhere to the *modes of operations* (*B*), then every time we shall get only $f(x+h)$ or $y_1 \approx f(x) + f'(x)h$ *.

Thus, $y_1 - y \approx f'(x)h = \frac{dy}{dx}h$. But only this $f'(x)$, that is also its symbolic equivalent $\frac{dy}{dx}$, i.e., $f'(x)h = \frac{dy}{dx}h$, every time in the other from $(B)_{1)}$. $(B)_{2)}$. and $(B)_{3)}$. $y_1 - y$, which $\approx$

* Since it is clear that here the issue is one of approximate equality, we took the liberty of using here, and later on, the modern sign of approximate equality.— Ed.

$= f'(x)h = \frac{dy}{dx}h$, retains in all the three equations, the *same form*, but has entirely *different values*, besides the connecting link between them is so small, as, for example, is that of the original function x^3 with a preceding one, from which it is deduced ; if, for instance, our original function was x^4, then

$(x+h)^4 = x^4 + 4x^3h + \cdots$

Here the first derivative $f'(x)$ is $4x^3$, and we shall get

$4(x+h)^3 = 4x^3 + 4\cdot 3x^2h + 4\cdot 3xh^2 + 4h^3$.

Having divided both the sides by 4 we shall have

$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$,

where x^3, the *original function*, from which we proceeded sub b), figures, in its turn as the *derived function*. As little it has disturbed us in respect of x^3 sub $(B)_{1)}$, so little it must disturb us in respect of $3x^2$ sub $(B)_{2)}$ or $6x$ sub $(B)_{3)}$. The "derivative" is deduced only relative to that function, from which we proceed, taking it as the original function.

Thus, in mode (B), $y_1 - y$ is not the sum of derived functions and that is why here it does not suit us.

On the other hand, if we turn to (A), where the derived functions are presented as *successively* derived from the original function and, hence, as a chain of derivatives, then

$$f(x+h) - f(x) \text{ or } y_1 - y = \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1\cdot 2} + \cdots$$

is at the same time a chain of the differences $y_1 - y$ [168].

Here the formula

$y_1 - y$ or $f(x+h) - f(x) = \cdots$

signifies, in respect of $y_1 - y$, only this, that through the successive differentiation of $f(x)$, that is, also of $f(x+h)$, we get this series, whence through differentiation :

of $y_1 - y$ we obtain $f'(x)h$ or $\frac{dy}{dx}h + \cdots$,

of $(y')_1 - y'$ we obtain $f''(x)h$ or $\frac{d^2y}{dx^2}h + \cdots$,

of $(y'')_1 - y''$ we obtain $f'''(x)h$ or $\frac{d^3y}{dx^3}h + \cdots$.

Thus, we treat the original function x [not] as one simply considered by itself, but as one potentially containing all its "derivatives"; owing to this, $f(x+h) - f(x)$ or $y_1 - y$ too, not only contains $y_1 - y$, but also $(y')_1 - y'$, $(y'')_1 - y''$ etc.

Here we consider the original function in x, for example $x^m = y$, as potentially containing in itself all the functions deducible from it; that is why, [we do the same for] the increment $f(x+h) - f(x)$ or $y_1 - y$, as expressible through these derivatives, in place of which there then appear the equivalent differential symbols, i.e., the symbolic differential coefficients corresponding to them.

After this there are a number of aditions to p. 1 (to "Taylor's theorem") in the manuscript, the first among which was written in several variants. Towards the end Marx re-wrote it under the title "Prelude to p. 1". Here he raised the problem of transition from algebra — where x and h are constants, to the differential calculus — where each of them may be viewed both as a constant and as a variable, besides every time the sense, in which it is being considered, is to be specified. Namely, in this connection Marx also discussed the transition from the mode of expressing the changes of the variable x through the difference $x_1 - x$, to its expression through the sum $x + h$.

All these variants of the first supplement (in our list these are parts II, V, III and VII) are being described below, in the main in the order in which they were written in succession by Marx. An exception has been made for part V, which is an insertion, made by Marx, to what has been written earlier.

Part II of this manuscript (see, the list on p. 264) begins as follows (sheet 4) :

AD P.1) ADDITIONALLY

Let us take as given, what follows below where $(x+h)^1$ is an ordinary binomial and that is why x is not viewed as a variable, that

$$\text{a) } (x+h)^{m+1} \text{ or } y = x^{m+1} + (m+1)x^m\frac{h}{1} + (m+1)\,mx^{m-1}\frac{h^2}{1\cdot 2} +$$
$$+ (m+1)\,m\,(m-1)x^{m-2}\frac{h^3}{1\cdot 2\cdot 3} + (m+1)\,m\,(m-1)\,(m-2)\,x^{m-3}\frac{h^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

Our method permits differentiation of this equation, i.e., the acceptance of x as a variable, while h is considered to be a constant, and not an increment of x, since $x_1 - x$ exists for us only in this difference-form, and not as some $x_1 - x = h$ and, that is why not as $x_1 = x + h$.

After this Marx differentiates the equation a) (sheets 4-5) "algebraically" (i.e., by his own method) : wherein he at first changes x into x_1 and correspondingly y into y_1, from the equation thus obtained he subtracts equation a) term by term, then takes the common factors out of the brackets in each binomial and then transforms the factor $(x_1^p - x^p)$ into the form $(x_1^p - x^p) = (x_1 - x)\,(x_1^{p-1} + x_1^{p-2}x + \cdots + x_1x^{p-2} + x^{p-1})$ $(p = m,\ m-1,\ \cdots 1)$, and finally divides both the sides of the equality by $x_1 - x$, assumes that $x_1 = x$ and as a result obtains $\frac{0}{0}$[i.e., $\frac{dy}{dx}$] $= (m+1)\,x^m + (m+1)\,mx^{m-1}\frac{h}{1} + (m+1)m(m-1)\,x^{m-2}\frac{h^2}{1\cdot 2}$ + etc.

Then he divides both the sides of the obtained equation by $(m+1)$, writing therein, the result obtained in the left hand side in the form of $\frac{0}{0 \times (m+1)}$. Here sheet 5 (p. 5 in Marx's numeration) comes to an end.

On sheet 10, under the title : "*Ad. p.5*" (i.e., in part V of the manuscript) Marx returns to the latter notation, and observes : "*In order to escape from this* $\frac{0}{0 \times (m+1)}$ *etc.*" After this he transforms the afore mentioned equation into the form

$$A)\ (m+1)(x^m + mx^{m-1}\frac{h}{1} + m(m-1)x^{m-2}\frac{h^2}{1 \cdot 2} + \cdots +) = (m+1)(x+h)^m$$

and writes, at first justifying this transformation :

But we have repeated this binomial $(x+h)^m$ (in its unexpanded form) and in its serial expansion $(m+1)$ times. The factor $(m+1)$ is the umbilical cord, which indicates the origin of the derivative from $(x+h)^{m+1}$. Exactly in the same way ... the factor $(m+2)$ would indicate the origin of the derivative from $(x+h)^{m+2}$.

If in A) we strike out the factor $(m+1)$, then the binomial $(x+h)^m$ will appear as independent as its initial equation $(x+h)^{m+1}$, and we shall get :

$$B)\ (x+h)^m = x^m + mx^{m-1}h + m(m-1)x^{m-2}\frac{h^2}{1 \cdot 2} + m(m-1)(m-2)x^{m-3}\frac{h^3}{1 \cdot 2 \cdot 3} + \\ + m(m-1)(m-2)(m-3)x^{m-4}\frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots .$$

The insertion in p.5, i.e., part V ends with the words:

Equation A) gives us the binomial $(x+h)^m$, $(m+1)$ times, deduced from $(x+h)^{m+1}$. As an example, it is more than enough; thus; we proceed to B).

Thus, it is clear, that Marx carried out all these calculations, only for the sake of an example. In one of the latter supplements he writes, that he did it for obtaining the expansion of $(x+h)^m$ with the help of differentiation, though in that case the binomial theorem of Newton, is all the same assumed to be already proved, and besides [proved] for $(x+h)$ in its $(m+1)$-th power.

On p.6 of his manuscript, Marx went over to the following analysis of the equation obtained by him :

This equation of m-th degree is one degree lower than the equation of $(m+1)$-th degree, from which it is derived. Nevertheless we can transform y *or* $f(x)$ *into* y_1 *or* $f(x_1)$, without having to change the algebraic composition of the equation by the breadth of a hair. For this it is enough to :

1) put $x^m = f(x)$ or $= y$, and here it is all the more justified, since after the deduction of x^m from x^{m+1} we at once represented all the following functions, at first as directly deduced from x^m, and then successively one from the other — that is [represent] all of them as functions *successively* deduced from x^m ; and

2) consider h, which was an ordinary constant magnitude in the deduction of our equation, like a in $(x+a)^m$ in algebra, as the *increment* (positive or negative) of x. We have

the right to do this also as : $x_1 - x = \Delta x$, and this Δx itself, instead of serving , as in our mode, as a simple symbol or a simple sign for the difference of the x-s, i.e., for $x_1 - x$, may also be treated as the magnitude of the difference $x_1 - x$, [itself] as indeterminate as $x_1 - x$ and, as changing as it (this magnitude). Thus, $x_1 - x = \Delta x$ or = the indeterminate magnitude h. Hence it follows, that $x_1 = x + h$, and $f(x_1)$ or y_1 turns into $f(x + h)$.

Thus, we get : $f(x) = x^m$,

$$A)\ f(x+h) \text{ or } y_1 = (x+h)^m = x^m + mx^{m-1}h + m(m-1)x^{m-2}\frac{h^2}{1\cdot 2} + \dots *$$

If now we examine both the sides of this equation, then the left hand side shows us, that x^m or $f(x)$ turned into $(x+h)^m$ or into $f(x_1) = f(x+h)$, as x increased by h, since the binomial $(x+h)^m$ was obtained from the monomial x^m, which, however, now appears as an expression of the variation of x^m, and not, as in an ordinary binomial $(x+a)^m$, as an expression of the sum of two constants raised in power. This may be said about the *general unexpanded expression* $(x+h)^m$ or $f(x+h) = y_1$.

In the expanded serial expression on the right hand side, the first term x^m is no more — as in the binomial theorem — simply the highest power of the first term of the binomial $(x+h)^m$; it is $f(x)$, since $y = x^m$, and all the remaining terms together represent only the increment, which $f(x)$ or y, or x^m attained, as x increased by h.

After this Marx once more proves, that in the given case the first term of the expansion of $f(x+h)$, is $f(x)$ or y (in the given case x^m); and then he writes (sheet 7) :

That is why, we may write the equation $A)$ also as :

$$B)\ f(x+h) = y_1 = y + mx^{m-1}h + m(m-1)x^{m-2}\frac{h^2}{1\cdot 2} + m(m-1)(m-2)x^{m-3}\frac{h^3}{1\cdot 2\cdot 3} + \dots$$

We were in need of all these preliminary twists and turns, because in our method y_1 is represented not as the sum of $f(x)$ + its derived terms, but, conversely as the difference between $f(x_1)$ and $f(x)$, expressed generally through $y_1 = A\ (x_1^m - x^m)$ where A may represent an arbitrary constant [169].

The differential method proper proceeds from $x_1 - x = h$ (i.e., $= \Delta x$) ; hence, $x_1 = x + h$; that is why x_1 figures from the very beginning as $x + h$, i.e., as a binomial of the first power, since $x + h = (x+h)^1$, owing to which its differential expression is $(x + dx)$. With the exception of the functions *in* x of the first degree, for all the remaining functions *in* x, as soon as x increases by h, the powers of the *binomials* are that is why computed, beginning with the second, and the expansion itself constitutes an application of the binomial theorem[170].

---

* This has been repeated twice, almost word for word; since at first Marx did not designate the obtained equation by the letter $A)$ (for further reference) ; here only the repetition is being reproduced. — Ed.

Hence, here it stands to reason, that the first term of the series, i.e., of the binomial expansion $= f(x)$ or y, all the remaining terms are = the increment attained by this function, owing to the fact, that x turned into $x + h$ and that, hence, the happy expression for the general formula of the binomial $(x + h)^m$ instantly appeared *in the form of the equation B*).

The fact, that though Newton's binomial theorem for $(x + h)^m$ was obtained with the help of differentiation, however, it was subject to the supposed validity of this theorem for $(x + h)^{m+1}$, was not liked by Marx. And he made an attempt to substantiate this assumption. The lower half of page 7 (part III in our list) constitutes the beginning of this attempt.

This beginning is being reproduced below, in full (sheet 7) :

All this be better begun as under :

I

Suppose that given :

$$f(x) = x^6,\ f(x_1) = x_1^6 .$$

We have shown earlier (see the first manuscript[171]), that if $f(x) = x^m$, $f(x_1) = x_1^m$, [then]

$$y_1 - y = f(x_1) - f(x) = x_1^m - x^m = (x_1 - x)\,(x_1^{m-1} + x_1^{m-2}x + x_1^{m-3}x^2 + x_1^{m-4}x^3 +$$

$$+ \text{ upto the } m\text{-th term } x_1^{m-m}x^{m-1}).$$

Dividing this by $(x_1 - x)$, we shall get

$$\frac{y_1 - y}{x_1 - x} \text{ or } \frac{f(x_1) - f(x)}{x_1 - x} = (x_1^{m-1} + \cdots + x_1^{m-m}x^{m-1}) .$$

Assuming $x_1 = x$, hence , $x_1 - x = 0$, we get

$$\frac{dy}{dx} \text{ or } \frac{(f(x_1) - f(x))}{(x_1 - x)} = mx^{m-1}.$$

Just as the first derivative of x^m was obtained, so may we obtain all the latter. All of them are found through *one and the same method*, based on the *algebraic* presupposition, that a difference of the form $x^m - a^m$ is always divisible by $x - a$ and, hence, can always be represented as $(x - a)\,P$.

(Continuation on p. 11^a) .

It is difficult to understand the content of part VII (sheets 19-23 in our list) that follows. Only this much is clear, that having found the successive derivatives y', y'', $\cdots$ of $y = x^m$, Marx wanted somehow to substantiate the necessity : 1) of multiplying y, y', y'', $\cdots$, respectively by h^0, h^1, h^2, $\cdots$, 2) of dividing the obtained products, beginning with $y'h$, by 1, 1·2, 1·2·3, ... respectively, and 3) finally of adding all these quotients in order to thus obtain the expansion for $(x + h)^m$, without leaning upon Newton's binomial theorem. But Marx could not realise this intention[172], and later on he turned away from both the variants of the supplement to p.1, contemplated by him. That is why, those portions of part VII are being reproduced below, which are of independent interest.

The first among them is related to Marx's substantiation of the fact, that for an integral and positive m, $(x+h)^m$ must be expandable in a series according to the powers of h. Marx justifies it as follows (sheet 20) :

We know from algebra , that in the binomial $(x+h)^m$ $(=(a+c)^m$, for instance, where m is an integral, positive index of power) expressions in x have as their multipliers, the latter term of the binomial, here h, in ascending powers, [since] $(x+h)^6$ [f.i] = $(x+h)(x+h)(x+h)(x+h)(x+h)(x+h)$. Here, whether x is a variable, or an unknown algebraic constant, or even an unknown like a in $(a+h)^6$, the latter, in the given case the constant term h in ascending, integral and positive powers $h^0(=1)$, h^1, h^2 etc., will always be (under the given conditions) made multipliers of the successive expressions [in x], which are obtained for x, as constants in algebra, through succesive multiplication by itself, but which appear as functions of the variable x, while differentiating the intermediate "derivatives".

Here we shall also reproduce Marx's general comments on the modes of expressing the changes of variables in the "algebraic" method of differentiation,which Marx placed in p. 11[a], in connection with the question of the value of the constant as an item or a factor in such differentation (sheet 19).

Notabene : here h is introduced, not only in the beginning,as an ordinary *constant,* like $(x+a)^6$, $(x+c)^6$, $(a+0)^6$, $(a+h)^6$ in algebra ; it must always remain a *constant* and, on the basis of the algebraic method of differentiation adopted here, by itself it can neither be a variable, nor the increment of a variable, since in this method of differentiation, the difference of the independent variables x_1-x (accordingly that of the dependent variables y_1-y or $f(x_1)-f(x))$ remains always in this initial form of it and, hence, x_1-x, as well as y_1-y, can never be assumed to be equal to some value of the difference, $=\Delta x$ or h, and hence, can never be represented in the form of $x_1-x=\Delta x$ or h, $x_1=x+\Delta x$ or $=x+h$, nor can y_1-y become $=\Delta y$ or k. If we write $\dfrac{y_1-y}{x_1-x}$ or $\dfrac{\Delta y}{\Delta x}$ as an equivalent of the *preliminary derivative*, then for us these are only signs for the indeterminate x_1-x and y_1-y and not [*fixated*] *values of differences*, such that $x_1-x=\Delta x$, as the value of an indeterminate difference, or $y_1-y=\Delta y$, as just the same. In the entire deduction [of the derivative] from $f(x)$, the augmented x always figures as x_1 and, on the other hand, the augmented y — as y_1, that is why, they can not at the same time figure as $x+\Delta x$ and $y+\Delta y$.

Having turned away from both the variants of the supplement to p.1, Marx decided to substantiate the transition from the "algebraic" form of Newton's binomial theorem to its "differential"form,in another way. The following part (part VIII according to our list) of the manuscript is devoted to this. It is being reproduced here in full.

Here, under the tittle : *"Prelude to p.1; Taylor's Theorem"*, Marx begins with the formula

I) y_1 or $f(x+h)=(x+h)^m=$

$$=x^m+mx^{m-1}\frac{h}{1}+m(m-1)x^{m-2}\frac{h^2}{1\cdot 2}+m(m-1)(m-2)x^{m-3}\frac{h^3}{1\cdot 2\cdot 3}+\cdots\cdot+x^{m-m}h^m,$$

and writes about it a few lines below (sheet 23) :

Considering sub I) the equation $y_1 = x^m + \text{etc.}$, to be independently appearing, we can prima facie take it as an ordinary algebraic expression of a binomial of the simplest form $(x+h)^m = x^m + \text{ etc.}$, and, namely, the same in integral and positive power, as it was assumed from the very beginning about the index of power m. We shall, at first, take it in this form as the point of transition to the differential method, where it is assumed that $x_1 - x = h$ [[hence, also $y_1 - y = k$, if it suits]], that is $x_1 = x + h$. This gives us the opportunity of direct [transition to the differential method] from the method applied so far, where $x_1 - x$, $y_1 - y$, appear only in this universal form of their difference and where, also, as soon as on the right hand side of the equation, x increases, it appears only as x_1, i.e., in the same indeterminate form, as the initial x, and never turns into $x_1 = x + \Delta x$ or $x + h$. But it would be a mistake to think, that it was possible to act along the opposite path, i.e., to assume, conversely, that $h = x_1 - x$, and to pass over from the differential method to ours.

We reproduce below the whole of the remaining text of part VIII (according to our list) in full (sheets 24-25) :

$$\left.\begin{matrix}\mathrm{I)}\\ \mathrm{a)}\end{matrix}\right\{ (x+h)^5 = x^5 + 5x^4h + 5\cdot 4x^3\frac{h^2}{1\cdot 2} + 5\cdot 4\cdot 3x^2\frac{h^3}{1\cdot 2\cdot 3} + 5\cdot 4\cdot 3\cdot 2x\frac{h^4}{1\cdot 2\cdot 3\cdot 4} + h^5 .$$

This is an ordinary algebraic equation; on the one hand, the unexpanded general expression of the binomial of two constants x and h, namely, in the left hand side $(x+h)^5$; on the other, the serial expansion, attained through the binomial theorem in the right hand side. If in the general expression $(x+h)^5$ we put $h = 0$, then instead of $(x+h)^5$ we shall get $(x+0)^5 = x^5$. If now we consider x as a variable, then x^5 will be a determinate function *in* x, namely $f(x)$, but x^5 is also a function of x, because the value of $f(x)$ changes with the variations of x; thus, now, viewing this function as a function dependent on x, we call it y; and the expression $f(x)$, when it is taken in the sense of this dependence, is equivalent to y. On the other hand, since $x^5 = f(x)$, $(x+h)^5$ turns into $f(x+h)$ or y_1. Thus, in place of I)a) we get :

$$y \text{ or } f(x) = x^5,$$

$$y_1 \text{ or } f(x+h) = (x+h)^5 = x^5 + 5x^4h + 5\cdot 4x^3\frac{h^2}{1\cdot 2} + 5\cdot 4\cdot 3x^2\frac{h^3}{1\cdot 2\cdot 3} + \cdots + h^5.$$

If we again put $h = 0$, then this series is reduced to x^5; in the left hand side $f(x+h)$ is reduced to $f(x)$ or y_1 to y. That is why, we have : y or $f(x) = x^5$. If we substitute this value of x^5 in the equation I)a), then it turns into equation b)*[see, below]. The first term of the series, in the ordinary binomial expansion of two constants remains in its own place in the first term, but it changes its character completely. It is no more the first term [of the expansion] of the ordinary binomial $(x+h)^5$, but [it is] the prototype function of the variable x — namely, x^5 — before it turned into $f(x+h)$, and hence x^5 too turned into $(x+h)^5$.

* After "b)" in the manuscript at first there was written : "(see, the opposite page)". Part of this statement was later on struck out. — Ed.

Thus, now the eqation has the form :

b) y_1 or $f(x+h)=y$ or $f(x)+\left(5x^4h+5{\cdot}4x^3\frac{h^2}{1{\cdot}2}+5{\cdot}4{\cdot}3x^2\frac{h^3}{1{\cdot}2{\cdot}3}+5{\cdot}4{\cdot}3{\cdot}2x\frac{h^4}{1{\cdot}2{\cdot}3{\cdot}4}+h^5\right)$.

On the other hand,from b) it follows that :

c) y_1-y or $f(x+h)-f(x)=5x^4h+5{\cdot}4x^3\frac{h^2}{1{\cdot}2}+5{\cdot}4{\cdot}3x^2\frac{h^3}{1{\cdot}2{\cdot}3}+5{\cdot}4{\cdot}3{\cdot}2x\frac{h^4}{1{\cdot}2{\cdot}3{\cdot}4}+h^5$.

Thus, at one stroke, having turned the first term x^5 into $f(x)$, the whole of the remaining bionomial series, beginning with its second term, was also turned into a series of terms, constructed from the successivly derived functions in x.

All the h, h^2, h^3 etc., which accompany these "derived" functions as multipliers, now turn into different powers of the difference between $x_1$and x or between $(x+h)$ and x — since $x_1-x=h$, or $x_1=x+h$ — from being the second terms of the ordinary binomial $(x+h)^5$, because $((x+h)-x)=h$ is a difference, which emerges owing to the fact that the variable x grew into x_1 or $x+h$. Thus, the *entire series* is now *the difference between the original function in x and the function of x grown into* $(x+h)$. This difference may also be positively represented as an increment, which the original function in x obtained, as x increased into x_1 or $x+h$. Thus, thanks to a very simple manoeuvre *the entire serial expansion of* $(x+h)^5$,

a) from the binomial expression of two constant magnitudes, turned into a series, whose first term is the original function of the variable x, and all the remaining terms are the sum of increments, attained by this original function,owing to the fact that, from $f(x)$ it has turned into $f(x+h)$. Hence, all these increments sprang up from its movement,and not as terms of an ordinary algebraic binomial expansion. Whence follows(see above) equation c).

[b)] However, the supposition of $h=0$ served us, not only in the metamorphosis of the ordinary binomial of two constants into the expansion for the function of one variable x,when this variable increases, from the very beginning it also indicated the *method* to be applied,in order, on the one hand, to free the ready-made successive derived functions in x from their surroundings, wherein they are situated in the serial expansion, and on the other,to produce the correspending symbolic differential coefficients.

A) It became possible to put the first term of the series $x^5=f(x)$, because : 1) it itself has the factor $h^0=1$, i.e., is free from h, and 2) the supposition of $h=0$ has already removed all the remaining terms and,thus the entire series has been reduced to x^5. That is why, so that the derived functions $5x^4$, $5{\cdot}4x^3$ etc. found themselves in a condition analogous to x^5, they are to be : 1) successively freed from the multipliers h, h^2 etc., which is possible only through successive division by h, and 2) when a "derivative" is freed from h, i.e., turns out to be freed as a "derived" function, then it is necessary, as it was with the first term, to assume that $h=0$, i.e., to preliminarily remove, at the same time, all its collateral terms from the path ; and the entire series is reduced to a freed "derivative", as it happened with the first term[173].

B) The supposition of $h = 0$, in the operation with the first term of the series turned the left hand side, from $f(x + h)$ or y_1, into $f(x + 0)$ or y, in other words into $f(x)$. But as the *derived functions* may be freed from their multipliers h only through a division by h, this division produces in the left hand side

$$\frac{y_1 - y}{h}\left(= \frac{y_1 - y}{x_1 - x}\right) \text{ or } \frac{f(x+h) - f(x)}{h}.$$

Hence, when, in the right hand side h turns into 0, so as to reduce the entire series into a freed "derivative", then the left hand side inevitably assumes the form

$$\frac{y_1 - y}{0} = \frac{0}{0} \text{ or } \frac{f(x+0) - f(x)}{0} = \frac{0}{0} = \frac{dy}{dx}.$$

Thus, the supposition of $h = 0$ generates, in the left hand side, the symbolic differential coefficient of the "derived" function *in* x.

Hence, now, it is *proved,* that the first operation,through which it is established that the first term of the series $= f(x)$ or y, gives two things at once :

1) transformation of the ordinary binomial $(x + h)^5 = x^5 +$ etc., into $f(x + h) = f(x) +$ a series of functions with the multipliers h, h^2 etc., deduced from $f(x)$, i.e., with the powers of the increment, which the independent variable x obtains, when from x it turns into x_1, i.e., when it obtains the increment $h = x_1 - x$;

2) the *method,* which frees the ready-made "derived" functions in x, as such (in fact it is obtained through the expansion of the ordinary binomial of the two constants x and h), and, along with this, counterposes them opposite their *symbolic* differential expression. Thus, we can, now, again return to the business.

$$\left.\begin{matrix}\text{I)}\\ \text{a)}\end{matrix}\right\{ (x+h)^5 = x^5 + 5x^4h + 5{\cdot}4x^3\frac{h^2}{1{\cdot}2} + 5{\cdot}4{\cdot}3x^2\frac{h^3}{1{\cdot}2{\cdot}3} + 5{\cdot}4{\cdot}3{\cdot}2x\frac{h^4}{1{\cdot}2{\cdot}3{\cdot}4} + h^5.$$

In fact, so far, in this manuscript only the other mode of representing the bionomial theorem of Newton, has been discussed, i.e., its translation, from the language of algebra to the language of differential calculus and, correspondingly the serial expansion of $(x + h)^m$ (m — integral and positive) according to the powers of h. But even in its formulation without the residual term, Taylor's theorem is valid for a much wider class of functions, the transition to which is connected with the transition from a finite polynomial to an infinite series. Marx discusses this transition, in one more addition to p.1 of the manuscript, which follows the "Prelude". Here he specially dwells upon the difficulties, connected with the extension of the expansion obtained for $(x + h)^m$ (when m is integral and positive) to a wider class of functions. This addition is being reproduced below in full. With this begins part IX of the manuscript (according to our list) (sheets 26-28).

AD P. 1 (TAYLOR'S THEOREM)

We deduced the initial equation

1) $(x + h)^m = x^m + mx^{m-1} h + \cdots\cdots$

itself, through *differentiation* from the equation

$(x + h)^{m+1} = x^{m+1} + (m + 1)\, x^m h + \cdots$

But thus we took the same *binomial theorem* as the already given starting point : we simply took as the starting point, the binomial $(x + h)$ in its $(m + 1)$-th power, so as to obtain through *differentiation* the binomial $(x + h)^m$, so that it could appear through differentiation, as the given starting point.

But the *algebraic binomial* is related only to the binomials of a determinate power. From the point of view of algebra , these are based only on constant magnitudes, $x + h$ itself is the binomial of first power = $(x + h)^1$. We can continue the series obtained for $(a + h)^m$ as far as we wish, and designate this continuation through + , as it happens in equations 3) and 4) *, because m is an algebraic magnitude with an *indeterminate* numerical value; but this is no hindrance to the finitude of the series. We can also write it with a beginning and an end :

$x^m + mx^{m-1}h + \cdots + mxh^{m-1} + h^m$.

Here y_1 or $f(x + h)$, which we use in the differential calculus, is and remains only a symbol for the binomial $(x + h)$ of a determinate though arbitrary power.

Thus, only *formally*, through the interruption with + etc., do we get an infinite series, whereas in fact it is expressible, as in the generalised form, with a beginning and an end, with the intermediate + etc., in the middle. But this is not all. Coefficients of the functions h^0 (or 1), h^1 , h^2 , h^3 etc. show that we represented $(x + h)^m$ as expanded in integral, positive and ascending [powers of h] . Thus we have put at the base, not only an algebraic [binomial], and that is why, essentially, some power of a given binomial, but a special form of the binomial theorem proper. This apart, we should not forget, that for obtaining h simply as a multiplier of the functions in x, we chose the form $(x + a)^m$ (since in an algebraic binomial h is a constant, analogous to a), where x is the first term, and h — the latter, instead of the form $(a + x)^m$, where they stand in the inverse order.

It is true that we could also have started from a binomial with negative or fractional power, $-m$ or $\frac{m}{n}$, and thereby could have obtained an infinite series.

But in this case too, not to speak of the other limitations, which will be stated later on, only a binomial of some determinate power is again put at the base, since $-m$ or $\frac{m}{n}$ are also determinate powers, just like m. Even in this case, the infinite series is an expansion of some general expression in a determinate power (like $(x + h)^{-m}$, $(x + h)^{\frac{m}{n}}$, [[in a determinate]] though negative or fractional [[power]].

---

* See, PV , 273 .

Here, the y_1 or $f(x+h)$, obtained by us, always remains only the symbol of a binomial of some power : positive, negative or fractional ; that is, here the series corresponding to such y_1 or $f(x+h)$, is also nothing but a generalised expression of a power of the binomial, [it is] in fact only the *generalised expression of an example of a binomial* of some determinate algebraic form.

Perhaps, this deficiency may be avoided. It may be possible for us to get rid of this limitation of the algebraic binomial, having recourse to the algebraic method of indeterminate coefficients. At first we shall take the expression in equation 2) back to its initial algebraic form, namely from :

$$2)\ f(x+h) \text{ or } y_1 = x^m + mx^{m-1}\frac{h}{1} + m(m-1)x^{m-2}\frac{h^2}{1\cdot 2} + m(m-1)(m-2)x^{m-3}\frac{h^3}{1\cdot 2\cdot 3} + \cdots$$

[to]

$$2\,a)\ f(x+h) \text{ or } y_1 = x^m + \left(\frac{m}{1}x^{m-1}\right)h + \left(\frac{m(m-1)}{1\cdot 2}\,x^{m-2}\right)h^2 + \left(\frac{m(m-1)m(m-2)}{1\cdot 2\cdot 3}\,x^{m-3}\right)h^3 + \cdots$$

Here the functions no more appear as *integral functions* [174], they appear, as it was initially the case in the *binomial expansion*, as

$$x^m + mx^{m-1}\,h + \frac{1}{2}\,m(m-1)\,x^{m-2}\,h^2 + \frac{1}{1\cdot 2\cdot 3}\left(\text{or } \frac{1}{6}\right)m\,(m-1)\,(m-2)\,x^{m-3}\,h^3 + \cdots\cdots$$

and this has been indicated [above] through the brackets.

After this formal modification, consisting of the fact, that h, h^2, h^3, etc. are freed from their denominators, and functions in x are represented in that form, which they had in their initial algebraic deduction, where only the second term mx^{m-1} appears as an integral function, the third as half of the integral function $m(m-1)\,x^{m-2}$ etc., we substitute the functions of x by the indeterminate coefficients A, B, C etc. and obtain for 3) (and hence, also for 4))

$$f(x+h) \text{ or } y_1 = f(x) \text{ or } y + Ah + Bh^2 + Ch^3 + Dh^4 + Eh^5 + \cdots$$

Here A, B, C, D, E etc. appear as functions of x (compare 2a)), which, however, are yet to be found out. This apart, the series in itself may be continued as far as one wishes. And now it can no more be reduced to a series, whose end, like its beginning will also lend itself to determination, since we can go on writing at our will, as far as $Fh^6 + Gh^7$ etc. etc., making the numerical coefficients $\frac{1}{2}$, $\frac{1}{1\cdot 2\cdot 3}$, $\frac{1}{1\cdot 2\cdot 3\cdot 4}$, which give away their origin, invisible in the indeterminate coefficients *A, B, C, D, E, F* etc., along with the derived functions themselves. While this manoeuvre is carried out in the right hand side, in the left $f(x+h)$ or y_1 turns from the symbol of a binomial of some power, into a general unexpanded expression of this infinite series, not having any power, though it does include every power.

$f(x)$ is [[a]] function of [[a]] variable x, open to infinite expansion ; $f(x+h)$ is the general unexpanded expression of this function [of] x, when the latter turns into $x+h$ from x.

Further, it is understandable in respect of the equation

$$3)\quad y_1 \quad \text{or } f(x+h) = y + f'(x)\,h + f''(x)\frac{h^2}{1\cdot 2} + \cdots,$$

that for the functions $f'(x)$, $f''(x)$ [etc.] we may not as simply substitute ~~their~~ symbolic equivalents, as it happens in equation 4), but that [these equivalents] are yet to be found out through differentiation [175].

The section of the manuscript, which follows this addition, and which is devoted to a critique of the proofs of the theorems of Taylor and MacLaurin, as they were known to Marx, show that according to Marx the justification of this entire manoeuvre was not at all self-evident. This section begins as follows (sheet 28) :

TAYLOR'S THEOREM

Taylor's initial equation is :

$$\text{I)}\quad y_1 \text{ or } f(x+h) = y + Ah + Bh^2 + Ch^3 + Dh^4 + Eh^5 + \cdots$$

To obtain the opportunity to work with this theorem, it is necessary to use the manoeuvre based on the same binomial [theorem], consisting of an application of this theorem to a polynomial expression, through its representation in the form of a binomial.

Let us take as an example $(x+a)^2$, and write $(x+a+b)^2$ instead of the latter. We can expand it : as $((x+a)+b)^2$ or as $(x+(a+b))^2$;

in the first case we shall have :

$$(x+a+b)^2 = (x+a)^2 + 2(x+a)\,b + b^2 ;$$

in the second case :

$$(x+a+b)^2 = x^2 + 2x\,(a+b) + (a+b)^2 .$$

The important thing in this example is the fact, that between the two right hand sides there is only a formal, though wittingly merely formal, difference; however, from the very beginning their identity is demonstrated through the identity of the left hand sides.

Having then written: "*And now to business*", Marx went over to an account of the proof of Taylor's theorem, according to Boucharlat. As we already know (see, for example, Appendix pp. 337-338) in Boucharlat's book this proof is preceded by a lemma (§§ 55, 56, pp. 34-36), asserting that

$$\frac{df(x+h)}{dx} = \frac{df(x+h)}{dh}.$$

Marx begins with this lemma. Having given an account of its proof according to Boucharlat, Marx comments (sheet 29) : "*It could have been proved much more simply*".

The explanation that follows is so sketchy, that here one can only surmise about the course of Marx's thought. We shall hazzard below one such conjecture.

In order to prove that the equality

$$\frac{df(x+h)}{dx} = \frac{df(x+h)}{dh}, \tag{*}$$

holds, one may argue as follows. If we attribute to x the augmented value x_1, then $x+h$ turns into x_1+h and $f(x+h)$ into $f(x_1+h)$, and we shall have

$$\frac{\Delta f(x+h)}{\Delta x} = \frac{f(x_1+h)-f(x+h)}{x_1-x} = \frac{f(x_1+h)-f(x+h)}{(x_1+h)-(x+h)}.$$

Analogously

$$\frac{\Delta f(x+h)}{\Delta h} = \frac{f(x+h_1)-f(x+h)}{h_1-h} = \frac{f(x+h_1)-f(x+h)}{(x+h_1)-(x+h)}.$$

If now to facilitate the argument we substitute $x+h$ by z (this Marx himself did not do), then we shall always have the opportunity to represent the augmented value of z, i.e. z_1 in the form of x_1+h, as well as in the form of $x+h_1$. Herein both the ratios

$$\frac{f(x_1+h)-f(x+h)}{(x_1+h)-(x+h)} \text{ and } \frac{f(x+h_1)-f(x+h)}{(x+h_1)-(x+h)}$$

may be represented as

$$\frac{f(z_1)-f(z)}{z_1-z},$$

and both will be equal to one and the same "preliminary" derivative according to z for $f(z)$ (i.e., for $f(x+h)$), if the latter exists. Whence the correctness of equation (*) follows very easily. The crux of the affair is again the fact, that z_1-z may be represented both in the form of x_1-x, as well as in that of h_1-h, so that $x_1-x=h_1-h$. The following words of Marx (sheet 29), go to show that the aforementioned conjecture really corresponds to Marx's ideas :

Since we have two expressions for the sublated difference (x_1-x), namely (x_1-x) and (h_1-h) [...], it is clear that these two are only formally different forms, of one and the same sublated difference, which are mutually equal [176].

The explanation omitted here, was placed by Marx within brackets. It contains such a slip of pen, which could not be removed without some sort of conjecture.

Further, Marx analogously treats the "proof" of Taylor's theorem according to Boucharlat, which is based on this lemma : at first he enunciates this proof and then criticises it.

Boucharlat's proof (see, § 57, pp. 36-37) consists of this : at first the aforementioned equation I) is differentiated in respect of h and in respect of x, which gives the equations

$$\text{II)}\quad \frac{dy_1}{dh} = A + 2Bh + 3Ch^2 + 4Dh^3 + 5Eh^4 + \cdots,$$

$$\text{III)}\quad \frac{dy_1}{dx} = \frac{dy}{dx} + \frac{dA}{dx}h + \frac{dB}{dx}h^2 + \frac{dC}{dx}h^3 + \frac{dD}{dx}h^4 + \frac{dE}{dx}h^5 + \cdots;$$

referring to the lemma, owing to which $\frac{dy_1}{dh} = \frac{dy_1}{dx}$, Boucharlat then equalises the coefficients of the same powers of h in the equations II) and III), and thus obtains Taylor's theorem in the form of the equation

IV) $y_1 = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\cdot2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\cdot2\cdot3} + \frac{d^4y}{dx^4}\cdot\frac{h^4}{1\cdot2\cdot3\cdot4} + \cdots$

Then Marx criticises this entire procedure as under (sheet 30) :

The question arises, how we could transform formula I), an algebraic binomial of the simplest form $(x+h)^m$, masked by indeterminate coefficients, into IV) ?

Didn't that adroit manoeuvre — which consisted of : first differentiating I) according to h, and then according to x, and thus obtaining for y_1 two differential equations, the general expressions [in the left hand sides] of which $\frac{dy_1}{dx}$ and $\frac{dy_1}{dh}$ are identical and that is why the respective serieses of expansion are equivalent, owing to which their separate terms may be equated with each other — have a decisive significance here ?

Not at all. The criterion indicating *which functions of x in the two equations may be equated*, is given by the factors $h^0\,(=1)$, h^1, h^2, h^3, h^4 etc. Only those functions may be *equated* whose factors happen to be the same powers of h. But where from did we get these very factors h^0, h^1, h^2, h^3, etc., stipulating the whole process ?

The initial equation I) is like this :

I) $y_1 = yh^0 + Ah^1 + Bh^2 + Ch^3 + Dh^4 + Eh^5 + \cdots$

It is true, that we have disguised the functions $f'(x)$, $f''(x)$, $f'''(x)$, etc. derived from the binomial, in the *indeterminate coefficients* A, B, C, D, E etc. However, we remember about the latter that they are after all functions of x, or else we could not have differentiated them, neither [according to] x, nor [according to] h ; but in our equation we grabbed the factors h^0, h^1, h^2, h^3 etc. in [their] pristine virgin form, as they were given to us by the binomial theorem; and by not fastening them in the equation by any trick, with the two identical but formally different translations of equation I) into differential forms, we could not seduce anybody.

After this Marx went over to discussing the attempts to prove, by the method, not only of indeterminate coefficients A, B, C, but also of indefinite indices of powers $\alpha, \beta, \gamma, \cdots$ of h, that h must enter into the expansion of $f(x+h)$, into a series according to the power of h, namely in the same way, as it has been presupposed in equation I. Here he adduces the proof of Taylor's theorem according to Hind (see, Hind, §74, pp. 83-84) and criticises it. Since an aquaintance with this proof is required, for understanding Marx's critique of it, its account, as given in the manuscript (sheets 31-32), is being reproduced below in full.

This is still more vividly striking in the later attempts to give the differential deduction from I) such a form, wherein not only would the functions of x, but also the *powers of h, appear* to be found *through differentiation*.

Namely, here the initial equation is written as :

I a) $y = y + Ah^\alpha + Bh^\beta + Ch^\gamma + Dh^\delta + \cdots$

And then we get :

1) $\frac{dy_1}{dh} = \alpha Ah^{\alpha-1} + \beta Bh^{\beta-1} + \gamma Ch^{\gamma-1} + \delta Dh^{\delta-1} + \cdots,$

$$2)\quad \frac{dy_1}{dx} = \frac{dy}{dx} + \frac{dA}{dx}h^{\alpha} + \frac{dB}{dx}h^{\beta} + \frac{dC}{dx}h^{\gamma} + \frac{dD}{dx}h^{\delta} + \cdots$$

Then we argue as follows : the left hand sides of both 1) and 2) are equal, hence, their serial expansions too are equivalent, and their correspondingly analogous terms may be equated; that is why, in the first place, $\alpha\ Ah^{\alpha-1} = \frac{dy}{dx}$. But since $\frac{dy}{dx}$ has the multiplier 1 ($= h^0$), so $h^{\alpha-1}$ must be $= h^0$, and, hence, $\alpha-1 = 0$, that is, $\alpha = 1$.

Thus we have killed three birds with a single shot : 1) since $\alpha = 1$, so now we know, that in the initial equation $Ah^{\alpha} = Ah^1 = Ah$; 2) thereby the values of the indeterminate indices of power β, γ, δ etc. in h^{β}, h^{γ}, h^{δ} etc., have also been *potentially* given to us; 3) since $\alpha\ Ah^{\alpha-1} = \frac{dy}{dx}$, i.e., $\alpha Ah^{\alpha-1} = 1\cdot Ah^{1-1} = A$, so $A = \frac{dy}{dx}$.

The second act of equating is as follows : $\beta Bh^{\beta-1} = \frac{dA}{dx}h^{\alpha}$. Hence, $h^{\beta-1} = h^{\alpha}$; that is $\beta - 1 = \alpha$; but since $\alpha = 1$, $\beta - 1 = 1$, i.e., $\beta = 2$ and $\beta Bh^{\beta-1} = 2Bh^{2-1} = 2Bh$.

Now we know firstly, that in the initial equation Ia) we can put Bh^2 instead of Bh^{β} ; hence, it is *deduced*, and not assumed from the very beginning.

Secondly, that $\beta Bh^{\beta-1} = \frac{dA}{dx}h^{\alpha}$ turns into $2Bh = \frac{dA}{dx}h$ (since $h^{\alpha} = h^1$), hence, $2B = \frac{dA}{dx}$, and $B = \frac{1}{2}\cdot\frac{dA}{dx}$; if here we substitute the value of A , then

$$B = \frac{1}{2}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{2}\cdot\frac{d^2y}{dx^2}.$$

There is no need to extend it further.

It may appear, it is true, that Marx's criticism of this proof is based on a misunderstanding. Hind assumes that $f(x+h)$ may be expanded into a series "according to the integral, positive and ascending powers of h " (see, Hind §74, p.83), and in fact in this presupposition it is demonstrated, that if the coefficient of $h^k (k > 0)$ is not identically equal to zero, then the coefficients of h^1, h^2, $\cdots$, h^{k-1} also can not be identically equal to zero. (The fact is this, that Hind also presupposes, though tacitly, that the coefficients of $h^{\alpha}, h^{\beta}, h^{\gamma}, \cdots$ are "finite", i.e., are different from 0, as well as from ∞). However, this proof is highly slipshod in form and may give rise to the impression : as if; following Lagrange, the author wants to demonstrate the expansibility ("in the general instance ") of *every* $f(x+h)$ into a series according to the powers h^0, h^1, h^2, $\cdots$; namely, thus did Marx understand it. Apart from this, Hind no where verifies the validity of the initial presupposition (regarding the properties of the indices, of power α, β, γ, $\cdots$) for any concrete function, i.e., as such, no where does he find it necessary. In sum, Marx has rightly reproached Hind, by saying, that he merely created an appearence of greater generality for his formulation of Taylor's theorem. According to Marx, Hind did not in vain directly formulate his initial assumption, that he is going prove Taylor's theorem for the functions $f(x+h)$, permitting its representation in the "binomial form", i.e., as

$$f(x+h) + f(x) + Ah + Bh^2 + \cdots$$

The entirety of Marx's criticism of Hind is being reproduced here in full (sheets 32-33)

This improved and more pretentious version of Taylor's expansion comes to the following. The equalisation of the first terms of both the equations, is the key to it all :

$\alpha\ Ah^{\alpha-1} = \frac{dy}{dx}$; and since $\frac{dy}{dx}$ has as its multiplier h^0, then $h^{\alpha-1}$ must be $= h^0$, and, hence, $\alpha - 1 = 0$, i.e., $\alpha = 1$ and $\alpha\ Ah^{\alpha-1} = Ah^{1-1} = A$.

But let *, in the initial equation

$$y_1 = yh^0 + Ah^{\alpha}\,[+\cdots]$$

α be a negative magnitude (and an indeterminate α may be any thing, as it is indicated by the binomial theorem in its general account), then Ah^{α} turns into $Ah^{-\alpha}$ and $\alpha Ah^{\alpha-1}$ into $-\alpha\ Ah^{-\alpha-1}$. Now if we put $\alpha = 1$, as it follows from the aforementioned account, then $-\alpha\ Ah^{-\alpha-1} = -Ah^{-1-1} = -Ah^{-2}$.

Since Marx did not have a sign for absolute magnitude, to express that α is a negative number, he substitutes α by $-\alpha$, where the latter α is positive..

Hence, according to our previous argument, we must conclude, that since

$$\frac{dy}{dx} = \frac{dy}{dx} \times 1 \quad (\text{or } h^0),$$

so it must be that $h^{-2} = h^0$; i.e., $-2 = 0$, or, in other words, since $\frac{dy}{dx} \cdot h^0 = -\alpha\ Ah^{-1-1}$, so it must be that $h^{-1-1} = h^0$ and, hence, $-1-1 = 0$, which results in $-1 = +1$. However, this ridiculous result only demonstrates, that we quietly proceeded from the presupposition, that h^{α} is only a masked expression for h^1, i.e., we presupposed the equality $\alpha = +1$, and our intention of not only establishing the expansibility into a series with the factors h^0, h^1, h^2, h^3 etc., but also of deducing the corresponding numerical powers of $h: 1, 2, 3$, etc., through differentiation, from the indeterminate indices of power α, β, γ etc., was from the very beginning a thoroughgoing fraud.

Further, in Ah^{α}, α could be less than 1, and then $\alpha - 1$ would also be a proper fraction, so that the general expression [for the power of h] would be of the form $h^{m/n}$. Exactly in the same way α could also be [irrational].

Here instead of an irrational α, Marx wrote $\log h$, and it is not clear whether it is meant as the *index* of the power of h, or as the power h..

However, herein the initial equation

$$y_1 = y + Ah^{\alpha} + Bh^{\beta} + Ch^{\gamma} + Dh^{\delta} + \cdots$$

is *either only an improved mask for* h^1, h^2, h^3, borrowed from the binomial, or if we actually consider α, for example, to be a *simple general symbol* of the index of power, i.e. in all possible forms of α, then the terms of the two equations deduced through differentiation, are not to be equated, i.e., they are not worth a farthing.

* The sense here is : "let us consider an instance, when ". —Ed .

But we do know, that there exist functions of x, which, when x increases by h, give negative or fractional powers of the latter.

Hence, the *result*, consists of this : that the [multipliers] of the different successive derived functions in x, borrowed from the binomial, namely, h^0, h^1, h^2 etc., must remain *presuppossed* ; that we got them through the binomial, and not through differential development; that, consequently, the binomial form, where from to proceed, is fully determined, where the multiplier h is expanded in *integral*, positive and ascending powers*.

Marx continues his critique further. Now he dwells upon the question of expressing the indeterminate coefficients A, B, C, D etc., through the derived functions of $f(x)$ and, correspondingly, upon the instances, where (according to Marx) a "finite" derivative does not exist. He writes (pp. 33-34) :

But this is not all. Even with the derived functions in x (see, 3)) $f'(x)$, $f''(x)$ etc., disguised in the indeterminate coefficients A, B, C, D etc, the affair is not fully Kosher**.

There we deduced : $f'(x) = \frac{dy}{dx}$, but there we also had $f(x) = x^m$, thanks to which $f'(x) = m\,x^{m-1}$, i.e. like all the functions in x, deduced latter on with the help of the binomial, $f'(x)$ was a *determinate and finite expression in* x ; and , we note, that the initial functions, which are expanded with the help of the binomial, always have, in their *general form*, a determinate power, as for instance, when $f(x) = x^m$, as well as, when it $= \frac{a^m}{x^m - a^m}$.

Whichever way the derived functions might be represented : through a finite or an infinite series, each term of such a series is determinate and, to that extent, a *finite* expression in x. For example, in x^4 (or x^m) $4x^3$ (or mx^{m-1}) is a determinate and finite expression, which does not in any way influence its possibility of being expanded as a function in the variable x. Owing to the fact that x again turns into $x + h$, $4x^3$ is tranformed into $4(x+h)^3$, though $4x^3$ is a *determinate* expression, and is to that extent *finite*.

The same for

$$\frac{a}{a-x} = 1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3} + \cdots$$

Each term of the series, $\frac{x}{a}$, $\frac{x^2}{a^2}$, $\frac{x^3}{a^3}$ etc., is a *finite, because determinate, function of* x, besides it is a matter of total indifference, whether I find the series by dividing a by $a - x$ or, through a shorter route — by successive differentiation of $\frac{a}{a-x}$.

But when we substitute the indeterminate coefficients A, B, etc., in place of $f'(x)$, $f''(x)$ etc., we face the same dilemma, as in h^0, h^1, h^2, etc., i.e., for the multipliers of the

* Here an expansion of the form $f(x+h) = A + Bh + Ch^2 + Dh^3 + \cdots$, is called the "binomial form". — Ed.

** Marx used this word ironically. In ancient Hebrew it means : strict observance of the religious prescription, prohibiting the use of one and the same utensils for meat and milk products. Evidently, here Marx wishes to stress, that even from the point of view held by the author, the arguments given here are unsatisfactory. —Ed.

derived functions in Taylor's initial equation, as a whole transferred from a special form of the binomial.

Either A, B, C etc. are indeterminate coefficients — only the other names of the *determinate* and *to that extent "finite" derived functions in* x, borrowed from the binomial, or as general symbols for the derived *functions* in x, A, B, C etc., must be the symbols of the derived functions, not only when the latter are determinate and finite, but also when they are $= 0$, $= \infty$, or $= -\infty$. And we already know, that in fact such is the case.

Since in the following pages of the manuscript (pp. *i*, *h*, again *i*), Marx sums up all that he has already said about Taylor's theorem (sheets 34-36) and gives a general outline of his attitude to Lagrange's attempt to substantiate the initial assumption, upon which the demonstration of Taylor's theorem is built, as per the books of Boucharlat, Hind etc. — the text of these pages of the manuscript has already been reproduced in the first part of the present volume (see,. pp. 93-94 and note [82]).

After this Marx went over to a more detailed account of Lagrange's demonstration of Taylor's theorem. He wrote (p. *i* continued and p. *j*); (sheets 36-37) :

If we take Taylor's initial formula :

$$y = f(x),$$

I) y_1 or $f(x+h) = f(x) + Ah + Bh^2 + Ch^3 + Dh^4 + Eh^5 + \cdots,$

then we can write it as :

$$= f(x) + h\,(A + Bh + Ch^2 + Dh^3 + Eh^4 + Fh^5 + \cdots).$$

If we call the entire expression within the brackets P, then

1) y_1 or $f(x+h) = f(x) + Ph$.

Lagrange says, that this is the expression, to which the entire serial expansion must be reducible, as soon as the variable x turns into $x+h$ and, hence, $f(x)$ turns into $f(x+h)$; because, putting $h = 0$, we shall get $f(x+h) = f(x)$, in other words, $f(x+h)$ is conversely reduced to its original expression. Thus, it demonstrates to us, that in the serial expansion of $f(x+h)$, its first term must be $= f(x)$ or y.

Let us now consider Ph more closely :

$Ph = h\,(A + Bh + Ch^2 + Dh^3 + Eh^4 + \cdots)$;

hence ,

$P = A + (B + Ch + Dh^2 + Eh^3 + \cdots)\,h.$

If we designate this A by p and the expression in brackets by Q, then

$P = p + Qh$.

If we now substitute this value of

$P = p + Qh$ in

1) $f(x+h) = f(x) + Ph,$

then we shall get :

$f(x+h) = f(x) + (p + Qh)\,h = f(x) + ph + Qh^2$;

i.e.

2) $f(x+h) = f(x) + ph + Qh^2$.

Lagrange at first considers only the second term ph. Since p has h, only as a multiplier *outside* itself (as distinct from Q, which again includes the function h *within itself*), and since in general, apart from x and h, we do not have the other formative elements in the series, so p must be a function of x, besides *its first derivative* is the minimal expansion of $f(x)$. Then he demonstrates that, p can not be $= 0$ or ∞ and can not have the multiplier h with negative or fractional indices of power [177]. The greatness of this proof lies in the following : just as in $f(x)$ the *variable* x is indeterminate and general, i.e., it never takes any particular value $= a$ [178] etc., and retains the capability of any increment, so does $f(x+h)$, containing this x in its serial expansion and in the general rule of expansion, exclude all particular instances, which appear in Taylor as its failures. [[Another streak of its greatness lies in this, that the theory of *derived functions,* which is interwoven into the serial expansion of $f(x+h)$, is at once applied, conversely, for more exact determination of the terms of this series. But here I shall not enter into the details.]]

Thus, till now we had :

$f(x+h) = f(x) + ph$ (or $f'(x)h$) $+ \cdots$

Having analysed Qh^2 in detail, we see that it

$= (B + Ch + Dh^2 + Eh^3 + Fh^4 + \cdots)\, h^2 =$

$= Bh^2 + h^3 (C + Dh + Eh^2 + \cdots)$.

If we designate B by q and the expression included within brackets by R, then $Qh^2 = (q + Rh)h^2$; hence, $Q = q + Rh$.

Having substituted this expression in 2), we shall get :

3) $f(x+h) = f(x) + ph + (q + Rh)h^2$,

$f(x+h) = f(x) + ph + qh^2 + Rh^3$,

arguing further in the same way, we shall get :

$R = r + Sh$, $S = s + Th$, $\cdots$,

$f(x+h) = y$ (or $f(x)$) $+ ph$ (or $f'(x)h$) $+ qh^2 + rh^3 + sh^4 + \cdots$

This series cannot be concluded, because every time a new expression will be obtained : after the last $S = s + Th$, an analogous $T = t + Uh$, where U again includes within itself the functions of x and h. Thus, this mode of expansion excludes any final completion of the series, whatever that might be.

We do not have the pages *l*, *m*, and *n* of the manuscript at our disposal, and we do not know whether Marx had them or not. Page *k* begins with a line and then comes the following section, devoted to MacLaurin's theorem *.

Here Marx writes (sheets 38) :

* This page of the manuscript was crossed out in pencil. —Ed.

MACLAURIN'S THEOREM

I) $f(x+h)$ or $f(x_1) = y_1$ is expanded according to Taylor's theorem; according to MacLaurin's — $f(x)$ or y, i.e., the function of x itself, is required to be expanded, not algebraically, but with the help of differential calculus; thus, in fact, it does not stand for anything other than the [assertion] that, the *constant coefficient of the* [powers of] x must be found through differentiation.

II) If we have a binomial, for example $(x+c)^4$, then its expansion may be written in two forms, depending upon, which term x or c, is taken as the first or as the latter :

a) $(x+c)^4 = x^4 + 4x^3c + 6x^2c^2 + 4xc^3 + c^4$,

b) $(c+x)^4 = c^4 + 4c^3x + 6c^2x^2 + 4cx^3 + x^4$.

In a) the function in x appears as expanding in series, and c in ascending, integral and non-negative * powers, i.e., c^0, c^1, c^2 etc. appear simply as factors.

In b), conversely, the function in c appears as expanding in series, and x in ascending etc. powers, as factors.

Evidently, equation b) is *equation* a) *inverted*, because, reading equation a) from right to left, I get

$$c^4 + 4c^3x + 6c^2x^2 + 4cx^3 + x^4.$$

Thus, if Taylor's initial equation is :

α) $y_1 = y + Ah + Bh^2 + Ch^3 + Dh^4 + \cdots$,

then, having designated c^4 by the letter A, $4c^3$ by B, $6c^2$ by C and $4c$ by D, [we shall obtain MacLaurin's equation] :

β) $y = A + Bx + cx^2 + Dx^3 + \cdot\cdot$

In α) the indeterminate coefficients A, B, C etc. are functions of x, in β) — they are functions of the constant c.

III) A) y or $f(x)$ or $(c+x)^m =$

$$= c^m + mc^{m-1}x + \frac{m(m-1)}{1\cdot 2}c^{m-2}x^2 +$$
$$+ \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}c^{m-3}x^3 +$$
$$+ \frac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}c^{m-4}x^4 + \cdots$$

Since only the *variables* can be differentiated, the expansion of the constant coefficients of the function [of] x through differentiation presents itself as a contradictio in adjecto **.

But we shall do what we did with Taylor's theorem. If we assume that $x = 0$, then

y or $f(x) = (c+x)^m$

turns into

* In the manuscript : "positive". —Ed.

** Here it means : finds itself in contradiction with the very definition of differentiation . — Ed.

y or $f(0)$ or $(c+x)^m = (c+0)^m = c^m$.

Since we have thus found the m-th power of c or c^m assuming $x = 0$ in $(c+x)^m$, analogously, all derivatives of c^m may also be so found : we shall seek the derivative through the differentiation of $(c+x)^m$, and then assume that $x = 0$; hence, as

y or $f(x) = (c+x)^m$

so

$$\frac{dy}{dx} = m(c+x)^{m-1} ;$$

if here we put $x = 0$, then $(c+x)^{m-1}$ turns into

$(c+0)^{m-1} = mc^{m-1} = f'(0)$.

In the same way we shall successively obtain the terms of the series.

Hence, we get :

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{1\cdot 2} + f'''(0)\frac{x^3}{1\cdot 2\cdot 3} + \cdots$$

$$\cdots = (y)_0 + \left(\frac{dy}{dx}\right)_0 x + \frac{1}{1\cdot 2}\left(\frac{d^2y}{dx^2}\right)_0 x^2 + + \frac{1}{1\cdot 2\cdot 3}\left(\frac{d^3y}{dx^3}\right)_0 x^3 + \cdots + + \frac{1}{1\cdot 2\cdot 3\cdots m}\left(\frac{d^my}{dx^m}\right)_0 x^m,$$

where the brackets $(\quad)_0$ indicate that, these symbolic differential coefficients correspond to the "derived" functions, in which it has been assumed that $x = 0$.

Here, part IX of the manuscript (acording to our list) comes to an end.

Then follws part X. It carries the title : "*Successive Differentiation*" (pp. o , p ; sheets 39-40). Here Marx returns to the question : how, after having the expansion of $f(x+h)$ according to the powers of h, one is to extract from it the successive derivatives of $f(x)$, by forming the differences $f(x+h) - f(x)$, then dividing them by h ("freeing " the first term of the expansion of difference from the factor h) and, finally by assuming that $h = 0$. He has already considered this question earlier, so far only in application to the first derivative, (see, p. 283 and note [173]). Now Marx examines the same question for successive derivatives. The issue is this : how, after already having the serial expansion of $f(x+h)$ according to the powers of h, these may be utilised for extracting from it, all the derived functions of $f(x)$, in Marx's words "potentially" contained in it. With this aim in view, Marx carries out those very operations with $f'(x+h)$, which were discussed above, in respect of $f(x+h)$. However, while obtaining the expansion for $f'(x+h)$, from the expansion of $f(x+h)$, Marx commits a mistake in calculation, which he himself notices later on. Since he discusses the same question once more, in the supplement to p.1, which follows (part XI in our list), and besides does so in a considerably clearer form, only this supplement is being reproduced below (sheet 41). Since this supplement also contains that very error in calculation (leading to a

situation where the derivatives appear to be coinciding with the coefficients of h in the expansion of $f(x+h)$), we omit here the latter part of this supplemennt (to p.1). It will not be difficult for the readers to complete these omitted calculations themselves.

AD P.1 (TAYLOR'S THEOREM)

I) y_1 or $f(x+h)$ or $(x+h)^m =$

$$= x^m + mx^{m-1}\frac{h}{1} + m(m-1)\,x^{m-2}\frac{h^2}{1\cdot 2} +$$

$$+ m(m-1)(m-2)\,x^{m-3}\frac{h^3}{1\cdot 2\cdot 3} +$$

$$+ m(m-1)(m-2)(m-3)\,x^{m-4}\frac{h^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

Putting $h = 0$, we shall get

$$f(x) = x^m.$$

Thereby the riddle is already solved. The first term [of the expansion for] $(x+h)^m$,i.e., x^m is not considered as the first term of a binomial expansion, but rather as the given function of the variable x, which, in the given instance, is x^m. Thus, the entire expression in the left hand side sub I) *minus* x^m *itself,* appears as produced by the fact that , in x^m the variable x grows and turns into $x+h$, and thereby x^m turns into $(x+h)^m$ or into what we have written [above].

But the method, *through which the successive "derived" functions already contained in the series* are sought, as such, in the system, where x_1 is treated as $x+h$, follows from the very first operation, through which we found $x^m = f(x)$; as its multiplier x^m has only h^0 (or 1); that is why, when we assume that $h = 0$, the other terms which have h as their multiplier, disappear ; and only x^m remains as the equivalent of y_1, when it has again turned into y .

Thus must the other functions of x be successively freed from their h, and then it must be assumed that $h = 0$. As in the first operation, by equating h with zero, $f(x+h)$ was turned into , $f(x+0)$, i.e., into $f(x)$ so also now, through an analogous operation in the left hand side, we obtain the form of symbolic differential coefficient corresponding to it.

Since

$$y_1 \text{ (or } f(x+h)) = y \text{ (or} f(x)) + mx^{m-1}h +$$
$$+ m(m-1)\,x^{m-2}\frac{h^2}{1\cdot 2} + \cdots$$

so

$$y_1 - y \text{ (or } f(x-h) - f(x)) = mx^{m-1}h + m(m-1)\,x^{m-2}\frac{h^2}{1\cdot 2} + \cdots$$

Dividing both the sides by h, we shall get

$$\frac{y_1 - y}{h} \left(\text{or } \frac{f(x+h) - f(x)}{h} \right) =$$

$$= mx^{m-1} + m(m-1)x^{m-2}\frac{h}{1\cdot 2} + m(m-1)(m-2)x^{m-3}\frac{h^2}{1\cdot 2\cdot 3} + \cdots$$

Here mx^{m-1} plays the same role as x^m earlier . . .

In part XII of the manuscript Marx again (for the last time) returns to p.1. Considering Taylor's theorem to be the most general operational formula of the differential calculus, logically as well as historically obtained through the latter, he proposes that it is essential to start with the deduction of the binomial theorem, with the means of the ready-made differential calculus, i.e., with the help of the operational formulae of the latter. With this aim in view in the fragment *"Towards Taylor's theorem, p.1"*, he first of all obtains the derivative of x^m (for integral and positive m), using the formula for the differential of product. Then he demonstrates this formula by the methods of Newton-Leibnitz, evidently, thinking, that since we have already operated upon the grounds of differential calculus, in other words, substantiated its methods, we may now use the methods of Newton and Leibnitz too : that is, we may consider them sufficiently substantiated.

In the fragment *"Ad Taylor's Theorem (p. 1)"* which follows, Marx compares the successive derivatives of x^m thus obtained, with the coefficients of the powers of h in the expansion of $(x+h)^m$ according to the binomial theorem and reveals, wherein lies their differences. We recall, that due to an error in calculation, notwithstanding the fact that Marx knew them well and wrote about them earlier—see, p. 286 and note [174]— Marx did not notice these differences in parts X and XI of the manuscript. That is why, it may be thought, that the aim of the present fragment was : removal of this error. Both the fragments are being reproduced below, in full (sheets 42-43).

TOWARDS TAYLOR'S THEOREM, P. 1

If

$$(x+h)^m = x^m + mx^{m-1}h + \frac{m(m-1)}{[1\cdot 2]}x^{m-2}h^2 + \cdots,$$

then the differential calculus has already proved, independently of Taylor, that if $f(x) = x^m$, then

$$\frac{dy}{dx} \text{ or } f'(x) = mx^{m-1},$$

$$\frac{d^2y}{dx^2} \text{ or } f''(x) = m(m-1)x^{m-2},$$

etc.

But how ?

As an example let us take the product of the variables xz.

If they vary, then this product turns into $(x+dx)(z+dz)$, which [in a certain sense] is a *binomial of second degree* and which differs from $(x+a)(x+a)$ or $(x+a)^2$ only in form, in this, that instead of the first $(x+a)$ we have $(x+dx)$, and $(x+dz)$ — instead of the second $(x+a)$, that is why instead of $x^2 + ax + ax + aa$ we get $xz + zdx + xdz + dxdz$; having taken away xz from here, we have $zdx + xdz + dxdz$; having struck out the last term, we get

$$zdx + xdz.$$

Since, it is obtained through the binomial that $d(xz) = zdx + xdz$, it may be applied to the product of any number of variables, for example :

$$\frac{d\,(xzuv)}{xzuv} = \frac{zuv\,dx}{xzuv} + \frac{xuv\,dz}{xzuv} + \frac{uzx\,dv}{xzuv} + \frac{xzv\,du}{xzuv}.$$

After cancellation we shall get :

$$\frac{d(xzuv)}{xzuv} = \frac{dx}{x} + \frac{dz}{z} + \frac{dv}{v} + \frac{du}{u},$$

$$\frac{d(x^4)}{x^4} = \frac{dx}{x} + \frac{dx}{x} + \frac{dx}{x} + \frac{dx}{x} = \frac{4dx}{x},$$

$$d(x^4) = 4\frac{dx}{x} \cdot x^4 = 4x^3dx,\ \frac{d(x^4)}{dx} = \frac{dy}{dx} = 4x^3\ *.$$

Imagine, that in the formation of the product, not 4, but m variable factors $x\,z\,u\,v\,t\,y$ etc. took part ; then

$$\frac{d(x^m)}{x^m} = \frac{mdx}{x};$$

hence:

$$d(x^m) = mx^m\frac{dx}{x},\ d(x^m) = mx^{m-1}dx$$

or

$$\frac{d(x^m)}{dx}\left(=\frac{dy}{dx}\right) = mx^{m-1},\ \text{and}\ \frac{d(mx^{m-1})}{dx} = m(m-1)\,x^{m-2}\ \text{etc.}$$

Hence, it was obtained only owing to the fact that

$d(x\,y) = ydx + xdy$,

found with the help of the binomial, was applied as an operational formula.

AD TAYLOR'S THEOREM (P.1)

Regarding equation IV)

$$y_1 = y\ (\text{ or } f(x)) + f'(x)\frac{h}{1} + f''(x)\frac{h^2}{1\cdot 2} + \\ + f'''(x)\frac{h^3}{1\cdot 2\cdot 3} + f^{IV}(x)\ \frac{h^4}{1\cdot 2\cdot 3\cdot 4} + \\ + f^{V}(x)\frac{h^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots$$

*The last two lines were written on the right hand side, in Marx's hand, in pencil.—Ed.

it could be said, that [the equalities] $f'(x) = mx^{m-1}$, $f''x = m(m-1)x^{m-2}$ etc. have been demonstrated, it is true, also *differentially* *; but the numerical factors $\frac{1}{1 \cdot 2}$, $\frac{1}{1 \cdot 2 \cdot 3}$, $\frac{1}{1 \cdot 2 \cdot 3 \cdot 4}$ etc. *have simply been borrowed from the binomial theorem.* For our purposes herein lies the crux of the affair; and this crux is no more astonishing than the circumstance, that the companions of these functions, beginning with the second term, the coefficients of the functions h, h^2, h^3, h^4, h^5 etc., in ascending powers, remain directly derived "from the binomial theorem".

The general binomial coefficient $= \frac{m(m-1) \cdots (m-r-1)}{1 \cdot 2 \cdots\cdots r}$ (the same, *as the one for the combinations from m elements, taken r at a time, without repetitions*), which is, as in our case, when $r = m$ (r can never be greater than m),

$$= \frac{m(m-1) \cdots (m-(m-1))}{1 \cdot 2 \cdots m} = \frac{m(m-1)(m-2) \cdots 2 \cdot 1}{1 \cdot 2 \cdots m},$$

proved through combinatorics ; the binomial theorem itself, for integral and positive powers, is only a particular application of it (*the number of combinations divided by the number of permutations*).

But, for the differential method, where $x_1 = x + h$, $f(x_1) = f(x+h)$ etc., it is important, that what is given by the binomial theorem turn out to be deduced in the differential calculus itself.

However, such deduction, as we saw it while obtaining mx^{m-1} from x^m etc., by the means of differential calculus, can, in its turn, be actualised, upon the foundations of binomial theorem alone.

The last fragment of part XII (in out list, sheets 44-45), differs even in form from the rest of manuscript 4302. It is written on seperate sheets, not lengthwise, but across the sheets and contains only some incomprehensible calculations, in places repeated twice. This fragment carries the title : "*Ad Taylor's Th., p.1, eq. III*". But neither p.1, nor the equation III (Roman "I" and "III") could be found in the manuscript. The attempt to identify equation III with equation 3) on p. 1 (PV, 287) does not lead us anywhere. The fragment begins with the formula

I) y_1 or $(x+h)^m = y + Ah + Bh^2 + Ch^3 + Dh^4 + Eh^5 +$ etc.,

in which, then, $h + z$ is substituted in place of h. The purpose of this substitution is not at all clear : in any case it was done not to seek the coefficients A, B, C, etc. of the binomial expansion, since the binomial theorem of Newton, is already assumed to be known — it has been applied even in the expansion of $(x+h+z)^m$. The remaining calculations are equally incomprehensible. They are not accompanied by any explanation. That is why, this entire fragment is not being reproduced.

In spite of its incompleteness and draft character, manuscript 4302 is of unquestionable interest. It alone finally permits us to ascertain Marx's points of view on the nature of differential calculus proper, on the concept of function, on the means of mathematical representation of motion, on

* That is, not through "algebraic " differentiation, but with the help of the operational formulae of the differential calculus. — Ed.

the methods of Newton and Leibnitz, on Lagrange's theory of analytical functions, on the deficiencies of the manuals at Marx's disposal and, especially, those of the proofs of Taylor's theorem given in these books, on that method of proving this theorem, which appeared to be the correct one, to Marx himself. It is clear from the manuscript that this proof, should have begun, according to Marx, from the particular case of the powered function x^m and it should have been then extended to any function $f(x)$, having the "binomial form", i.e., to such, that

$$f(x+h) = A + Bh + Ch^2 + \cdots, \tag{1}$$

where A, B, C, etc., are functions only of x. From this expansion Marx further extracted the successive derivatives of $f(x)$, "contained already in it, in a ready-made form", and thus obtained Taylor's series from series (1). We note that such a proof of Taylor's theorem — with the corresponding specifications, concerning the regions of convergence of the serieses for $f(x+h)$, $f'(x+h)$ etc — is provided also in the modern courses of mathematical analysis and, especially, in those of the theory of serieses *.

---

* See, for example, 1) E.P. Natanson, Proizvodniye, integraly i riady (Derivatives, integrals and serieses), in the "Entsiklopedia elementarnoi matematiki", vol. III, M .- L., 1952, pp. 347-348, 455-457 and 469-470 (Taylor's theorem for the function x^k, theorems of term by term differentiation of a powered series and, Taylor's series); 2) K. Knopp, Theorie und Anwendung der unendlichen Reihen (Theory and application of infinite series), 2nd ed., Berlin, 1924, pp. 173-174. — Ed.

APPENDIX

ON THE CONCEPT OF "LIMIT" IN THE SOURCES CONSULTED BY MARX

At first we shall give an account of the definition of "limits" (together with the examples explaining it), as well as of the modes of using the word "limit", contained in the courses of Hind and Boucharlat. Marx had these books and studied them critically. This account will give the reader, who is accustomed to the modern use of the term "limit" in mathematics, an opportunity to correctly understand Marx's critical remarks regarding this concept, and his special way of interpreting it.

Hind's book was written according to d'Alembert's method, i.e., in it the derivative was defined through the concept of limit. That is why, the introductory chapter of the book is devoted to the "method of limits". However, neither this chapter, nor any other place in the text-book contained any definition of "limit". There we find only the definitions of "limits" of a variable, in some sense of the exact upper and lower boundaries of the set of its values. (In particular, this set could also contain an "infinitely large" value of the variable, designated by the sign ∞. But the rule for operating with this sign has not been specified : the concept of absolute magnitude is not present there, the signs $+\infty$ and $-\infty$ are also absent, it has been considered to be simply self-evident, that for any $\alpha \geq 0$, $\infty + \alpha = \infty$, that for any "finite" (i.e., different from 0, as well as from ∞) a, $a \cdot \infty = \infty$ and $\frac{a}{\infty} = 0$.)

As to what the function is — it is possible only to surmise, from the examples. The concept. of limit has in fact been silently introduced in the introductory chapter— we suspect— through the identification of the limit of a function (at a point, coinciding with the exact upper or lower bound of a set of values of the argument) with one of the two "limits" (with the exact upper or exact lower bound) corresponding to the set of values of a function. In so far as only monotonic or piecewise monotonic functions have been considered in Hind's book, such a "limit", in practice, turned out to be coinciding with (one-sided) limit in the more common place sense of the word. And Hind in fact used it in this sense, throughout the rest of his book. However, he used it in such a manner, that this concept, which was supposed to "improve" upon the method of actual infinitesimals, to wit, failed to attain this goal, and turned out, in general, to be redundant.

In fact, Hind could, of course, have substituted the search for one-sided limit of a piecewise monotonic function $f(x)$, determinate at the interval (a, b), when, for example, x tends to $+a$, by the solution of the two following problems:

1. To find out the number α, so that when $a < x < \alpha$, the function is monotonic (in the wider sense, i.e., non-diminishing or non-increasing ; let us assume for the sake of definitiveness, that herein the function turned out to be monotonically non-diminishing).

2. To find out the exact, in our assumption lower, bound of the set of values of the function, at the interval (a, α), i.e., for $a < x < \alpha$. It is clear ; that it will be the $\operatorname{Lim}_{x \to +a} f(x)$ sought.

But Hind did not proceed that way. Following Newton (see, Appendix, "On the Lemmas of Newton cited by Marx", pp.313 - 315), he treated limit as something actual, i.e., as the "last"

value of the function, for the "last" value of the argument. In other words, he was in search of $\underset{x \to +a}{Lim} f(x)$ as the exact lower bound of the values of the function, not in the interval $a < x < \alpha$, but in the segment $a \leq x \leq \alpha$, i.e., he assumed that the "last" value of $f(a)$ has some how already been determined, all by itself; and in that case the entire procedure described above lost all sense: the number a could be taken for α, and the exact lower bound of the set of values of the function, consisting of only one number $f(a)$ could be sought and that would turn out to be the same $f(a)$.

Apparently, this is what Marx wanted to say, when he remarked, having Hind's definition in view, that there is no sense in treating $3x^2$ as the limiting value of the same $3x^2$, when x tends to zero, characterising this sort of treatment as "banal tautology" (see, pp. 96-98 and notes [90-92]), and when he called, the actual approach to limit — the presupposition, that a function actually attains its limiting value, as its "last" value for the "last" value of the argument — in general "infantile", the emergence of which "should be sought in the first mystical and mystificatory method of calculus"(see, p. 98).

The fact that the actual approach to limit by no means solved the problems connected with the actual infinitesimals, becomes especially clear, when the "last" value of the argument has to be "infinitary".Thus, in particular, when the sequence $\{a_n\}$ is at issue, then the limit has to be that term of this sequence, in which $n = \infty$, i.e., the limit is considered as the end (the last term) of the infinite (i.e., endless) sequence of terms. It is hardly surprising, that a concept like the "actual limit" appeared to be no more clearer, than the concept of the "actual infinitesimal", which Marx called"mystical".

It is well known, that the definition of the limit of a function, without presupposing the completion of the infinite number of steps, and permitting an exact formulation in the terminology of variables and of parameters, having only finite values, finally entered into mathematical use only from Cauchy's time, to be more precise, only from the 70s of the last century. But even at that time, it was not fully clear to the authors of many text books, which were in wide use, that the limit is not to be interpreted actually; that even in those cases, where the function is continuous at a point a, i.e., when the limit of the function $f(x)$ as $x \to a$ equals $f(a)$, it must turn out to be equal to $f(a)$, namely, under the condition, that though x tends as close to a as possible, nevertheless it never reaches a.

In connection with Marx's mathematical manuscripts, it is of special importance to us, that if the value of $f(a)$ is not defined, and the limit of $f(x)$ exists when $x \to 0$ (correspondingly, when x tends to $+a$ or to $-a$), then it is possible simply to redefine the function $f(x)$ at the point a, having assumed, according to the definition, that $f(a)$ is equal to this limit. Such a redefinition of the value of a function is called its redefinition *at continuity*. In that case, the limit of the function $f(x)$ as $x \to 0$, will be the value of the function already re-defined at $x = a$. This, however, has nothing in common with the treatment of the value of $f(a)$ as the univocally defined value of the function $f(x)$ itself, but attainable only *at the end* of an infinite process of approximation of x to a, as far as possible. Namely, such a redefinition"at continuity", Marx had in view, apparently, when he called the limit of the ratio $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$, the "absolutely minimal expression" of this ratio (see, p. 97);

evidently, herein, for the sake of clarity it is held, that the limit of this ratio as $\Delta x \to +0$ and subject to the condition, that there exits a number α, such that for $0 < \Delta x < \alpha$, the ratio $\frac{\Delta y}{\Delta x}$ decreases with the dimunition of Δx.In fact, Lacroix used this very mode of redefining functions, in his examples(see below, pp. 310-311). But while constructing analysis, Lacroix proceeded from Leibnitz's metaphysical "principle of continuity", which he used as a self-evident axiom; hence he did not consider any other redefinition of the values of a function, at all possible. See pp. 28-29 and note [18], on the fact that to all appearence, Marx also admitted of the other modes of redefining the ratio $\frac{\Delta y}{\Delta x}$ when $\Delta x = \Delta y = 0$.

Now we shall give an account of those statements of Hind, which may be of use while studying Marx's manuscripts. The aforementioned conclusions follow from these statements.

Hind's introductory chapter : "On the method of limits", begins with definition 1, which reads:

"Limits of the quantities, which admit of changes in their magnitude, are such magnitudes, between which are included *all* the values, which it may have, in *all* its changes; beyond which it may never go and from which it may be made to differ by a quantity, lesser than any, which may be indicated in finite terms" (i.e., without using the signs ∞ and 0. — Ed.) (see, Hind, p.1, our italics. — Ed.).

A series of examples followed this definition ; however, in them the definition was not used even once in a clear form : not even once was it demonstrated, that the "limits" indicated by the author, do actually satisfy the demands, formulated in definition 1. The first of these examples reads:

"The quantity ax, where x admits of all possible values from zero, or 0 to infinity, or ∞, becomes 0 in the first instance and ∞ in the last; and, hence, the limits of the algebraic expression ax are 0 and ∞ : the first is called the *lower*, and the last the *upper* limit"(ibid). (Evidently, here it has been assumed that $a > 0$.)

This very first example must have led the reader to bewilderment. In fact, how to make the quantity ax, other than ∞, a "quantity, lesser than any, which may be indicated in finite terms "? For Hind, it is evident, that so long as x remains a finite magnitude, the difference $\infty - ax$ is equal to ∞. If $x = \infty$, then also $ax = \infty$ and the difference $\infty - \infty$ is undefined.

In the second example (one should think, of course, under the same conditions relative to x and a) the lower and upper limits of the expression $ax + b$, were found; these turned out, respectively to be b and infinity.

In the third example the lower limit $\frac{b}{a}$ of the fraction $\frac{ax+b}{bx+a}$ was found through the simple substitution of 0 in place of x in the expression of this fraction, the upper limit $\frac{a}{b}$ was obtained

by substituting ∞ in place of x in the fraction $\dfrac{a+\frac{b}{x}}{b+\frac{a}{x}}$. Here it was not explained: under what sort of conditions in respect of a and b, the values found are actually the lower and upper limits (respectively). There is not even a hint to the effect that, it should be verified whether or not these values at all satisfied the given definition of "limits" (to be sure, in particular, about the monotonic nature of the function considered). As though the reader "was prepared" to find out the limit of a function by substituting in its expression (or in its transformed expression, not bereft of meaning, also in that case, when the expression directly giving the function is indeterminate), the limiting value of the argument.

We are reproducing below, in full, the fourth and the sixth examples, as well as the example specially marked out in point 2 of the introductory chapter — in which a gradual transition takes place, from the concepts of lower and upper limits of a function to a concept of limit which is closer to the customary one; and Hind explained the actual character of the latter. From these accounts it will also be sufficiently clear: how far confused, in general, is Hind's account of this section of limits.

"Ex. 4. Sum of the geometric series $a+\frac{a}{x}+\frac{a}{x^2}$ etc., continued upto the n-th term, is expressed by the quantity

$$\frac{a\left(\frac{1}{x^n}-1\right)}{\frac{1}{x}-1}=\frac{ax\left(1-\frac{1}{x^n}\right)}{x-1};$$

now, if $n=0$, then, evidently, the lower limit $=0$; but if $n=\infty$ then $\frac{1}{x^n}$ becomes equal to 0, and that is why the upper limit is $\frac{ax}{x-1}$; it is usually called the sum of the series, continued upto infinity"(p.2). (Here, of course, it has been silently assumed that the argument of the function $\dfrac{ax\left(1-\frac{1}{x^n}\right)}{x-1}$ is n; a and x are such parameters, that $a>0$, $x>1$.)

"Ex. 6. If a regular polygon is inscribed in a circle and the number of its sides is successively doubled, then it is evident, that its perimeter will more and more approach equality with the periphery of the circle and that finally, their difference must become less than any quantity whatsoever; that is why, it is clear that the circumference of the circle is the limit of the perimeters of polygons "(pp. 2-3). Here, one of the "limits" of the sequence considered, is no more spoken of, not even its upper limit — as was natural, following definition 1, but simply the limit, and that too in its usual sense.

"2. *To prove that the limits of the ratios occuring between sine and tan of an arc of a circle, and the arc itself, are ratios of equality.*

Let p and p' be the perimeters of two right polygons with n sides, the first is inscribed in, and the second is circumscribed about a circle with radius 1 and circumference $= 6\,{}^{\cdot}28318$ etc. $= 2\pi$; then (trig.)

$$p = 2n \sin\frac{\pi}{n} \text{ and } p' = 2n \tan\frac{\pi}{n};$$

hence,

$$\frac{p}{p'} = \frac{2n \sin\frac{\pi}{n}}{2n \tan\frac{\pi}{n}} = \cos\frac{\pi}{n},$$

and if it is assumed that the value of n is indefinitely increasing, then the value of $\cos\frac{\pi}{n}$ is 1 and that is why $p = p'$; but, evidently, the periphery of the circle lies between p and p', and that is why in this case it is equal to either of them; hence, in this presupposition the n-th part of the perimeter of the polygon is equal to the n-th part of the periphery of the circle, i.e.,

$$2 \sin\frac{\pi}{n} = \frac{2\pi}{n} = 2 \tan\frac{\pi}{n}, \text{ or } \sin\frac{\pi}{n} = \frac{\pi}{n} = \tan\frac{\pi}{n},$$

or the sin and the tan of an arc of a circle, in their last or limiting condition find themselves in a ratio of equality with the arc itself"(p.3).

Here the word"limit"("limits") is met with only in the formulation of the theorem, but, in order to understand this formulation, it became necessary to surmise from the proof, that the issue here is about the usual limit of the ratio $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$, when x tends to 0. However, in this case Hind's proof is hardly satisfactory, even for his time.From it only this much follows, that the author wished to obtain the equality

$$\sin\frac{\pi}{n} = \frac{\pi}{n} = \tan\frac{\pi}{n}, \text{ when } n = \infty. \tag{1}$$

But, asserting, that for $n = \infty$, $\cos\frac{\pi}{n} = 1$, he used the fact that $\frac{\pi}{n} = 0$, when $n = \infty$; but in that case $\sin\frac{\pi}{n} = \sin 0 = 0$, and $\tan\frac{\pi}{n} = \tan 0 = 0$, i.e. ,for obtaining the equalities (1) , from which, in themselves, incidentally, the theorem about the limit of the ratio $\frac{\sin x}{x}$ when $x \to 0$, does not at all follow. The arguments through which the author arrives at these equalities, are not at all required.

In fact it remains incomprehensible, how such a confused account could pretend to be one about the essential advantage of the enunciated method of limits, interpreted actually, in comparison to the method of actual infinitesimals, in the given case to the simple identification of the infinitesimal arc of a circle, with its chord.

In the text-book by Boucharlat (see, Boucharlat, p. VII) too the method of limits, was considered to be more precise than the method of infinitesimals:"correcting, that which may be imperfect in the latter". However, in Boucharlat's book, there is no attempt to define, the meaning of the expression to "tend to some limit" (how one may make sure, that such and such magnitude may tend to such and such limit). The concept of limit — also "actual" — appeared in it, firstly in connection with the search for the derivative of the function $y = x^3$. We shall reproduce it in full, since Marx's critical remarks in the manuscript *On the non-univocality of the terms "limit"and "limiting value"*, are related to it.

"While examining the second term of the equation

$$\frac{y'-y}{h} = 3x^2 + 3xh + h^2, \tag{2}$$

we find, that this ratio diminishes along with the dimunition of h and that, when h becomes zero, this ratio is reduced to $3x^2$. Hence, the term $3x^2$ is the limit of the ratio $\frac{y'-y}{h}$: it tends to this term, when we compel h to diminish.

In so far as, in the presupposition that $h = 0$, the increment of the magnitude y also becomes 0, $\frac{y'-y}{h}$ is reduced to $\frac{0}{0}$, and, hence, equation (2) turns into

$$\frac{0}{0} = 3x^2 . \tag{3}$$

There is nothing absurd in this equation, because algebra teaches us, that $\frac{0}{0}$ may represent all sorts of quantities. We may note, though, that since by dividing both the terms of a fraction by one and the same number, its value is affected in no way and that, consequently, it may remain the same, even when its terms arrived at the last stage of smallness, i.e., turned into zero" (pp. 2-3).

It is also of importance to note, for understanding the aforementioned manuscript of Marx, that in Boucharlat's account the limiting transition from an equality of the form $\frac{\Delta y}{\Delta x} = \Phi\ (x_1, x)$ (where $y = f(x)$) to an equality of the form $\frac{dy}{dx} = f'(x)$, appeared to have been actualised separately, in the left hand and right hand sides of the first of these equlities : from $\frac{\Delta y}{\Delta x}$ to $\frac{dy}{dx}$ and from $\Phi\ (x_1, x)$ to $f'(x)$. (Herein, by the limit of the ratio $\frac{\Delta y}{\Delta x}$ (correspondingly, $\frac{y'-y}{h}$) he had in view an expression of the form $\frac{0}{0}$, turning into $\frac{dy}{dx}$.)

Thus, having obtained the equality $\frac{y'-y}{h}=1$, while searching for the differential of x, Boucharlat concluded : "Since the quantity h does not enter into the second term of this equation, we see, that in order to go over to the limit, it is enough to change $\frac{y'-y}{h}$ into $\frac{dy}{dx}$, which gives $\frac{dy}{dx}=1$ " (p.6).

Boucharlat treated the instance of the limit turning to be equal to zero, as one indicating the non-existence of limit. Thus, investigating the derivative of $y=b$ and having obtained the equality $\frac{dy}{dx}=0$, he concluded, that "neither is there a limit, nor a differential" (p.6).

Boucharlat, in essence, obtained the limit of the ratio $\frac{\sin x}{x}$ as $x\to 0$, the way Hind did, though in a more accessible form. At first he demonstrated — approximately in the way it is now done in the text-books — that the "arc is greater than the sin and smaller than the tangent"(p.29). But herein, it was not even mentioned, that hence it follows, that

$$\frac{\sin x}{\tan x}<\frac{\sin x}{x}<\frac{\sin x}{\sin x}\left(0<x<\frac{\pi}{2}\right),$$

i.e., that the ratio $\frac{\sin x}{x}$ lies between $\cos x$ and 1. Instead like Hind, Boucharlat wrote :

"From what has been said above, it follows that the limit of the ratio of sin to the arc is one; for when the arc h ···· becomes zero, owing to which the sin merges with the tan, then the sin merges all the more with the arc, which is placed between the tan and the sin ; hence, in the instance of limit we have $\frac{\sin h}{\operatorname{arc} h}$ or, better still, $\frac{\sin h}{h}=1$" (p.29). The fact, that when $h=0$, the ratio $\frac{\sin h}{h}$ "turns" into $\frac{0}{0}$, i.e., becomes in general indeterminate, and the conclusion made thereupon, only to the effect that "the sin merges with the arc", when the latter turns into 0, did not disturb Boucharlat either, as it did not disturb Hind.

The informations provided here, on the treatment of the concept of limit in the books of Hind and Boucharlat, are, to all appearence, enough for understanding those places in the manuscript *On the non-univocality of the terms "limit" and "limiting value"*, in which Marx criticises these authors for their actual approach to limit (notes[90-92] are related to these places).

For understanding the other places of the manuscripts, characterizing Marx's relation to the more modern treatments of limit, informations regarding the treatment of this concept in the other sources at Marx's disposal, are essential. Here first of all a mention must be made of Lacroix's large "Treatise" on the differential and integral calculus, 1810.

Following Leibnitz, Lacroix thought, that every function is subordinate to the "rule of continuity" in its changes, and that the transition to the limit is an expression of this rule, "i.e., of the rule, which is obeyed by motion in its linear description and according to which, the successive points of one and the same curve follow each other, without any interval"(p.XXV). But since change of magnitude can not be studied without a consideration of its two different values, between which, wittingly, there is an interval, so the rule of continuity must be expressed as follows : "the smaller this interval, the closer are we to this rule and only the limit corresponds to it fully" (ibid). This role of continuity, in Lacroix's mathematical analysis, explains why he found it expedient "to use the method of limits" (p. XXIV), while constructing a systematic course of mathematical analysis.

Lacroix thought, that the concepts of "infinite" and "infinitesimal" are defined only negatively, i.e., as "excluding every bound, in the sense of greatness as well as in that of smallness, which gives only a number of negations, but would never form a positive concept" (p.XIX). In a foot-note in this page he adds : "The infinite is essentially that, about which it is asserted, that its bounds *les limites* cannot be attained by any indicated value of it". In other words, Lacroix did not admit of any actual infinity : neither the actually infinitely large, nor the actually infinitely small.

Lacroix introduced the concept of "limit" as follows :

"Let us consider at first a very simple function $\frac{ax}{x+a}$, in which we assume that x is positive and indefinitely increasing ; the result $\frac{a}{1+\frac{a}{x}}$, obtained by dividing both the terms of this fraction by x, evidently shows that the function always remains less than a , but that it constantly approaches a, since the $\frac{a}{x}$ part of its denominator diminishes more and more and, may be made as small as one wishes. The difference between a and the proposed fraction, which may in general be expressed through

$$a-\frac{ax}{x+a}=\frac{a^2}{x+a},$$

becomes as small, as great x is, *and can be made smaller than any given magnitude, however small , so that the proposed fraction may come as close to a as we wish* : hence, a is the *limit* of the function $\frac{ax}{x+a}$, in respect of an indefinitely increasing x.

The properties just formulated, also include the true meaning, which should be attributed to the word *limit*, so as to include in it, all that may be required here" (pp. 13-14).

In Lacroix there are no assumptions about the monotonic or piece-wise monotonic functions. And usually his limit is not one-sided : the variable can approach its limiting value, in any way. In place of the concept of absolute magnitude Lacroix uses, though not systematically, the expression "magnitude without sign", however, its meaning still remains imperfect. He also stresses, that a function may not only attain its limiting value, but also

"cross" it, may, in general, oscillate around it. But Lacroix still did not explicitly formulate the bounds, consisting of the fact, that while approaching its limiting value α, the argument in respect of which the transition to limit takes place, it is not assumed to be attaining α, i.e., that the limit is not understood actually. Since the functions with which he was concerned, were continuous, i.e., the limits considered by him coincided with the values of the function for the limiting value of the argument, he sometimes permitted himself to speak, as would speak a person who thinks, that in the limiting transition, the approach of the argument to its limiting value must be completed in its attainment of this value.

It should also be mentioned, that Lacroix used one and the same word *limité* for designating *limit* — the term, which, as we saw, he understood in a much more general and precise sense, close to its modern meaning, as compared to the meaning which this concept had in the text-books of Hind and Boucharlat, criticised by Marx, as well as for designating the approximate values of functions, in certain cases.

All these informations on the concept of limit in Lacroix's large "Treatise" — which, as we know, Marx always used as the most reliable (from among those at his disposal) source of information on the basic concepts of mathematical analysis, like the "function", "limit" etc. — are, apparently, enough for understanding what Marx had in view, when he briefly observed, about the concept of limit in Lacroix's treatise, that "this category, which has found wide use in [mathematical] analysis, mainly in that of Lacroix, acquires an important significance, as a substitute for the category of "minimal expression" (pp.61-62). First of all it is clear, that Marx in fact understood the concept of "absolutely minimal expression", used by him, in connection with the "non-univocality" of the term "limit", in that very sense, in which we now understand the concept of limit. It is also clear, that he foresaw, that later on when Lacroix's understanding of the concept of limit will, evidently, fully supplant the less satisfactory concept of "limit", then that will make the introduction of a special — new — concept of "absolutely minimal expression" unnecessary. In other words Lacroix's concept of limit will be a substitute for the latter.

In connection with the just-referred paragraph of Marx's manuscript and also some other places in it, we should perhaps adduce Lagrange's words regarding the concept of limit, from his introduction to the "Theory of Analytical Functions" (Lagrange's Works, vol. IX, Paris, 1881).

Speaking of Euler's and d'Alembert's attempts to view the infinitesimal differences as absolute zeros, only the ratios of which actually enter into the calculus, which are, besides, viewed as the limits either of the finite differences or of the indefinitely large (*limes on indéfinies*) differences, Lagrange wrote: "But it should be noted, that this idea, though in itself valid, is not clear enough to serve as the [initial] principle of a science, whose trustworthiness is to be based on being obvious, and especialy owing to the fact that it is being proposed as the first [principle] " (p. 16).

Lower down he observes — in connection with the Newtonian method of the ultimate ratios of evanescent quantities — that "this method, like the method of limits, about which we spoke above, and which is in essence only its algebraic translation, has a big deficiency, in that it considers quantities in such a condition, when they cease, so to say, to be quantities; for, though we very well understand the ratios of two quantities, till they remain finite, reason

fails to connect any clear and precise idea with these ratios, as soon as its terms, the one and the other, at the same time become zeros" (p.18).

And here Lagrange went over to the attempts of the "skillful English geometer" Landen, to cope with these difficulties. He highly evaluated these attempts, though he thought that Landen's method was too cumbersome (see, Appendix, "The Residual Analysis" of John Landen).

About himself Lagrange wrote, that even in 1772 he held," that the theory of serial expansion of functions contains the true principles of differential calculus, freed from all considerations of infinitesimals or limits"(p. 19).

Thus, it is clear, that Lagrange did not consider the method of limits to be more perfect, than the method of actual infinitesimals, and that this idea of his was connected with the fact, that the limit discussed in analysis, was also understood actually : as the "final" value of the function for the "final" ("evanescent") value of the argument.

ON THE LEMMAS OF NEWTON CITED BY MARX

Marx mentioned, in a separate sheet attached to the rough draft of his essay on the historical course of development of the differential calculus : the scholia to Lemma XI of the 1st book and, lemma II of the 2nd book of Newton's "Principia", devoted to — the basic concepts of mathematical analysis, used by Newton — the concepts of *limit* and *moment.*

In obtaining (the scholia) to lemma XI of the first book of his "Principia Mathematica Philosophia Naturalis", Newton attempted to explain the concepts of "limiting ratio" and "limiting sum" with the help of very non-precise considerations of an ontological character, "metaphysical, non-mathemtical assumptions", as Marx characterised them. Namely, Newton writes : "An objection is raised, that the "limiting ratio" does not exist for evanescent quantities,for the ratio which they have before disappearence, is not limiting, and after disappearence, there is no ratio. But under such strained arguments it will appear, that for a body reaching some place, where movement stops, there can not be a "limiting" speed, for the speed which the body has earlier, before it had reached this place, is not "limiting", and when it has reached there, there is no speed. The answer is simple : by "limiting" speed should be understood that with which a body moves, neither before reaching the extreme place, where movement ceases, nor after that, but when it arrives there, i.e., namely, that speed, possessing which the body arrives at the extreme place and wherein the movement stops. Just like this, what must be understood by the limiting ratio of evanescent quantities, is the ratio of quantities not before their disppearence, and not after, but while they are disappearing. Exactly in the same way, the limiting ratio of emergent quantities is that with which they are born. The limiting sum of emergent or evanescent quantities constitutes their sum, when they, by increasing or decreasing, only begin or cease to be. There exists such a limit, which a speed may attain at the end of a movement, but cannot cross — it is the limiting speed. Such is the reason for the existence of the limit of emergent or evanescent quantities and proportions" (I. Newton, Metamaticheskie Nachala Naturalnoi Filosofii. Translated by A. N. Krylov. Transactions of the Nikolaevsky Naval Academy, Sankt-Peterburg, 1915, p. 64).

In modern mathematics the "speed of a body at a given moment t_o", is defined through the mathematical concept of limit, and it may lead to many observations, including those having an ontological character, in favour of the naturalness of this definition. However, the naturalness of the definition of the speed, of a body at a given moment t_o, through some limit of a ratio of evanescent quantities cannot as yet serve, either as a proof of the fact, that the corresponding limit exists, or more so, as a justification for defining this limit as the "ratio of some quantities, not before their disappearence, and not after, but while they are disappearing", i.e., as a ratio of zeros, the value of which (the ratio) must somehow be defined, as a body must have a velocity also at that moment, when it arrives at that extreme place, where movement stops. It is clear, however, that from such a "definition", the means of mathematical computation of limit, is not to be extracted and that, here, in fact we have a circle before us : *the speed at the moment* t_o is in fact understood as a *limit,* conversely,the *limit* is defined through the *speed at the moment* t_o, the existence of which in this case,does in fact appear as a "metaphysical, non-mathematical assumption" (consisting of the fact, that the reflection is assumed to be the object reflected upon : an abstract mathematical concept formed by our thought for cognitive purposes is taken to be a really existing ideal object).

Lemma II of the second book of the "Mathematical Principles of Natural Philosophy" (ibid, pp. 296-298) contains the following explanation of the concept of "moment" (or infinitesimal) : "Here I consider ... quantities as indeterminate and changing and, as though increasing or diminishing out of constant movement or flow, and by the word *moments* I understand their instant increases or decreases, such that the increases are considered as positive, or addable moments, the decreases — as subtractable, or negative. But we must see to it, that the finite particles are not taken for these. Finite particles are not moments, but are themselves quantities orginating from the moments. This should imply, that this is only the barely emergent beginning of finite magnitudes. That is why in this lemma, magnitudes of the moments are never considered, but only their initial ratios are considered. The same is obtained, if instead of the moments we take either the speeds of the increases or that of the decreases, or any other finite quantities whatsoever, but proportional to these speeds" (ibid, pp. 296-297). It is natural, that Marx would first of all be interested in this explanation — in which Newton again had recourse to the "metaphysical, non-mathematical assumptions", this time about the essence of the differentials ("moments").

But this lemma could also have drawn his attention, because it contains Newton's well known attempt to prove the formula for the differential of the product of two functions, without having recourse to the dismissal of the infinitesimals of higher orders.

This (unsuccessful) attempt consists of the following. Let $A - \frac{1}{2}a$ be the value of the function $f(t)$ at the point t_0, $B - \frac{1}{2}b$ — the value of the function $g(t)$ at the same point t_0, a and b — the increments respectively of the functions f and g, at the segment $[t_0, t_1]$. (Herein below we shall also designate them by Δf and Δg respectively.) Then the increment of the product $f(t) \,.\, g(t)$ at the segment $[t_0, t_1]$ is

$$\left(A + \frac{1}{2}a\right)\left(B + \frac{1}{2}b\right) - \left(A - \frac{1}{2}a\right)\left(B - \frac{1}{2}b\right);$$

i.e., $Ab + Ba$, which Newton takes as the differential ("moment") of the product of the functions f and g at the point t_0. But here $Ab + Ba$, is not $f(t_0)\,\Delta g + g(t_0)\,\Delta f$, but $\left(f(t_0) + \frac{1}{2}\Delta f\right)\Delta g + \left(g(t_0) + \frac{1}{2}\Delta g\right)\Delta f$, i.e.,it differs from $f(t_0)\,\Delta g + g(t_0)\,\Delta f$ by that very magnitude $\Delta f \cdot \Delta g$, the dismissal of which, Newton wanted to avoid. However, by identifying $Ab + Ba$ with $f(t_0)\,\Delta g + g(t_0)\,\Delta f$, Newton carried out, namely, this very dismissal (though silently).

As it appears from the first drafts of his work on the differential (see, PV, 41), Marx at first wanted to throw light upon the historical course of the development of differential calculus, by using the history of the theorem about the differential of a product as an example. There is, that is why, no doubt about the fact, that lemma II must have drawn Marx's attention in this connection.

As the sources from which Marx took his extracts,do not specially mention lemma XI of the first book and lemma II of the second book of the "Principia",there is all the ground to think,that Marx singled them out, having turned directly to Newton's "Principia".

Since the definition of the limit of the ratio of evanescent quantities through the speed of a body at a given moment t_0 , does not contain the means of computing this limit, Newton in fact could not use this definition for such computation. For that he had to use some other presuppositions about the limits,permitting the reduction of the computation of the limits of ratios of evanescent quantities into the computation of such limits, whose numerical value was fully and besides, quite naturally, determinate. The role of such a presupposition plays first of all : Newton's lemma I, in the first section of the first book of his "Principia" : "On the method of first and last ratios, through which the following is proved". In his comments on the history of differential calculus, Marx mentions this lemma along with the scholia to lemma XI (see, PV, 67).

Lemma I states : "The quantities as well as the ratios of quantities, which in the course of any finite time constantly tend to equality, and before the end of this time come closer to each other — closer than any given difference — will be equal at the limit " (I. Newton, Matematicheskie Nachala Naturalnoi Filosofii, SpB., 1915, p. 53).

However, in the proof of this lemma, the existence of limit as actually attainable *at the end* of the interval of time considered, was in fact silently presupposed. In fact this proof consisted of denying the fact,that the value of the quantities attained "at the end of this time"(their "limits") may differ from each other.

Thus, Newton always understood *limit* actually and that is why he hardly surpassed — in respect of mathematical exactitude and validity — the Leibnitzian *actual infinitesimals* or the *moments* corresponding to them, which, as is well known, Newton too used in practice.

ON LEONHARD EULER'S CALCULUS OF ZEROS

An acquaintance with Euler's attempt to construct the differential calculus as a calculus of zeros, is essential for understanding those places of Marx's manuscripts, wherein the ratio $\frac{dy}{dx}$ is viewed as a ratio of zeros, as something exactly equal to the value of the derivative of y in respect of x for any value of the variable x, and at the same time as something with which one may operate as one does with the ordinary fractions — for example, while equating the product $\frac{du}{dv} \cdot \frac{dv}{dx}$ with the "fraction" $\frac{du}{dx}$ ("cancelling" dv). This attempt should be highlighted also in connection with the fact,that in the list of literature,attached to the drafts of his essay on the history of differential calculus (see, PV, 66), Marx made a special mention of chapter III of Euler's "Differential Calculus", devoted to an account of this attempt. Further, it is also important, in view of the fact, that Marx called Euler's calculus "rationalist".

The "Differential Calculus"of the great mathematician,member of the Academy of Science of Peterburg,Leonhard Euler,was published by the Academy of Peterburg,in 1755. This work is based on an attempt to consider the differentials as exactly equal to zero in magnitude,but at the same time also as different zeros : zeros with "histories" of their origins, fixated in the differences of notation (dx, dy etc.),which permits them to be considered as such zeros, whose ratio $\frac{dy}{dx}$, where $y = f(x)$, is distinguished by the fact that it is the derivative $f'(x)$ and, that it may be operated upon as an ordinary fraction.

Euler undertook this attempt with the aim of freeing mathematical analysis from the treatment of differentials as actually infinitesimal magnitudes, having an explicitly contradictory character (being in a certain sense, both zeros and not-zeros at the same time). Euler thought, the assertion to the effect that "pure reason recognises the possibility of the thousandth part of a cubic foot of matter to be bereft of all extension", to be "quite insufficient" (in the given context it means "impermissible", see : L. Euler, Differential Calculus, M.- L., 1949, p. 90). "An infinitesimal quantity is nothing but an evanescent one, and that is why it is exactly equal to zero. This is in agreement with that definition of the infinitesimals,according to which they are smaller than all possible given quantity. In fact, if a quantity is so small, that it is smaller than any possible given quantity, then, wittingly, it cannot but be equal to zero; for had it not been equal to zero, then it would be possible to posit a quantity equal to it, but that will contradict the presupposition. Thus, if someone asks, what is an infinitesimal in mathematics, then we shall answer, that it is exactly equal to zero. Hence, this concept has no secrets hidden in it, the like of which are usually attached with it and which makes the infinitesimal calculus highly suspicious in the eyes of many" (ibid, p. 91).

Since a simple assertion of the identity of the differentials with zeros, does not as yet produce the differential calculus, Euler introduced "different" zeros, installing two types of equations for them : "arithmetic" and "geometric". In the "arithmetic" sense all zeros are equal to each other, and for an a not equal to any zero, $a+0$ is always equal to a,

independently of what sort of "zero" is added to a. In the "geometric" sense only two such zeros are equal to each other, the "ratio" of which is equal to one.

Here Euler does not explain, what he means by the "ratio" of two zeros. Only this much is clear, that he extended the usual properties of the ratios of magnitudes other than zero, upon these "ratios" and, that by the ratio of two "zeros": dy and dx, he in fact understood that which is expressed as $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, in the language of modern mathematical analysis. And that is why, Euler's theory of "zeros"could not free mathematical analysis from the need of introducing the concept of limit (and the difficulties connected with this concept).

In so far as Euler's zeros could be different "zeros" (in the "geometric" sense not equal to one another), it was necessary to have different signs for them. Euler wrote : "Two zeros may have any geometric relation between them,though from the arithmetic point of view their relation is that of equality. Thus, in so far as there may be any ratio between zeros, different symbols have been used on purpose, to express these different ratios, especially when the geometric ratio of two different zeros, is required to be defined. But in the calculus of infinitesimals, only the ratio of different infinitesimals is sought. That is why, if we do not use different signs for them, then there will be a great confusion and there will be no way out of it" (p. 91).

If by so interpreting dx and dy as "different" zeros, whose ratio is equal to $f'(x)$, we go over from $\frac{dy}{dx} = f'(x)$ to $dy = f'(x)\,dx$, then we obtain an equality, whose left hand side and right hand side are equal to each other, both in the "arithmetic"and in the "geometric" sense. In fact, in the left as well as in the right hand side "zeros" will stand, and, as it has already been noted,all zeros are mutually equal in the "arithmetic" sense. Since the ratio of dy to dx is equated *fully* with $f'(x)$ — i.e., in the "arithmetic" as well as in the "geometric" sense [the ratio $\left(\frac{dy}{dx}\right) : f'(x)$, where $y = f(x)$, is considered equal to one, even if $f'(x) = 0$], and the rules for operating with ordinary ratios are extended upon the "ratios" of zeros, we get $dy : f'(x)\,dx = \left(\frac{dy}{dx}\right) : f'(x) = 1$, or, in other words, dy and $f'(x)\,dx$ are equal to each other, in the "geometric" sense too.

To all appearence, it is namely this "complete" equivalence of the equalities $\frac{dy}{dx} = f'(x)$ and $dy = f'(x)\,dx$, not only in the sense of the possibility of transition from one of them to the other, but also herein (and owing to this),in the treatment of the "ratio" of the "differential particles"dy and dx as an ordinary ratio (as a fraction), in spite of the fact,that as "differential particles" dy and dx are zeros ("different" zeros, to be designated differently)— which Marx had in view,when he transformed the first of these equalities into the second (see, PV, 69).

For more detailed informations about Euler's zeros and the history of ideas connected with them the reader may refer to : A.P. Juschkewitsch, Euler und Lagrange über die

Grundlagen der Analysis, in : Sammelband zu Ehren des 250 Geburtstages Leonard Eulers, Berlin, 1959, pp. 224-244.

Here we shall limit ourselves to two more remarks of Euler. These may be of use while studying Marx's manuscripts. The first one is related to the concept of the differential as the principal part of the increment of a function. This concept played an important role in mathematical analysis, especially in its applications. Euler introduced it as follows : "Let an increment obtained by the variable x, be very small, so that in the expression [for the increment Δy of the function y of x, i.e., in] $P w + Q w^2 + R w^3 +$ etc.*, the terms $Q w^2 + R w^3$, and more so the rest, become so small,that in an expression,where a great exactitude is not required, they may be neglected,in comparison to the first term. Then, having found out the first differential Pdx, we know, of course only approximately, also the first difference, for it will be equal to Pw; this is of great help in many cases, where analysis is applied to solve practical problems" (ibid, p. 105). In other words,having substituted in the expression of the differential of the function y of x (i.e., in Pdx, where P is the derivative of y in respect of x)the differential dx,equal to Euler's zero,by the finite increment w of the variable x,we shall get that very concept of the differential as the principal part of the increment of a function, which is the point of departure of the modern courses of mathematical analysis.

There is an analogous concept of the differential as the principal part of the increment of a function in Marx's manuscripts (see, description of manuscript 2763, pp.141-142).

Another remark [of Euler] is associated with the question of choice of notations, specific to the differential calculus, i.e., for the differentials and derivatives. Above all what is of interest here is the fact that Euler interpreted Newton's pointwise notations as signs of differentials (and not of derivatives). In fact he wrote : "The name "fluxia", which was initially used by Newton to designate the speed of increment, was by analogy transferred to the infinitesinal increments, which a quantity admits,when it, as it were, flows" (p.103). And lower down there we read : "The differentials, which they (the English—Ed.) call fluxia, are designated by the points adopted by them, which they put over letters, so that $\dot{y}$ designates for them the first fluxia of y, $\ddot{y}$— the second, $\dddot{y}$— the third fluxia etc."

However, Euler was not satisfied with this mode of notations, and he continues : "Since the mode of designation depends upon will,these notations may not be rejected, if the number

*Euler's "Differential Calculus" began with the calculus of finite differences and the theorem, which reads : "if the variable x admits an increment, equal to w, then some function of x emerging as a result of this increment may be expressed as $P w + Q w^2 + R w^3 +$ etc., wherein, this expression is either finite,or it continues endlessly" (p. 103, see also, p. 61). The proof of this theorem was based upon the fact,that the class of functions herein considered by Euler consisted of powered functions, polynomials and elementary transcendental functions,expanded in infinite powered serieses; he permitted himself to treat them as finite polynomials.

of points is not large, so that they may be easily counted. However, if many points are to be superscribed, then this mode creates great confusion and a lot of discomfiture. In fact, the tenth differential or the tenth fluxia, may thus be designated extremely awkwardly as : $\overset{\vdots\vdots}{\ddot{y}}$; whereas our mode of designation $d^{10}y$, is easily understood. There are occasions, when differentials of much higher and even indeterminate orders are required to be expressed ; in such cases the English mode will become most unsuitable "(pp.103-104). Marx also spoke of an analogous identification (in certain cases), by Newton and his successors of the "fluxia" $\dot{x}$, $\dot{y}$ etc.with the "moments" (i.e., differentials) $\tau\dot{x}$, $\tau\dot{y}$ etc. (where τ is "an infinitesimal part of time"). Here Marx observed, that "Newton's τ plays no role in his analysis of basic functions and perhaps that is why it has been omitted" (p. 68), and that Newton himself gladly casts away τ (ibid). In consonance with this, when Marx spoke of the method of Newton, he also used such expressions, as "the differentials of y in the form of $\dot{y}$, of u in the form of $\dot{u}$ and of z in the form of $\dot{z}$ " (see, p. 69).

We also note that Marx especially stressed the advantage of the Leibnitzian symbols of differential calculus, over the symbols adopted by Newton and his successors (see, p.79).

"THE RESIDUAL ANALYSIS" OF JOHN LANDEN

In a number of Marx's mathematical manuscripts we find a mention of Marx's intentions to get acquainted with the works of John Landen, in the British Museum (see for instance, p. 39)

Marx saw in Landen, a possible predecessor of Lagrange, who strove to return to the strictly algebraic foundations of the differential calculus (p. 90), and assumed, that Landen's method must have been analogous to the method of "algebraic differentiation" proposed by him; but he had doubts, as to whether Landen understood the essential difference of this method from those of the others.

To get convinced about the fairness of these assumptions, Marx wanted to see in the Museum, "The Residual Analysis" of Landen.

Marx could have obtained informations about this book, from two sources at his disposal : Hind's book (2nd ed., p. 128) and Lacroix's big "Treatise" (vol. I, pp. 239-240). Incidentally, these two sources are almost indentical, as Hind, in essence, only translated the corresponding place from Lacroix, into English. In Hind we read : "However the idea of constructing a calculus of this type [i.e., differential calculus] upon purely algebraic foundations was, apparently, first concieved by John Landen, the famous English mathematician, whose creativity flourished in mid 18th century. The first task of his "*The Residual Analysis*" was to obtain an algebraic expansion of the difference of two identical functions of x and x', divided by the difference of these very magnitudes, or an expansion of the expression $\frac{f(x')-f(x)}{x'-x}$, and to find out, what is called *the special value* of the result obtained, when x' is assumed to be $= x$ and when, that is why, all trace of the divisor $x'-x$ has already vanished". (In Lacroix more perecisely : "... when this quotient $[f(x')-f(x) \ / \ x'-x]$ is already so obtained, that in it no trace of the divisor $x'-x$ is retained, [then] in it, it is assumed that $x'=x$, so that the ultimate aim of the calculus consists of arriving at a certain *special value* of the above mentioned ratio".)

To all appearence, Marx could not realise his intention of going through Landen's book in the British Museum. However, an analysis of the content of this book fully confirms the aforementioned assumptions of Marx, which he himself considered to be "highly likely".

The complete title of Landen's book is : *The ResidualAnalysis*, a new branch of the algebraic art, of very extensive use, both in Pure Mathematics, and Natural Philosophy, Book I. By *John Landen*. London, printed for the author, and sold by L.Haws, W.Clarke, and R.Collins, at the Red Lion in Pater-noster Row, MDCCL XIV.

Its preface begins with the words : "Having remained busy for sometime past with a new and simple method of investigations on the binomial theorem, with the help of a purely algebraic process, I wished to examine, whether the means which helped me to investigate this theorem, may turn out to be of use for investigating other theorems, and I soon found, that a computational technique based on those means, may be applied in many investigations ... I named this special technique — *the Residual Analysis,* since in all those investigations, where it is applied, the

principal means, through which we arrive at the desired conclusions, are such quantities and algebraic expressions, which the mathematicians call the residuals".

Later on the author criticised Newton's calculus of fluxions and Leibnitz's analysis of infinitesimals, because these are based upon the introduction of some unjustified new "principles" into mathematics. He thought, these are the explanations of the values of the new terminologies, introduced into the theory with the help of, in reality non-existent but nevertheless assumed (to be self- understood), *imaginary movements* and *figurative continuous flows,* which do not bring into mathematics any clear and precise idea, but compel us to talk, for example, about such — at least incomprehensible — things, like the *speed of time, speed of speed* etc., which are not even necessary for the explanations (and that is why, conversely, these are not capable of serving as the means for defining certain exact mathematical concepts) — [such are the new "principles"] of Newton's calculus of fluxions. In Leibnitz's analysis of infinitesimals, he considered, the introduction of the *infinitesimal magnitudes* and of the magnitudes *infinitesimally smaller, than the infinitesinal magnitudes* unjustified as new "principles" ; their dismissal (when obtaining of approximate results are not at issue) is "a very unsatisfactory (if not erroneous) mode of getting rid of such quantities" (p. IV). Landen proposes, that mathematics is not in need of such principles, which are alien to it, and that his "Residual Analysis", "not admitting of any principle, apart from those, that are accepted in Algebra and Geometry since antiquity", "is no less useful (if not more), than the calculus of fluxions or the differential calculus" (p. IV).

The starting point of Landen's residual analysis is the formula

$$\frac{a^r - b^r}{a-b} = a^{r-1} + a^{r-2}b + \cdots + b^{r-1} \tag{1}$$

(r is an integral and positive number), and the following formulae deduced from it, and with its help :

$$\frac{v^{\frac{m}{r}} - w^{\frac{m}{r}}}{v-w} = v^{\frac{m}{r}-1} \frac{1+\frac{w}{v}+\left[\frac{w}{v}\right]^2+\cdots+\left[\frac{w}{v}\right]^{m-1}}{1+\left[\frac{w}{v}\right]^{\frac{m}{r}}+\left[\frac{w}{v}\right]^{\frac{2m}{r}}+\cdots+\left[\frac{w}{v}\right]^{\frac{(r-1)m}{r}}}, \tag{2}$$

$$\frac{v^{-\frac{m}{r}} - w^{-\frac{m}{r}}}{v-w} = -v^{-1}\cdot w^{-\frac{m}{r}} \frac{1+\frac{w}{v}+\left[\frac{w}{v}\right]^2+\cdots+\left[\frac{w}{v}\right]^{m-1}}{1+\left[\frac{w}{v}\right]^{\frac{m}{r}}+\left[\frac{w}{v}\right]^{\frac{2m}{r}}+\cdots+\left[\frac{w}{v}\right]^{\frac{(r-1)m}{r}}} \tag{3}$$

(where m and r are integral and positive numbers, $\left.\frac{w}{v}\,\right]$ corresponds to our $\left(\frac{w}{v}\right)$ *).

Landen obtained the derivative of the powered function x^p, for an integral and fractional (positive or negative) index of power p, as a "special value" of the ratio

$$\frac{x^p - x_1^p}{x - x_1}$$

when $x = x_1$. In other words, he so redefined the ratio $\frac{x^p - x_1^p}{x - x_1}$ when $x = x_1$, that even when $x = x_1$, the strength of the equalities, corresponding to the formulae (1), (2) and (3), were retained.

Landen designated the "special value" of the ratio $\frac{y - y_1}{x - x_1}$ (where $y = f(x)$, $y_1 = f(x_1)$) when $x = x_1$, through $[x \,\underline{'}\, y]$.

He carried out the transition to the irrational index of power, in the light of examples; to begin with, by finding out the "special value" of the ratio $\frac{v^{4/3} - w^{4/3}}{v - w}$, when $v = w$ (i.e., the derivative of $v^{4/3}$ in respect of v) in two different ways : once according to formula (2), for $m = 4$ and $r = 3$; next time, according to the same formula, but — "since $\frac{4}{3} = 1{\cdot}3333$ etc." — successively applied to the pairs : $(m = 13\,333,\ r = 10\,000)$, $(m = 133\,333,\ r = 100\,000)$ etc. Landen escaped from the difficulties connected with the fact, that this process is infinite, by observing, that the "ultimate value" of

$$\frac{1 + 1 + 1 + 1\ \cdots\ (13\,333 \text{ etc. times})}{1 + 1 + 1 + 1\ \cdots\ (10\,000 \text{ etc. times})}$$

"is evidently equal to $\frac{4}{3}$, the magnitude, from which [the number] $1{\cdot}333$ etc. was obtained (by division)" (p. 7).

---

* To obtain formula (2), using (1), it is enough to note that

$$\frac{v^{\frac{m}{r}} - w^{\frac{m}{r}}}{v - w} = \frac{v^m - w^m}{v - w} : \frac{v^m - w^m}{v^{\frac{m}{r}} - w^{\frac{m}{r}}} = \frac{v^m - w^m}{v - w} : \frac{(v^{\frac{m}{r}})^r - (w^{\frac{m}{r}})^r}{v^{\frac{m}{r}} - w^{\frac{m}{r}}}.$$

Formula (3) is easily deduced from formula (2). In fact

$$\frac{v^{-\frac{m}{r}} - w^{-\frac{m}{r}}}{v - w} = \frac{(vw)^{\frac{m}{r}}\,(v^{-\frac{m}{r}} - w^{-\frac{m}{r}})}{(vw)^{\frac{m}{r}}\,(v - w)} = -\frac{w^{\frac{m}{r}} - v^{\frac{m}{r}}}{(vw)^{\frac{m}{r}}\,(w - v)}.$$

After this he went over to that instance, where $\frac{m}{r} = \sqrt{2} = 1{\cdot}4142$ etc., treating it according to the second mode, i.e., as he himself observed, "approximately", but so that, every time it may be made "more proximate"; he again concluded, that the "ultimate value" of

$$\frac{1+1+1+1 \cdots \text{(14 142 etc. times)}}{1+1+1+1 \cdots \text{(10 000 etc. times)}}$$

"is equal to $\sqrt{2}$, the magnitude from which [the number] 1·4142 etc. was obtained (by extracting the root)" (p. 8).

It is not surprising that Landen could not construct his "Residual Analysis" without falling back upon the concept of *limit*, in some form. However, he spoke of limit, precisely in the manner of Newton, treating limit as the "ultimate value" (as the end) of an infinite (i.e. endless) sequence. Naturally, he did not use this definition in practice, but he had to have recourse to such means of precisely estimating the approximations and convergence (or divergence) of the processes for attaining them successively, which were prompted by the concrete content of the problems considered by him.

Like the other mathematicians of his time, Landen thought that it was possible to freely use the divergent serieses in formal transformations of expressions with the help of infinite serieses, if the latter play, therein, only the role of an intermediate stage in the transformation. If a series must express the value of some magnitude to be computed, then, for it to be used, the series must be convergent. Here Landen did not think, that it was necessary to explain, what he meant by *convergence* (or *divergence*) of a series, but having expanded a function into a series (with the help of some formal transformations), he usually indicated the radius of convergence of the series thus obtained, and also mentioned the methods, permitting an "improvement" upon its convergence (its substitution by another series, which converges "faster" to the same limit). Thus, among the "principles", which "were accepted in algebra and geometry since antiquity", Landen, evidently counted some forms of transitions to the limit, with which he could somehow cope in practice (when approximate calculation was at issue). But he did not have a precise general concept of "convergence" and "limit". Nor did he have the methods for computing the limits (or detecting their absence), applicable to a sufficiently large class of expressions. That is why Landen sought such a definition of the derivative (of the "special value"), which would directly contain within itself an algorithm for finding it.

Like Newton, herein, he too, discussed the functions of x as certain analogies of the concept of real number. Just as every natural number may be considered as a sum (finite or infinite) of the powers of base 10; each being multiplied by one of the numbers 0, 1, 2, ..., 9, so also every function of x, according to Newton, must be representable in the form of a sum of the (finite or infinite) powers with base x, each being multiplied by numbers (coefficients) — i.e., in the form of a powered series. (The series was considered to be "representing"the given function, given with the help of a finite "algebraic" expression, if it was obtained through formal transformations from the expression, which posed the function. Thus, the series $1 + x + x^2 + \cdots + x^n + \cdots$ is considered to be "representing" the function $\frac{1}{1-x}$, in so far as it

could be obtained by dividing 1 by $(1-x)$ according to the method of dividing the polynomials.) That is why the problem of finding out the derived function of $f(x)$ could be represented as reducible to an analogous task for x^p and to the problems : having known the derivatives of summands (or factors), of having to find out the derivative of the sum (or product). These are the problems which Landen solves first of all in his "Residual Analysis". Extension of these methods to the functions of several variables and partial derivatives of different orders entail a number of technical difficulties and Landen managed to cope with them with the help of — sometimes very clever — formal calculations.

Herein, it was usually presupposed silently, that a function is *univocally* represented by its corresponding powered series, i.e., if two powered serieses must represent one and the same function of x, then the coefficients of the identical powers in them must be equal (hence the wide use of the method of so-called "indeterminate coefficients").

As an example, illustrating the application of these methods by Landen, we shall adduce here a proof of Newton's binomial theorem in the general instance of an arbitrary real index of power of the binomial — as proposed by Landen (with certain specifications now in use). Landen's proof may also be of interest to us, as Marx attached a special importance to this theorem of Newton, first of all in connection with the theorems of Taylor and MacLaurin (see, PV, 88 & 93). Let

$$(a+x)^p = A_1 + A_2 x + A_3 x^2 + A_4 x^3 + \cdots, \tag{1}$$

where p is any real number, A_1, A_2, A_3, A_4 are indeterminate coefficients, assumed to be not dependent upon x. Assuming $x=0$ in both the parts of the equality, we shall get $A_1 = a^p$. Differentiating the equality (1) in respect of x, term by term (of course, Landen did not speak of the derivatives according to x, but spoke of the corresponding "special values", which he could already find out for $A\,x^r$, where A does not depend on x and, for any real r) we shall have

$$p(a+x)^{p-1} = A_2 + 2A_3 x + 3A_4 x^2 + \cdots \tag{2}$$

Now multiplying both the sides of equality (1) by p, and of equality (2) by $(a+x)$, we shall get, further,

$$p(a+x)^p = pA_1 + pA_2 x + pA_3 x^2 + pA_4 x^3 + \cdots, \tag{1'}$$

$$p(a+x)^p = aA_2 + \left.\begin{matrix} 2aA_3 \\ A_2 \end{matrix}\right\} x + \left.\begin{matrix} 3aA_4 \\ 2A_3 \end{matrix}\right\} x^2 + \cdots, \tag{2'}$$

whence, owing to the presupposed univocality of the expansions of the expression $p(a+x)^p$ in series, according to the powers of x,

$$aA_2 = pA_1, \qquad \text{or } A_2 = \frac{p}{a}A_1 = pa^{p-1},$$

$$2aA_3 + A_2 = pA_2, \qquad \text{or } A_3 = \frac{p-1}{2a}A_2 = \frac{p(p-1)}{2}a^{p-2},$$

$$3aA_4 + 2A_3 = pA_3, \qquad \text{or } A_4 = \frac{p-2}{3a}A_3 = \frac{p(p-1)(p-2)}{2\cdot 3}a^{p-3},$$

. . . .

i.e., $(a+x)^p = a^p + \frac{p}{1}a^{p-1} + \frac{p(p-1)}{1\cdot 2}a^{p-2}x^2 + \frac{p(p-1)(p-2)}{1\cdot 2\cdot 3}a^{p-3}x^3 + \cdots$

and this is the binomial theorem of Newton.

Though John Landen's residual analysis did not become a working instrument of mathematicians : Landen's notations were very cumbersome, and — perhaps that is why — he did not arrive at the theorems of Taylor and MacLaurin ; it should not however be thought, that the works of Landen did not exert any influence upon the development of mathematics. Landen himself wrote, that a number of theorems from his "Residual Analysis", "drew the attention of Mr. De Moivre, Mr. Stirling and of the other prominent mathematicians"(p.45). Lacroix informs us in his "Treatise" (vol. I, p. 240), that he used Landen's methods in the "Supplements to the Elements of Algebra" for proving the binomial formulae, and for serial expansion of the exponential and logarithmic functions.

However, apparently, it was Lagrange who drew Lacroix's attention to Landen, upon whose "Theory of Analytical Functions" Lacroix based his "Treatise". Lagrange wrote, in the introduction to his book, referring to the difficulties bedeviling the basic concepts of Newton's analysis, that : "To get rid of these difficulties, a skilful English geometer, having made important discoveries in analysis, recently proposed to change that method of fluxions, which was so far steadfastly followed by all the English geometers, by another purely analytical method, which is analogous to the differential method, but in which, instead of using only the infinitesimals or the differences of variable quantities equal to zero, at first the different values of these quantities are used, which are then mutually equated with each other, since with the help of division, the factor — which this equalisation would turn into zero — is forced to disappear. Thus the infinitesimals and the evaneseent quantities are actually got rid of ; but the methods and applications of calculus become complicated and less natural, and one must agree to it, that this mode of making the principles of differential calculus more precise, robs it of its basic advantage — the simplicity of method and the ease of operations". (Apart from the "Residual Analysis", Lagrange also refers to Landen's "A discourse of the residual analysis", published already in 1758; see, "Oeuvres de Lagrange", T. IX, Paris, 1881, p.18).

Apparently, Lagrange's afore mentioned observation is connected with the fact, that Landen used highly cumbersome notations and hence could not arrive at the concept of differential and at a calculus operating with differential symbols.

In contradistinction to Lagrange, Lacroix observed, that Landen's method may "in essence, be reduced to the method of limits" ("Treatise .. ", p. XVII).

PRINCIPLES OF DIFFERENTIAL CALCULUS ACCORDING TO BOUCHARLAT

From among the books on mathematical analysis at Marx's disposal, apparently, Boucharlat's "Elements of differential and integral calculus" is of the greatest significance, for an understanding of Marx's mathematical manuscripts. Marx got acquainted with it, through the English translation of its third French edition (of 1826). This translation was done by Blakelock and it was published in 1828.

This text book enjoyed great popularity and saw many editions. In 1881, its 8th edition was published in Paris, with notes by M.P. Laurent. It was also translated into a number of foreign languages, including Russian.

An alumnus of the Polytechnical School, professor of "transcendental" (higher) mathematics, author of a number of texts books of mathematics and mechanics, Boucharlat, Jean-Louis (1775-1848) was at the same time a poet and was, from 1823, professor of literature in the Atheneum of Paris. It seems that the literary merit and clarity of exposition of this book also furthered its popularity in a large way.

Thus it is clear that, Marx was not accidentally drawn to Boucharlat's course.

At the same time, in spite of its author's pretentions about the greater strictness of the exposition and also to the effect, that he has managed to improve upon the "algebraic" method of Lagrange, with the help of the theory of limits (see, introduction to the 5th edition, 1838, p. VIII), the mathematical level of this course was not very high. Even in its 5th (1838) edition, and not only in the 3rd — the English transtation of which was used by Marx, the concepts of variable, function, derivative and differential were indroduced as follows *:

"1. It is said, that *a variable is a function of another variable,* when the first is equal to an analytical expression, constitued out of the second ; for example, y is a function of x in the following equations :

$$y = \sqrt{a^2 - x^2}\,,\ y = x^3 - 3bx^2\,,\ y = \frac{x^2}{a}\,,\ y = b + cx^3\,.$$

. . . .

3. Let us now take the equation

$$y = x^3 \qquad (1)$$

and assume, that y turns into y', when x becomes $x + h$; hence, we have

$$y' = (x + h)^3$$

and, carrying out the indicated operation,

---

*Marx not only took notes from this text book, in a number of places of his manuscripts and polemised against its author about his basic methodological directions, but also put in a lot of labour for the factual verification of the latter. That is why, it is hardly possible to do without an acquaintance with this text-book. Here we shall give a detailed account of the first twenty paragraphs of this book. These paragraphs are specific for this book, and Marx's critical observations are especially directed towards them. Wherever in the manuscripts, an acquaintance was required of these paragraphs, there a mention has been made of the corresponding pages of the appendix, containing their account. — Ed.

$$y' = x^3 + 3x^2h + 3xh^2 + h^3 ;$$

if from this equation we subtract equation (1), then we shall have

$$y' - y = 3x^2h + 3xh^2 + h^3,$$

and, dividing by h,

$$\frac{y' - y}{h} = 3x^2 + 3xh + h^2. \tag{2}$$

Let us see, what this result teaches us :

$y' - y$ is the increment of the function y, when x obtains the increment h, that is why this difference $y' - y$ is the difference between the new condition of the variable magnitude y and its initial condition.

On the other hand, since the increment of the variable x is h, it follows that, the expression $\frac{y' - y}{h}$ is the ratio of the increment of the function y to the increment of the variable x. Examining the second element of equation (2), we see that this ratio diminishes along with the dimunition of h and that, when h becomes zero, this ratio turns into $3x^2$.

Hence, the term $3x^2$ is the limit of the ratio $\frac{y' - y}{h}$; it tends to this term, when we compel x to diminish.

4. Since in the presupposition, that $h = 0$, the increment of y also becomes zero, so $\frac{y' - y}{h}$ turns into $\frac{0}{0}$, and that is why from equation (2) we get

$$\frac{0}{0} = 3x^2 . \tag{3}$$

There is nothing absurd about this equation, since algebra teaches us that $\frac{0}{0}$ can represent any magnitude whatsoever. On the other hand it is clear, that since the division of both the terms of a fraction by one and the same number does not change the value of the fraction, we may conclude, that the smallness of the terms of a fraction does not in the least influcence its value and that, hence, it may remain the same even when its terms attain the last stage of smallness, i.e., turn into zero.

The fraction $\frac{0}{0}$ in equation (3) is a symbol, that has substituted the ratio of the increment of the function to the increment of the variable ; since in this symbol no trace of this variable remains, we shall represent it by $\frac{dy}{dx}$; then $\frac{dy}{dx}$ will remind us that, y was the function, and x — the variable. But due to this, dy and dx will not cease to be zeros, and we shall have

$$\frac{dy}{dx} = 3x^2 . \tag{4}$$

$\frac{dy}{dx}$, to be more precise, its value $3x^2$ is the differential coefficient of the function y.

We note, that since $\frac{dy}{dx}$ is a sign, representing the limit $3x^2$ (as is shown by equation (4)), dx must always be under dy. However, to facilitate algebraic operations, here we must get rid of the denominator in equation (4), and we shall get $dy = 3x^2\,dx$.

This expression $3x^2dx$ is called the differential of the function y" (pp. 1-4).

In §§ 5-8 Boucharlat finds out the dy for

$$y = a + 3x^2\,,\ y = \frac{1 - x^3}{1 - x}\,,\ y = (x^2 - 2a^2)\,(x^2 - 3a^2).$$

In all these cases the expression for the augmented value of y, i.e., (in the notations of Boucharlat) for y', is equal to $f(x+h)$, if $y = f(x)$ is represented in the form of a polynomial, ordered according to the powers of h (with coefficients in x), after which the ratio $\frac{y'-y}{h}$ is easily repesented by a polynomial of the same kind. The assumption of $h = 0$ in the latter gives $\frac{dy}{dx}$, the multiplication of which by dx completes the search for the expression of the differential dy.

"9. The expression dx itself is the differential of x, for let $y = x$, then $y' = x + h$, hence, $y' - y = h$ and, that means, $\frac{y'-y}{h} = 1$. Since the quantity h does not enter into the second element of this equation, we see that for the transition to limit, it is enough to change $\frac{y'-y}{h}$ into $\frac{dy}{dx}$, which will give $\frac{dy}{dx} = 1$, hence, in our supposition,

$$dy = dx\text{ ".}$$

" 10. We shall also find, that the differential of ax is adx; but if we had $y = ax + b$, we would also have got adx as the differential, whence it follows, that the constant b (not accompanying the variable x) gives us no term when differentiated, or, in other words has no differential at all.

However, we may note, that if $y = b$, then before us we have the case, where a is zero in the equation $y = ax + b$ and where, that is why, since $\frac{dy}{dx} = a$ is now reduced to $\frac{dy}{dx} = 0$, there is neither a limit, nor a differential" (p. 6).

Thus we see that in Boucharlat's book :

1) There is no definition either of the limit, or of the derivative and the differential. All these concepts are only explained in the light of examples, and besides only such, that the ratio $\frac{f(x+h) - f(x)}{h}$ is represented in the form of a polynomial, ordered according to the powers of h, with coefficients in x. Herein, the search for the limit of this ratio when $h \to 0$ is treated as the assumption of $h = 0$ in the polynomial obtained. The questions, as to whether there are other

instances, and whether it is possible to "differentiate" in those cases, and if possible, then how, are not even mentioned here.

2) The transition from the derivative $\frac{dy}{dx} = \varphi(x)$ to the differential $dy = \varphi(x)\,dx$ is considered to be an illegitimate operation, carried out only with the aim of "facilitating" algebraic calculations.

3) From

$$\frac{f(x+h)-f(x)}{h} = \varphi(x, h), \tag{A}$$

when $h \neq 0$, it is concluded that also when $h = 0$, i.e., when $\frac{f(x+h)-f(x)}{h}$ loses its meaning (turns into $\frac{0}{0}$), the equality (A) retains its strength, i.e., we must get

$$\frac{0}{0} = \varphi(x, 0). \tag{B}$$

In other words, it is held, that $\varphi(x, h)$ must be defined (and continuous) at $h = 0$ and that the equaltiy (B) logically follows from the equality (A) (though the expression $\frac{0}{0}$ is devoid of any sense).

4) The equality of zero with the limit or the differential is evaluated as a testimony to the effect, that "there is neither a limit, nor a differential", though at the same time dy and dx are always zeroes (if $\varphi(x) \neq 0$, then the differential — equal to $\varphi(x)\cdot 0$ — exists ; if $\varphi(x) = 0$, then it does not). But, evidently, here the question, as to which of the zeroes are considered to be "existing", and which are not, does not even arise.

It is not surprising, that Marx was not satisfied with such a mode of treating the basic concepts of differential calculus. And in fact, even his first conspectus of the initial paragraphs of Boucharlat's book (see, PV, 153) contains critical remarks directed at its author. But what Marx especially did not like, is the fact, that a basic concept of differential calculus — the concept of *differential* — turned out to be unsubstantiated and its introduction was justified only by saying, that it "facilitates algebraic operations" (see, the manuscript "On the Differential", p. 33)

§ 11 of Boucharlat's book is devoted to the observation, that "sometimes the increment of the variable happens to be negative ; in that case it is necessary to substitute x by $x - h$ and to operate as earlier". Thereby in the example $y = -ax^3$, it is obtained that $dy = -3ax^2dx$, and the following conclusion is drawn : "we see, that it is reduced to the presupposition of a negative dx in the differential of y, computed under the assumption of a positive increment". But for Boucharlat dx is 0. However, the question as to what does a "negative zero" mean did not dawn on him. (In the manuals of that period the concept of "absolute magnitude" was not present.)

Since the following three paragraphs 12-14 are especially characteristic of Boucharlat's book and since a number of places of Marx's manuscripts are related to them, here their text is being reproduced in full.

"12. Before proceeding farther,we shall make an important observation, namely, that if in an equation, the second term of which is a function of x and, that is why, which may be represented by us in the general form of $y = f(x)$, we substitute x by $x + h$ and, having then ordered it according to the powers of h, obtain the expansion

$$y' = A + Bh + Ch^2 + Dh^3 + \text{etc.}, \tag{C}$$

then we must always have $y = A$. In fact, if we put $h = 0$, then the second element is reduced to A; as to the first element, since we marked y by a stroke only to indicate that y has undergone a definite change, when x turned into $x + h$, it suits us to remove the stroke of y, when h will be zero, and the equation (C) is then reduced to

$$y = A.$$

"13. This will give us the opportunity to generalise the procedure of differentiation. In fact, if in the equation $y = fx$, in which we may consider the expression represented through fx to be known, we substituted $x + h$ for x and, having ordered it according to the powers of h, we could then obtain the following expansion :

$$y' = A + Bh + Ch^2 + Dh^3 + \text{etc.},$$

or, better still, if according to the previous paragraph [we obtained],

$$y' = y + Bh + Ch^2 + \text{etc.},$$

then we shall have

$$y' - y = Bh + Ch^2 + \text{etc.},$$

whence,

$$\frac{y' - y}{h} = B + Ch + \text{etc.}$$

and, in transition to the limit, $\frac{dy}{dx} = B$. This teaches us, that the differential coefficient is equal to the coefficient of the term, containing the first power of h, in the expansion of $f(x + h)$, ordered according to the ascending powers of h.

"14. If instead of one function y, changing its condition due to the increment imparted by the variable x, which it contains, we have two functions y and z of the same variable x and are able, in particular, to find out the differentials of each of these functions, then it will be easy to deduce from there the differential of the product zy of these functions, through the following proof. In fact, if we put $x + h$ for x in y and in z, then we shall have two expansions, which, having been ordered according to the powers of h, may be represented as :

$$y' = y + Ah + Bh^2 + \text{etc.}, \tag{5}$$

$$z' = z + A'h + B'h^2 + \text{etc.}, \tag{6}$$

in transition to the limit, we shall find :

$$\frac{dy}{dx} = A, \quad \frac{dz}{dx} = A', \tag{7}$$

multiplying the equations (5) and (6), each by the other, we shall get

$$\begin{aligned} z'y' = zy &+ Azh + Bzh^2 + \text{etc.} + \\ &+ A'yh + AA'h^2 + \text{etc.} + \\ &\qquad + B'yh^2 + \text{etc.}, \end{aligned}$$

whence

$$\frac{z'y' - zy}{h} = Az + A'y + (Bz + AA' + B'y)h + \text{etc.};$$

in transition to the limit and indicating which expression is to be differentiated by a point, we shall get

$$\frac{d{\cdot}zy}{dx} = Az + A'y\,;$$

putting in place of A and A', their values given by equations (7) we shall have :

$$\frac{d{\cdot}zy}{dx} = \frac{z\,dy}{dx} + \frac{y\,dz}{dx}.$$

and, removing the common divisor dx,

$$d{\cdot}zy = z\,dy + y\,dz.$$

Thus to obtain the differential of the product of two variables, it is necessary only to multiply each of them by the differential of the other and to add these products".

In § 15 this rule is used to find out the differential of the product of three variables, in § 16 — for obtaining the differential of the quotient $\frac{y}{z}$.

In § 17 the differential of the power $y = x^m$, for an integral and positive m, is obtained from the formula

$$\frac{d\cdot xyztu\ \text{etc.}}{xyztu\ \text{etc.}} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} + \frac{dt}{t} + \frac{du}{u} + \text{ etc.} \tag{9}$$

by assuming that x, y, z, t, u etc. are equal to x and taking their number to be m.

§ 18 contains a formulation of the rule for differentiating power.

In § 19 this rule is demonstrated for fractional and negative indices of power, through formal operations with the signs of differentials (permitting reduction of the problem into instances already considered).

In § 20 the differential of power is obtained directly, with the help of the expansion of $(x + h)^m$, according to the binomial theorem of Newton.

In the third edition of Boucharlat's book, the English translation of which was used by Marx, there was a note (Note 2), the beginning of which figured under the title : "Considerations, demonstrating that the principles of differentiation are based upon the bionomial theorem". Since this note especially drew the attention of Marx, here we reproduce its text :

"With the exception of the differentials of circular functions, which, as we saw, are easily found with the help of trigonometric formulae, all the other one-term differentials, like, for example, the differentials of the functions x^m, a^x etc., were deduced from the binomial theorem

alone. It is true, that while defining the constants A in the exponential formulas, we had recourse to MacLaurin's theorem, but we could have done without it".

Later on it has been shown, how, namely, this could have been done, with the help of formal calculations involving infinite serieses — but these calculations are not at all substantiated ,from the modern point of view. After that Boucharlat concludes :

"Hence, it follows, that all the principles of differentiation are based upon the binomial theorem alone ; and since this theorem was demonstrated with all possible strictness, in the Elements of Algebra, we may conclude, that our principles rest upon a solid basis".

Thus, it is clear, that Boucharlat held the point of view of "algebraic" differential calculus of Lagrange, which he attempted to improve, with the help of the concept of limit. However, his "improvement" amounted to this, that while Lagrange wanted to avoid using the still unsubstantiated concept of limit, and defined the derivative of fx simply as the coefficient of the first power of h in the expansion

$$f(x+h) = f(x) + Ah + Bh^2 + Ch^3 + \cdots, \tag{1}$$

where $A, B, C, \ldots.$ are functions in x, Boucharlat "divined" the same derivative ("differential coefficient") through a transition to limit, consisting only of the fact, that he assumed, that $h = 0$ in the expansion

$$\frac{f(x+h) - f(x)}{h} = A + Bh + Ch^2 + \cdots, \tag{2}$$

which he obtained purely formally from the expansion (1). Herein, Boucharlat did not give any definition of the concept of "limit" or any interpretation of the latter. He confined himself to the hints, that the limit is the last value of the variable indefinitely tending towards it (i.e., *without* having a last value). It is not surprising that such a definition of the concept of limit could not satisfy Marx.

THEOREMS OF TAYLOR AND MACLAURIN AND LAGRANGE'S THEORY OF ANALYTICAL FUNCTIONS IN THE SOURCES CONSULTED BY MARX

1) These theorems and Lagrange's theory of analytical functions especially drew Marx's attention. Quite a few of the most important manuscripts are devoted to them (see, manuscripts 4000, 4001, 4300, 4301, 4302). For an understanding of these manuscripts and, especially for an understanding of the criticism to which Marx has subjected the proof of Taylor's theorem, as it was found in the manuals at his disposal, an acquaintance with these proofs and with the corresponding ideas of Lagrange, is a must. However, before we pass over to them, let us dwell a little upon the history of the theories of Taylor and MacLaurin*.

Taylor's theorem is contained as the 7th proposition, in the book : "Methodus incrementorum directa et inversa", by the English mathematician Brook Taylor (1685-1731). It was published in London, in the year 1715. Already in 1712, Taylor informed his teacher J. Machin about this result in writing. Condorcet was the first to call it "Taylor's theorem" in 1784, in his article "Approximations", in the 1st volume of the French Encyclopaedia (Encyclopédie méthodique). In 1786, Simon Lhuilier also used this name in the book "Exposition élémentaire des principes des calculs supérieure", which received a prize of the Academy of Sciences of Berlin (the theme was proposed by the Academy for a competition). Since then this theorem entered into all the manuals of mathematical analysis and was never called otherwise. However, now it is known that the Scottish mathematician James Gregory (1638-1675) was in possession of it already in 1671/72**.

Both Gregory and Taylor arrived at "Taylor's theorem", proceeding from the finite differences. While so doing, Taylor deliberately set before himself the direct task, of examining the highly confused account given by Newton, of his interpolation formula. He obtained his theorem, by first assigning a (finite) increment, other than zero, to the independent variable, and then — after a number of transformations — by turning the latter into zero, "by dividing it into an infinitely large number of parts". If we substitute the exceptionally cumbersome notations of Taylor by more modern ones, then its proof will look like this.

Let $y=f(x)$, where x is a variable, which changes, as he says, "uniformly", i.e., by successively obtaining the values x, $x+\Delta x$, $x+2\Delta x$, $\cdots, x+n\Delta x = x+h$. And let the corresponding values of $f(x)$ be y (or y_0), y_1, y_2, $\cdots$ y_n. Let the successive differences

* Here the following books serve as our sources : M. Cantor, Vorlesungen über Geschichte der Mathematik, 2nd ed., vol. III, pp. 378-382; D. D. Morduhai-Boltovsky, Kommentarii k "Metodu raznostiei", in the bk.: Isaac Newton, Matematicheskie raboty, M.-L., 1937, pp. 394-396; M.Ya. Vygodsky, Fstupitelnoe slovo k "Differentsialnomu ischisleniu " L. Eulera, in the bk.: L. Euler, Differentsialnoe ischislenie, M.-L., 1949, pp. 10-12; G. Vileitner, Istoriya matematiki ot Dekarta do serediny XIX stoletia, M., 1960, pp. 138-140; O. Becker und J.E. Hofmann, Geschichte der Mathematik, Bonn, 1951, pp. 200-201, 219; G. G. Tseiten, Istoriya matematiki v XVI i XVII vekah, M.-L., 1938, pp. 412, 445; D. J. Struik, Kratkii ocherk istorii matematiki, M., 1964, pp. 153-154. For a fuller treatment, see : M. Cantor's book, pp. 378-382. — Ed.

** Now we also know that Newton himself was in possession of the theorems "of Taylor and MacLaurin". On this see : *Yushkevich A.P.*, Mathematics and its History in Retrospective // Special Supplement to the present volume, part three, first article. — Tr.

(differences of the first order) between y_{k+1} and y_k $(k=0, 1, \cdots, n-1)$ be Δy, Δy_1, $\cdots$, Δy_{n-1}; the differences between these differences (differences of the second order) : $\Delta^2 y$, $\Delta^2 y_1$, $\cdots$, $\Delta^2 y_{n-2}$ etc. For the sake of visual clarity let us write all this in the form of the following circuit :

$$\begin{array}{ccccccccccc}
x & & x+\Delta x & & x+2\Delta x & & x+3\Delta x & \cdots & & & x+n\Delta x \\
y & & y_1 & & y_2 & & y_3 & \cdots & & & y_n \\
& \Delta y & & \Delta y_1 & & \Delta y_2 & & \cdots & & & \Delta y_{n-1} \\
& & \Delta^2 y & & \Delta^2 y_1 & & & \cdots & & & \Delta^2 y_{n-2} \\
& & & \Delta^3 y & & & & \cdots & & & \Delta^3 y_{n-3}
\end{array}$$

. . . .

Then it is clear, that

$$y_1 = y + \Delta y\,;$$
$$y_2 = y_1 + \Delta y_1\,,\ \ \Delta y_1 = \Delta y + \Delta^2 y\,,$$
$$y_3 = y_2 + \Delta y_2\,,\ \ \Delta y_2 = \Delta y_1 + \Delta^2 y_1\,,\ \ \Delta^2 y_1 = \Delta^2 y + \Delta^3 y\,,$$

. . . .

Hence we get further :

$$f(x+\Delta x) = y_1 = y + \Delta y\,,$$
$$f(x+2\Delta x) = y_2 = (y+\Delta y) + (\Delta y + \Delta^2 y) = y + 2\Delta y + \Delta^2 y\,,$$
$$f(x+3\Delta x) = y_3 = (y+2\Delta y+\Delta^2 y) + (\Delta y + \Delta^2 y) + (\Delta^2 y + \Delta^3 y) = y + 3\Delta y + 3\Delta^2 y + \Delta^3 y\,,$$

. . . .

Having noticed the general regularity, Taylor consequently concludes, that

$$f(x+n\Delta x) = y + n\,\Delta y + \frac{n\,(n-1)}{1\cdot 2}\,\Delta^2 y + \frac{n\,(n-1)\,(n-2)}{1\cdot 2\cdot 3}\,\Delta^3 y + \cdots + \Delta^n y\,, \tag{1}$$

and this is the interpolation formula of Newton (for interpolating through equal intervals). Its resemblance with Newton's binomial theorem, above all the fact that the coefficients of the expansion according to Δy, $\Delta^2 y$, $\cdots$, $\Delta^n y$ are exactly the same, is quite evident.

Assuming $n\,\Delta x = h$ (in Taylor it is v, not h), we shall have :

$$n = \frac{h}{\Delta x}\,,\ \ n-1 = \frac{h-\Delta x}{\Delta x}\,,\ \ n-2 = \frac{h-2\Delta x}{\Delta x}\,,\ \cdots\cdots,\ \ n-(n-1) = \frac{h-(n-1)\,\Delta x}{\Delta x}\,.$$

Putting these values of n, $(n-1)$, $(n-2)$, $\cdots$, in formula (1) Taylor obtained (in our notations) :

$$f(x+h) = y + h\,\frac{\Delta y}{\Delta x} + \frac{h\,(h-\Delta x)}{1\cdot 2}\,\frac{\Delta^2 y}{\Delta x^2} + \frac{h\,(h-\Delta x)\,(h-2\Delta x)}{1\cdot 2\cdot 3}\,\frac{\Delta^3 y}{\Delta x^3} + \cdots, \tag{2}$$

wherein he did not write out the last term

$$\frac{h\,(h-\Delta x)\,(h-2\Delta x)\,\cdots\,(h-(n-1)\Delta x)}{1\cdot 2\cdots n}\,\frac{\Delta^n y}{\Delta x^n}.$$

Then he imagined h to be fixated, n actually infinitely large, and Δx to be an actual infinitesimal ("zero"), thinking, that herein $\frac{\Delta y}{\Delta x}$ turns into the first fluxion $\dot{y}$ ($\frac{dy}{dx}$ according to Leibnitz), $\frac{\Delta^2 y}{\Delta x^2}$ — into the second fluxion $\ddot{y}$ ($\frac{d^2y}{dx^2}$ according to Leibnitz) etc. Thereby formula (2) turned into

$$f(x+h)=y+\dot{y}\,h+\ddot{y}\,\frac{h^2}{1\cdot 2}+\dot{\ddot{y}}\cdot\frac{h^3}{1\cdot 2\cdot 3}+\cdots,$$

i.e., into Taylor's series.

Thus, even after beginning with the finite differences and only then "removing" them, Taylor still operated entirely in the style of Newton and Leibnitz with the actual infinities, actual infinitesimals and the symbolic formulae of the calculus of fluxions, without pondering upon whether these have any "real equivalent" or not, and of course, without caring for the convergence of the series thus obtained (and besides, namely, for $f(x+h)$). Here it should also be mentioned, that though Taylor was an ardent supporter of Newton in the controversy with Leibnitz, which is why he did not use the notations of the latter and no where referred to him, it was not accidental, that still Euler enunciated his proof in the Leibnitzian language*. As has been observed by D. D. Morduhai-Boltovsky, in essence, Taylor approached the Newtonian fluxions from the Leibnitzian, and not the Newtonian side, namely, from the finite differences (see, the "Kommentarii" mentioned above, p. 396).

Regarding the history of MacLaurin's theorem, it should be noted, first of all, that it is already there in Taylor, in the form of a particular instance of his theorem, when $x=0$. It is true, that in contrast to MacLaurin, who obtained the expansions for a^x, $\sin\frac{x}{a}$; $\cos\frac{x}{a}$ already known at that time, more simply with the help of this theorem, Taylor nowhere uses "MacLaurin's series".

Further, in connection with the manuscripts of Marx, who specially observed, that he borrowed the "algebraic expansion" directly from MacLaurin himself, it should be mentioned, that the demonstration of MacLaurin's theorem given in the text-books of Boucharlat and Hind (through the method of indeterminate coefficients), in fact belonged to MacLaurin himself. Such a direct borrowing from the author, whose name the theorem carries, could of course have taken place, also in the case of Taylor's theorem. The bibliographical list, which Marx composed in connection with his preparatory studies for the historical essay, is to all appearance, a testimony to the effect, that Marx intended to get acquainted with Taylor's work in the original ; but this wish of his remained unfulfilled.

2) In accordance with the order in which Marx criticises the demonstration of Taylor's theorem in manuscript 4302, we shall begin with Boucharlat's book (J.-L. Boucharlat, Èlémens

* Euler demonstrated Taylor's theorem as per Taylor as well. See, L. Euler, Differential calculus, ch. III ("On finding out the finite differences"), §§ 44-48, pp. 240-241. — Ed.

de calcul différentiel, 5th ed., Paris, 1838 ; Marx had an English translation of another edition of it).

Having enunciated in § 30 (pp. 19-20) the question of successive differentials —where, incidentally, having obtained $6a$ as the third derivative of ax^3, he observes (p. 20) : "Here we can no more differentiate, as $6a$ is a constant" — Boucharlat goes over to MacLaurin's theorem (§ 31, pp. 20-21). In his book the demonstration of MacLaurin's theorem precedes that of Taylor's theorem (the latter is demonstrated in §§ 55-57, pp. 34-37). As has already been observed, Boucharlat demonstrated MacLaurin's theorem according to MacLaurin himself. However, he did not read latter's work. In fact, in the note to the title "MacLaurin's theorem", Boucharlat wrote : "As Peacock has observed, this theorem was already discovered by Stirling in 1717 and hence, before MacLaurin made use of it", and as has already been mentioned, MacLaurin fully acknowledged, that the theorem was already there in Taylor. In Boucharlat's demonstration, no where are the questions of legitimacy of his assumptions, not to speak of those about the convergence of the serieses under consideration, raised in any way. Here we are giving an almost word for word translation of Boucharlat's demonstration of MacLaurin's theorem.

"Let y be a function of x ; let us expand it in respect of x and suppose that

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{ etc.} ; \tag{16}$$

we shall get, by differentiating and dividing by dx :

$$\frac{dy}{dx} = B + 2Cx + 3Dx^2 + 4Ex^3 + \text{ etc.},$$

$$\frac{d^2y}{dx^2} = 2C + 2\cdot3Dx + 3\cdot4Ex^2 + \text{ etc.},$$

$$\frac{d^3y}{dx^3} = 2\cdot3D + 2\cdot3\cdot4Ex + \text{ etc.}$$

. . . .

We shall designate by (y) that, which turns into y, when $x = 0$,

by $\left(\frac{dy}{dx}\right)$ that, which turns into $\frac{dy}{dx}$, when $x = 0$,

by $\left(\frac{d^2y}{dx^2}\right)$ that, which turns into $\frac{d^2y}{dx^2}$, when $x = 0$;

. . . .

the preceding equations will give us

$$(y) = A, \ \left(\frac{dy}{dx}\right) = B, \ \left(\frac{d^2y}{dx^2}\right) = 2C, \ \left(\frac{d^3y}{dx^3}\right) = 2\cdot3D ;$$

whence we shall extract

$$A = (y), B = \left(\frac{dy}{dx}\right), \ C = \frac{1}{2}\left(\frac{d^2y}{dx^2}\right), \ D = \frac{1}{2\cdot3}\left(\frac{d^3y}{dx^3}\right) ;$$

substituting these values in equation (16), we shall have

$$y = (y) + \left(\frac{dy}{dx}\right) x + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right) x^2 + \frac{1}{2 \cdot 3}\left(\frac{d^3y}{dx^3}\right) x^3 + \cdots; \tag{17}$$

and this is MacLaurin's formula".

In the following §§ 32-34 (pp. 21-23), in accordance with MacLaurin's formula, expansions are found for

$$y = \frac{1}{a+x}, \; y = \sqrt{a^2+bx}, \; y = (a+x)^m.$$

Thus, in the third example, the binomial theorem is deduced from MacLaurin's theorem. In the first appendix of the 5th edition of Boucharlat's book, which we have, entitled "Demonstration of Newton's formula with the help of differential calculus", a direct proof (by the same method of indeterminate coefficients) of Newton's binomial theorem (for integral and positive indices of power) has been given, with the help of successive differentiation. It reads as under.

Boucharlat begins with the expansion of $(1+z)^m$, then from it he obtains the expansion of $(a+x)^m$, necessary for him, by substituting $z = \frac{x}{a}$. He said, let

$$(1+z)^m = A + Bz + Cz^2 + Dz^3 + Ez^4 + \cdots \tag{1}$$

Assuming $z = 0$, he gets $A = 1$ and, hence,

$$(1+z)^m = 1 + Bz + Cz^2 + Dz^3 + Ez^4 + \cdots$$

Differentiating both the sides of this equation in respect of z, he gets further

$$m\,(1+z)^{m-1} = B + 2Cz + 3Dz^2 + 4Ez^3 + \text{ etc.}$$

Referring to the fact, that this equation occurs for any z, Boucharlat assumes that $z = 0$ and thus obtains $m = B$. Differentiating once more and again assuming $z = 0$, he obtains

$$m\,(m-1) = 2C,$$

whence

$$C = \frac{m\,(m-1)}{2},$$

and after that he concludes : "All the remaining coefficients are so defined, and by substituting their values in equation (1), this equation is turned into

$$(1+z)^m = 1 + mz + \frac{m\,(m-1)}{1 \cdot 2} z^2 + \frac{m\,(m-1)\,(m-2)}{1 \cdot 2 \cdot 3} z^3 + \text{ etc." (pp. 491-492).}$$

3) Boucharlat also proved Taylor's theorem by the method of indeterminate coefficients. Therein he not only assumed, that every function of some variable may be expanded into a series, according to the powers of any of these variables, but also thought, that this expansion must be unique, i.e., the coefficients of any two such expansions (in respect of the powers of one and the same variable) must be equal. This will give him the opportunity of applying the method of indeterminate coefficients.

In order to get such an opportunity, i.e., of being able to equate the coefficients of two expansions of one and the same function, Boucharlat begins with a lemma, stating that the

derivatives of $f(x+h)$ in respect of x, and in respect of h, are equal. Since in manuscript 4302 (see, p. 287) Marx expressed dissatisfaction with the demonstration of this lemma in Boucharlat's book, and since pp. 41-42 (see, note [117]) of manuscript 3888 cannot even be understood without an acquaintance with this demonstration, here we reproduce it in full.

§§ 55 (pp. 34-35) is devoted to it, wherein we read :

"If in a function y of x, the variable x changes into $x+h$, then we get one and the same differential, both when x is a variable, and h — a constant, and when h is a variable, and x — a constant.

If to prove this, in the equation $y=fx$ we substitute $x+h=x_1$* for x, we shall have $y_1=fx_1$; the differential of fx_1 will be equal to some other function of x_1, represented by φx_1 and multiplied by dx, hence, $dy_1=\varphi x_1\, dx_1$ or, if we substitute for x_1, its value $x+h$,

$$dy_1=\varphi(x+h)\, d(x+h).$$

But the only change, which the hypothesis, that x is a variable and h is a constant, introduces into this differential, is related only to the factor $d(x+h)$, which is reduced to dx, when x is a variable, and h — a constant; hence, in that case we have

$$dy_1=\varphi(x+h)\, dx,$$

whence

$$\frac{dy_1}{dx}=\varphi(x+h). \tag{35}$$

Conversely, if we make x a constant, and h a variable, then the factor is reduced to dh, and we shall have

$$dy_1=\varphi(x+h)\, dh,$$

that is,

$$\frac{dy_1}{dh}=\varphi(x+h); \tag{36}$$

equating these two values of $\varphi(x+h)$, we shall get

$$\frac{dy_1}{dx}=\frac{dy_1}{dh}\text{ ''}.$$

In the following § 56 Boucharlat extended this lemma to the derivatives of higher orders and, in § 57 demonstrated Taylor's theorem with its help. The words with which he begins this "demonstration", speak of what he thought to be — as Marx calls it — his "initial equation" (37), applicable to any function. He begins : "Let y be a function of $x+h$; let us assume, that when we expand this function according to the powers of h, we shall have

$$y_1=y+Ah+2Bh^2+3Ch^3+\text{ etc.}, \tag{37}$$

where A, B, C, ... are unknown functions of x, which are to be determined".

Differentiating equation (37) in respect of h and in respect of x and having thus obtained

* Though Boucharlat used the Lagrangian notations for the derived functions, he designated the augmented values of x and y (i.e., $(x+h)$, and $f(x+h)$) by x', y'. We have changed these notations into x_1 and y_1. — Ed.

$$\frac{dy_1}{dh} = A + 2Bh + 3Ch^2 + \text{ etc.},$$

$$\frac{dy_1}{dx} = \frac{dy}{dx} + \frac{dA}{dx}h + \frac{dB}{dx}h^2 \text{ etc.},$$

Boucharlat then equated, referring to the lemma, the coefficients of the same powers of h in the last two equations and thus obtained the expressions for the coefficients $A, B, C, \ldots,$ required by him, through y and its successive derivatives. Marx gave an account of this demonstration in manuscript 3888 (sheets 54-55, pp. 50-51 in Marx's numeration), where he compared it with the aforementioned proof of MacLaurin's theorem. In manuscript 4302 he criticised this demonstration, mainly for its unsubstantiated initial assumptions.

The §§ 58-61 (of Bourcharlat's book) contain examples of expansions of $f(x + h)$ according to Taylor's formula, for the instances, where $f(x)$ is $\sqrt{x}$, $\sin x$, $\cos x$, $\log x$. The question of convergence of the serieses obtained, is not even mentioned anywhere. The cases of inapplicability of Taylor's series, have been considered only in the last paragraphs, printed in brevier, of the first part of the book (devoted to the differential calculus).

The final § 62 of the section on Taylor's theorem and its applications, is devoted to a deduction of MacLaurin's theorem from Taylor's theorem. A full account of this deduction has been given by Marx in manuscript 3888 (see, sheets 55-56; pp. 51-52 in Marx's numeration).

NOTES
AND
INDEXES

NOTES

1 This manuscript was completed by Marx in 1881, for Engels. This is the first work in the cycle of manuscripts planned by Marx, devoted to a systematic account of his ideas on the nature and history of differential calculus. In this article he introduced his concept of algebraic differentiation and the corresponding algorithm for finding out the derivative for certain classes of functions. The envelope, attached to this manuscript,carries the heading in Marx's hand : "For the General". That is how Engels was called by the Marx family, for his articles on military affairs.

Upon getting acquainted with this manuscript, Engels wrote to Marx his letter dated the 18th of August 1881 (see, K. Marx and F.Engels,works ,vol.35,pp.16-18) [and, special supplement to the present volume, part one,letters (excerpts), fifth letter — Tr.]. The present text follows the fair copy made by Marx. Some of the materials preparatory to it (drafts and supplements),are being published in p. 248 of the present volume.Variations in the text,along with their unpublished rough drafts have been indicated in the footnotes.A part of this manuscript was at first published in 1933, in Russian translation,in the collection "Marksizm i estestvoznanie" [Marxism and Natural Science], M., Partyzdat, 1933,pp. 5-11,and in the journal "Pod Znamenem Marksizma" [Under the Banner of Marxism],No.1, 1933,p.15ff.

The German (original) has been published for the first time in the 1968 edition. —Tr.

2 In order to avoid confusion over the notations of derivatives, here and everywhere afterwards in analogous situations,Marx's notations x', y', $\cdots$ for the new values of variables, have been substituted by x_1, y_1, $\cdots$

The sources used by Marx,did not as yet have the concept of absolute value.That is why Marx often (apparently,for the sake of determinateness) considered only the increase in the value of the variable,but sometimes(see,for example, PV, 88, 274) he also spoke of an "increase " of x "by the positive or negative increment h ".

3 In keeping with the terminology adopted in the sources used by Marx,by finite difference,is meant, always, a difference other than zero.

4 In every equality Marx distinguishes two sides (now called : two parts) — the left and the right, which do not always play a symmetrical role in his writings.

On the left hand side of an equality he often places two different,but synonymous expressions, joining them by the connective "or".

5 In the mathematical literature at Marx's disposal,the term "limit" (of a function)did not have an univocal meaning.Most often it was understood as the value of a function,actually attained by it at the end of an infinite process of approximation of the argument to its limiting value (see,Appendix,pp. 303-305). Marx's manuscript devoted to a criticism of these shortcomings, entitled *On the non-univocality of the*

terms "limit" and "limiting value", has come down to us only in the form of a rough draft (see,pp. 96-98).

In the present manuscript, Marx has used the term "limit" in a special sense: as an expression,redefining those values of the argument, in which it is not defined. For Marx, such expressions,in need of redefinition,were the ratios $\frac{\Delta y}{\Delta x}$ (when $\Delta x = 0$, turning into $\frac{0}{0}$) and $\frac{dy}{dx}$, interpreted as the symbolic expression for the ratio of "removed,or extinct,differences", i.e., for $\frac{0}{0}$. In application to the ratio $\frac{\Delta y}{\Delta x}$, Marx understood "limit" — in some connection with the definitions of this concept in Hind and Lacroix (see,Appendix,pp. 303-305) — as an expression identically equal to this ratio,when $\Delta x \neq 0$, but redefining it at continuity,when it turns into $\frac{0}{0}$. Hence ,here the "limit" had to be the "preliminary derivative" (on this see, p. 20-21 and note [7]).In accordance with this Marx writes (on p.21) (regarding the ratio $\frac{\Delta y}{\Delta x}$, where $y = ax^3 + bx^2 + cx - e$) :

"The preliminary "derivative " $a(x_1^2 + x_1 x + x^2) + b(x_1 + x) + c$ is here the limit of the ratio of finite differences,i.e., however small we may take these differences to be,the value of $\frac{\Delta y}{\Delta x}$ will be given by this "derivative" ". Lower down (see,p. 21-22) Marx speaks about the fact, that the assumption $x_1 = x$, i.e., $\Delta x = 0$,"takes this limit to its minimal value ",which gives the "final derivative".

Analogously, by the "limit of a ratio of differentials", in this manuscript Marx understands the "real"("algebraic",see:note [6]) expression,giving the value of this ratio,in other words,the derived function. However, elsewhere Marx writes,that in the equation $\frac{dy}{dx} = f'(x)$ " neither of the two sides,is the limiting value of the other.They are situated, not in a limiting relation to each other,but in an equivalence relation "(see, p. 98).But here the concept of "limit"("limiting value") has been used in another sense:it is closer to the concept,with which we are now accustomed.

In a sense,even more close to the modern concept of limit,Marx uses the term "absolutely minimal expression "(see, p. 97); about which elsewhere (see, pp. 61-62) he wrote,that the category of limit,in the sense which Lacroix imparted to it and in which it has an important significance for mathematical analysis,is a substitute for it (for Lacroix's definition see, Appendix, pp. 309-3012).

6 By "algebraic" Marx understood any expression, not containing the symbols of the derivatives and the differentials.Such an use of the term "algebraic expression", was characteristic of the mathematical literature of early 19th century.

Marx often made a distinction between the concepts: "function of x " and "function in x ", i.e., between a function as a correspondence and a function as an analytical expression (see,p. 269). In the present manuscript he did not strictly adhere to this distinction, and often spoke simply of the "function x ", perhaps, because he always had in view only the functions, given by some "algebraic expression". He indicated the correspondence, the value of the dependent variable y in respect of the value of the independent variable x, with the help of the equation $y = f(x)$,where y is the dependent variable, and $f(x)$ is the analytical expression , considered in respect of the appearence of the variable x in it.

7 The essence of Marx's method of algebraic differentiation consists of this : he redefines the ratio

$$\frac{f(x_1)-f(x)}{x_1-x} \tag{1}$$

of the finite differences (having a meaning only when $x_1 \neq x$) at continuity for $x_1 = x$. With this aim in view he seeks the function $\varphi(x_1, x)$, Which coincides with the ratio (1), when $x_1 \neq x$, and, which is continuous when $x_1 \to x$.

Such a function $\varphi(x_1, x)$ Marx calls : *the preliminary derived function of the function* $f(x)$, and he calls the function $\varphi(x, x)$,obtained from $\varphi(x_1, x)$ by assuming $x_1 = x$: *the derivative of the function* $f(x)$. If the latter exists (which is the case for the class of functions here considered), then it coincides with the modern concept of the derivative :

$$\lim_{x_1 \to x} \frac{f(x_1)-f(x)}{x_1-x} = f'(x) .$$

At the same time Marx was aware of those functions for which the operation of differentiation was not defined (see, PV, 93-94).

8 Here Marx reproduced a formal expansion of a function into a series,keeping aside the questions of convergence of the series obtained and of coincidence of the values of the function with the limits of partial summations.This was characteristic of the mathematical books at his disposal.

9 $\therefore$ — a symbol,that has been substituted by the word "hence",in written proofs.Now this symbol is not in use.

10 The text of the entry entitled: "Additionally",consists of the contents of a separate sheet appended to the manuscript.It has its own independent page numbers: 1 and (on the other side) 2.

11 To all appearence, by finite difference equation,Marx implies an expression of the form $f(x_1) - f(x) = (x_1 - x)\ \varphi(x_1, x)$. See , note [7].

12 Here S.Moore wrote in pencil : "That is not the case, these factors are $x_1 - x - 1$, $x_1 - x - 2$ etc.". Apparently,here Marx did not mean the factor $(x_1 - x)$, but the expression $(x_1 - x)$ and wanted to say, that the turning into zero of the difference $x_1 - x$, retained in the expression of the preliminary derivative,does not rob the latter of its meaning.

13 This manuscript also belongs to 1881.The envelope attached to this manuscript carries the heading : "II . For Fred ". Marx called it the "second instalment " (see,p. 39). In it he continued to give an account of his conclusions, at which he arrived in the process of his mathematical studies. Engles showed this manuscript to S. Moore and sent latter's comments to Marx with his own letter dated the 21st of November 1882 (see,Works, vol.35, pp. 92-93 ; and special supplement to the present volume, part one, letters (excerpts), sixth letter — Tr.).

This manuscript "On the Differential " was published for the first time, in part, in Russian,in 1933: in the collection "Marxism and Natural Science ", pp.16-25, and in the journal "Under the Banner of Marxism", 1933,No. 1.

14 Thus, here Marx assumes, that the functions u and z are given by, as it will be clear from what follows, the equations $u = f(x)$ and $z = \varphi(x)$, where $f(x)$ and $\varphi(x)$ are expressions " in the variable x ". They are differentiable functions of x. This situation, where for the justification of the theorem about the differential of the product of two no other information is required about the form of the functions $f(x)$ and $\varphi(x)$ apart from this,finds its reflection in Marx's picturesque expressions about $\frac{du}{dx}$, $\frac{dz}{dx}$ as "shadows without the bodies which have cast them, symbolic differential coefficients without real differential coefficients, i.e., without the corresponding equivalent "derivatives"" (see,p. 30). Marx also specially stipulated them in the drafts of his work on the differential .Here,as well as everywhere later on,Marx's brief notation duz, has been substituted by the notation $d(uz)$.

15 That is from the symbolic expressions specific to the differential calculus : the signs of derivatives and dif rentials.

16 In the literature of the 18th-19th centuries, the derivative was often called the "differential coefficient ", having,evidently, in view the definition of the derivative as the coefficient of first power of the increment h of the independent variable x in the expansion of $f(x + h)$ into a series according to the powers of h.

The adjective "real" is associated with the fact,that in the expression for $f'(x)$ there are no symbols specific to the differential calculus.

17 The mode of expression,according to which,as a result of multiplication by zero "the variables z and u themselves become equal to zero", is explained by the fact,that even during Marx's time, the representation of mathematical operations over numbers

as something which changes those very numbers, was wide spread : addition of the positive number b to a "increases the number a", multiplication of a by 0 "turns the number a into zero" etc.Only in the 20th century,these representations were subjected to [greater] scientific specification.

18 Apparently the words "as we can arbitrarily start the nullification either from the numerator or from the denominator ",signify that the redefinition of an expression of the form $\frac{f(x)}{g(x)}$, which at $x = 0$ turns into $\frac{0}{0}$ and that is why loses meaning,may be carried out differently for $x = 0$. If,while redefining, we wish to retain that property of the ordinary fraction according to which,when the numerator is equal to zero,it is itself equal to zero,then the value of $\frac{f(a)}{g(a)}$ must be 0. In that case "to start nullification from the denominator " also means: to posit $\frac{f(a)}{g(a)}$ as equal to zero.In so for as a fraction with 0 as its denominator does not exist, "to start nullification from the denominator" can no more mean : to retain,while redefining,some property of the ordinary fraction with 0 as its denominator. But if,when $x \neq a$, $\frac{f(x)}{g(x)} = \varphi(x)$ and $\varphi(x)$ is continuous at the point a (i.e., $\lim\limits_{x \to a} \varphi(x) = \varphi(a)$), then it is natural to posit $\frac{f(a)}{g(a)}$ as equal to $\varphi(a)$, having thus retained,also for $x = a$, the equality $\frac{f(x)}{g(x)} = \varphi(x)$. If due to this the numerator turns into zero, as a result of the fact, that the denominator was made equal to zero,then the words "to start nullification from the denominator " may naturally be interpreted as signifying : to redefine in the aforementioned manner, i.e., "by the continuous approach ". In the books used by Marx ,even in Lacroix's big "Treatise", retention of the equality $\frac{f(a)}{g(a)} = \varphi(a)$ for $f(a) = g(a) = 0$ was considered, in general, to be independent of any "arbitrariness" whatsoever : it was an inevitable consequence of the metaphysical law of continuity "of all that is real".

19 Here is a slip of pen in the text : instead of $x = a$, here he wrote $x^2 = a^2$. Apparently, instead of correcting it, Moore made the following insertion in pencil : " and since $x^2 = a^2$, $x = \pm a$, [that is why $P(x + a)$] $= 2Pa$ or 0 ". However , such an interpretation does not fit into the entire context.

20 Here Marx calls the expression $\frac{dy}{dx}$, obtained in the transition from the finite differences to the derivative, *the symbolic differential expression* for $\frac{y_1 - y}{x_1 - x}$ $\left(\text{corresponding to } \frac{f(x_1) - f(x)}{x_1 - x}\right)$.

21 To all appearence,what is being discussed here is the case,when the choice of the independent variable is not so fixed , that any one of the variables u and z may be taken as the independent variable . In general,if u and z may be viewed as functions formed from one and the same independent variable, then the choice of the value of one from u , z determines the value of the independent variable, and that is, also the value of the other.In other words,here we have in view the invariance of the symbolic operational equation obtained,in respect of the choice of the independent variable .

22 Apparently the words "to you", in the phrase "known to you" (retained in the draft) were dropped while copying.It may be assumed,that here the French mathematician L.B. Francoeur is being referred to. Engels wrote to Marx about him in his letter dated the 30th of May 1864 (see,Works,Eng. ed.vol.41,p.532).

The word "elegant" within quotation marks is related to Engels' comment : "some one very elegant " — and expresses Engels' ironical attitude to the person being referred to. Like Boucharlat, Francoeur too — but somewhat differently — tried to connect Lagrange's "algebraic" method (see,p. 33) with Leibnitz's differential calculus,which operated with the symbols for the differential.Marx's ironical "clearly" refers to the way this was done,by Boucharlat, as well as by Francoeur.The first one did it to "facilitate algebraic operations" (see,the next paragraph of the text),by deliberately introducing a wrong formula; the second person asserted, that the differential " is a synonym for the derivative and differs from it only in notation", and accordingly wrote there itself,that "the derivative of x is $x' = 1$, or $dx = 1$" (see, L.B.Francoeur, vol II, p.253).

23 The text within quotation marks is a translation from the 5th edition of 1838,of J.-L.Boucharlat's book, p.4.

24 Evidently by the expression "reduced to its absolute minimum ", what is meant here is: the redefinition of the said ratio by the continuous approach,when $x_1 = x$, i.e., in essence,the transition to limit when $x_1 \rightarrow x$.

25 See, Appendix, " On Leonhard Euler's Calculus of Zeros".

26 Here Marx is making a distinction between the differential particles dx and dy appearing as the "sublated " differences Δx and Δy, and the differential dy, defined *by* the equality

$$dy = f'(x)dx . \tag{1}$$

The latter may be treated as an operational symbol permitting the derivative $f'(x)$ to be found out,in accordance with the differentials dy and dx already found,in transition to the equality

$$\frac{dy}{dx} = f'(x) , \tag{2}$$

equivalent to (1) (see,note [24]).

27 Marx's argument against treating the inversion of method,as originating already in the "algebraic" differentiation of the simplest functions of first power,consists of the following : 1) the step of assuming $x_1 = x$ is unnecessary,as here the preliminary derivative already coincides with the final one, i.e., the specificity of the "algebraic" method of differentiation is not revealed ; 2) an extension of the observations regarding the differentiation of the functions of the first power on to the general instance could lead to the clearly false conclusion, that all the derivatives of higher orders, starting with the second, must be equal to zero.

28 That is, if we consider $\frac{dy}{dx}$ as a ratio of infinitesimals,as it was done by Leibnitz and Newton.

29 That is,if the derivative of y in respect of x is to be found,considering y as a function of x, jointly given by the equations :

$$1)\ y = 3u^2, \qquad 2)\ u = x^3 + ax^2.$$

30 Here Marx assumes, that the right to operate with the differentials,in the manner of operating with the ordinary fractions, has already been established (see,pp.32-33, and also Appendix,pp. 327-328).

31 Here, in the manuscript Moore made the following remark in pencil : "On p. 12(5) this is proved for the concrete case there investigated. Should it not have been proved also for the general case , and should not its validity be assumed ?" This remark is based on a misunderstanding. The "developments observed in concrete functions" consisted of the fact, that as a result of their differentiation the symbolic expressions of the form $\frac{dy}{du}$ and $\frac{du}{dx}$, were obtained.Since Marx had already assumed the right to operate with such expressions,in the manner of operating with ordinary fractions,it was natural [for him] to observe in conclusion,that $\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}$.

32 Marx did not write this section III, apparently, because he could not actualise his intention of looking up J.Landen's book in the British Museum (see,Appendix,p.320)

33 Under this title ,three drafts of the different sections of the work "On the Differential" and a few drafts of the additions to it,have been joined together. For greater details see, PV, 241, 244, 251, 252.

34 This draft has been taken from the notebooks,to which Marx gave the titles "A.I" and "B (continuation of A).II"(see,PV,241,244). It begins on the last page of the note-book "A.I."(not numbered by Marx) and is spread over different places in the

note-book "B"(which Marx indicated by special marks). A part of this draft was published for the first time in 1933, in Russian (see,"Pod Znamenem Marksizma", No. 1,as well as "Marksizm i estestvoznanie", pp.34-43).

35 Marx everywhere calls an expression containing the symbols dx, dy etc.,which are specific to the differential calculus — symbolic (as distinct from the algebraic, see note [6]). He calls an expression of the same function, not containing the said symbols — real.

36 Here those symbolic expressions are called operational formulas of the differential calculus, which indicate (see, in the text below): which operations are to be carried out upon definite functions, for obtaining the real value of this or that derivative.

37 Here note-book "A.I" comes to an end. At the end of this page,it is written in Marx's hand: "see,further note-book II,p.9".It refers to the note-book "B (continuation of A)".

38 On the character of this sort of redefinition by the continuous approach and about the possibility of other redefinitions, satisfying this or that demand, see note [18], as well as the Appendix, pp. 304-305 .

39 That is, when we go over from the region of ordinary algebra to that of the functions (dependent variables), for which the ratio

$$\frac{f(x_1)-f(x)}{x_1-x},$$

turning into the indeterminate form $\frac{0}{0}$ when $x_1 = x$, is to be redefined.

40 Usually Marx calls the expressions, not containing the symbols specific to the differential calculus — "algebraic" (see,note [6]) or "real " (see,note [2]).

41 Manuscripts of the second and third drafts are available only in rough copies.They contain many cancellations and insertions. The first four pages of the second draft are missing (that is why we begin it with a dotted line).

With certain cuts,these two drafts were published for the first time in 1933,in Russian ("Pod Znamenem Marksizma", No. 1, and also "Marksizm i estestvoznanie",pp.26-34). See, "Preliminary drafts and variants of the manuscript on the differential", point a), p. 250.

42 This entire paragraph (starting from the words "if the variables grow") is Marx's translation,into German, of the corresponding place of Hind's book (see, T.Hind, 2nd ed.,Cambridge, 1831, p.108). Here the second draft comes to an end.

After this paragraph Marx left about half- a page blank ; apparently, not finding the essential informations in Hind,he set aside his intended investigation, evidently, having decided to return to it later on.

The materials about the differentiation of product by Leibnitz's and Lagrange's methods were available in the text-books by Hind and Boucharlat (see, Appendix, pp.329-330). However, these text-books did not throw any light upon Newton's method.

43 This quotation is from Boucharlat's book (see, the 7th ed. of his book, Paris, 1858,pp. 3-4).

44 Here Marx resorted to a somewhat different numeration of the sections of his work.It differs from what he did earlier.

He wanted to place in section III, the material that was placed in section II of the second draft.In section IV, he wanted to throw light upon the historical course of the development of differential calculus with the help of the example of the history of the theorem of differential of a product.

45 In connection with this paragraph see: note [5]; the Appendix "On the concept of "limit" in the sources consulted by Marx", pp. 309-310 (where informations have been provided to the effect,that in Boncharlat's book both the sides of the equality $\frac{dy}{dx} = f'(x)$ were treated as limits); pp. 310-312 (where the concept of limit has been discussed according to the big "Treatise" of Lacroix and Marx's words in the present paragraph about this concept have been referred to). It is still not clear,what Marx had in view,when he treated the symbolic expression as the limit of $f'(x)$.(Perhaps he had only that situation in view,where the derivative is obtained by assuming $x_1 = x$, i.e., when the numerator and the denominator of the ratio $\frac{\Delta y}{\Delta x}$ attained their limiting values "zero", and that is why the expression $f'(x)$ had to correspond not to $\frac{\Delta y}{\Delta x}$, but to $\frac{dy}{dx}$.) About Marx's reference to Lagrange's relation to Newton's understanding of the concept of limit , see there, pp. 311-312.

46 Marx intended to make some additions to the manuscript "On the Differential". Four drafts of these have been preserved (for the details see,PV, 252-259, where a number of fragments from these drafts,have been reproduced).These draft are incomplete in nature; and here we reproduce only two relatively unbroken (and comprehensible) fragments out of these.

47 Marx gave this heading to section A) of the second draft of his supplements to the manuscript " On the Differential ". Here only its point 1) is being published. It contains a brief resume of his principal work on the differential. The important addition

here is the reference to the geometric application of the operational formulae. For the details see, PV, 252.

48 This is § A) of the third draft . The title belongs to Marx.Here only its point 3) is being published. In it Marx gives an account of an example of the application of the theorem about the differential of a product, as an operational formula for seeking the differential of a quotient (in his characteristic literary manner).

49 Marx finished his manuscript "On the Differential" with a promise to write a special section, devoted to the historical course of development of the differential calculus. In the drafts, preceding this letter, he expressed an intention to throw some light upon the history of differential calculus through the example of the history of the theorem on the differential of product. To all appearence, none of these intentions was actualised in full. Only some rough drafts, contained in the note-book "B (continuation of A)" have been preserved. There they alternate with the calculations carried out by Marx in connection with his work on the differential.

In accordance with Marx's initial intensions, these drafts begin with a discussion of the methods of Newton and Leibnitz, in the light of the example of the theorem on the differential of a product. Then follows only the beginning of the unfinished section devoted to d'Alembert's method.Later on Marx went over to a more detailed discussion and criticism of the methods of Newton and Leibnitz, in general. This leads him to a general periodisation of the history of differential calculus.

It is divided into there periods : 1) the mystical differential calculus of Newton and Leibnitz, 2) the rational differential calculus of d'Alembert and, 3) the purely algebraic differential calculus of Lagrange.The second part of the preserved drafts of the essay on the history of differential calculus, consists of a discussion of the characteristics of this third period. Apparently, Marx wanted to develop this part as the third letter to Engels. The concluding part of these drafts of historical character, is devoted to a more detailed account of the general ideas contained in the first part. Save some omitted paragraphs, these are contentwise related to the work "On the Differential" ; these drafts are being published here in full.

50 In a number of cases this bibliographical list, contains references to those places of the quoted source, in which the basic concepts and methods have been discussed. The text-books at Marx's disposal do not contain these references. That is why it may be assumed, that these places were picked out by Marx, after he went through them in the corresponding works (evidently, in the British Museum). The fact that Marx put the name of John Landen within a box,indicates that he specially wanted to get acquainted with J.Landen's "The Residual Analysis". For the details see, Appendix, pp. 320-325. It is not known, from where Marx collected the years of birth and death, mentioned in the list .Only this much is clear,that this source was silent about Lagrange's year of death.

51 In the scholia to lemma XI of the first book of "Principia Mathematica" and in lemma II of its second book, Newton explained the basic concepts of differential calculus, corresponding to our concepts of the "derivative" and the "differential".

For the details of these lemmas of Newton, see : Appendix, pp. 312-314.

52 For Marx's conspectus of this work(along with his critical remarks) see : pp.126-131 of the present volume.

53 d'Alembert's "Traité de fluides" does not contain any material on the basic concepts of differential calculus. d'Alembert's ideas on the basic concepts of differential calculus are to be found in his essays in the Encyclopaedia and in the "Opuscules mathématiques". It is not clear as to what namely drew Marx's attention to d'Alembert's "Traité de fluides".

54 The third chapter of part I of L.Euler's "Differential Calculus"is devoted to the question "of infinities and infinitesimals". For the details see, Appendix, pp. 315-318.

55 Abbe Moigno composed this book,"according to the methods and works of Cauchy, published and unpublished ". The first volume of Moigno's "lectures" were published in 1840 ,the second volume — in 1844.

56 This conclusion (according to Newton) requires explanation : "Since the numerical magnitudes of all possible quantities may be represented by straight lines", so that the change of every magnitude may be represented in the form of a rectilinear movement with variable speed. And since in course of an infinitesimal interval of time, the speed of movement may be considered to be unchanging , so the path traversed by a point,corresponding to this interval of time (that is,the corresponding change in our magnitude), is equal to the derivative of this speed (fluxion) during the infinitesimal interval of time τ. That is why the "moments, or infinitely small parts of the generated magnitudes = the products of their speeds and of the infinitely small parts of time". On the metaphysical character of Newton's attempts to substantiate the concepts of "fluent", "fluxion" and "moment", corresponding to our concepts of "function","derivative" and "differential", which were defined in the terminology of mechanics, see : Appendix, pp. 313-315.

57 In note [49] it has been pointed out that Marx still wished to return to the history of development of the differential calculus, in the light of the history of the theorem on the differential of product. That is why he left a blank space after the incomplete extract from Hind's book. Here, after repeating once more the same extract, the theorem on the differential of product — treated according to Newton — is enunciated as an example. (In Hind's book, this theorem constitutes example 3, on p. 109).

58 In Hind's book the theorem on the differential of product has not been used to illustrate the method of Leibnitz. That is why Marx turned to Boucharlat's book. The present paragraph is an extract from his book (see, Boucharlat, p. 165).

59 This sentence is from the text-book by Hind quoted above (Hind, p. 106). However, Marx did not further enunciate Hind's account of the theorem on the differential of product. After this, in Marx's note-book there are 5 pages (pp. 16-20), which we have dropped. These are in the main full of calculations, related to the theorems about the differentiation of the quotient and of the composite function,and also to the solution of the problem of the tangent to a curve, taking the parabola $y^2 = ax$, as an example. We have reproduced only the comments, written on the blank spaces on pp. 16-18, in which Marx stressed the fact, that Newton and Leibnitz began directly with the operational formulae of differential calculus.

Then under the rubric "Ad Newton" Marx criticised these methods of Newton and Leibnitz, stressing that such methods,in spite of a number of their advantages, inevitably entail the introduction of actual infinitesimals and the difficulties connected with them. Here again,the theorem about the differential of product has been used as the main example.

60 By $\dot{x}$, $\dot{y}$, $\dot{z}$ Newton and his successors usually designated the rate of change (fluxion) of the variables x, y, z (fluent), i.e., the derivatives of x, y, z in respect of the variable,playing the role of "time"; by $\tau\dot{x}$, $\tau\dot{y}$, $\tau\dot{z}$ they indicated the "moments", corresponding to the Leibnitzian differentials or infinitesimal increments. However,the notations $\dot{x}$, $\dot{y}$, $\dot{z}$ were often used by the Newtonians also for the "moments" or differentials (see, Appendix, pp. 318-319).

61 Here the following heuristic generalisation is being discussed : in the formula

$$\dot{y} = a\dot{x} \qquad (1)$$

y is treated only as a function $f(x)$, and the constant a — as the derivative of this $f(x)$, as a new function $f'(x)$; herein formula (1) becomes a particular case of the more general formula

$$\dot{y} = f'(x)\,\dot{x}. \qquad (2)$$

However, while treating $\dot{x}$, $\dot{y}$ as increments,though infinitesimal, the factor $f'(x)$, is a function not only in x, but also in $\dot{x}$; in formula (2) the "derived" function $f'(x)$, independent of $\dot{x}$, does not occur. It is this situation (which forced the Newtonians to forcibly remove the terms containing $\dot{x}$, though the latter had to be other than zero, so that formula (2) became comprehensible), which lies at the root of Marx's criticism—a few lines below— of the Newtonian definition of the derivative of the function $y = f(x)$ as the ratio $\frac{\dot{y}}{\dot{x}}$.

62 That is, obtained in the form of a "real" expression, not containing differential symbols.

63 Here a few lines have been omitted. Their meaning is not at all clear.

64 If $\dot{y} = \dot{x}$ and y is x itself, then, to approach the equality, in which there is a side not containing the differential symbol $\dot{x}$, it is enough to divide both the parts of the equality by $\dot{x}$.

65 Evidently here "the accretion in x" signifies a new function in x, obtained from the initial function x^2 — so to say, in addition to it — with the help of the binomial theorem: as the coefficient of dx in the expansion of $(x + dx)^2$.

66 Evidently what is being referred to here is the fact, that not $dy = 2x\,dx$, but $dy = 2x\,dx + dx^2$ is obtained directly with the help of the binomial theorem. But only as the consequence of a false premise, does the latter equality appear to be mathematically correct.

67 Meaning of the expression "twofold result" remains unclear. Then follows point a), but there is no point b). Perhaps, here the "twofold result" consists, firstly of the fact, that in the left hand side the difference $\frac{\Delta y}{\Delta x}$ turns into $\frac{dy}{dx}$ (and is not identified from the very beginning with $\frac{dy}{dx}$), and, secondly, of the fact that, in the right hand side, now the terms $3xh + h^2$ are removed with the help of a correct mathematical operation,and not by sleight of hand .

68 The expression within quotation marks is from Hind, § 99, pp.128-129.

69 Evidently here the following fact is being referred to : Taylor's theorem was published in his "Methodus incrementorum", in 1715, i.e., during the life time of Newton,in whose works there is no mention of this theorem. See also : Appendix, p. 332 .

70 For materials about the theorems of MacLaurin and Taylor see :PV, 88,93 214,231,261,264

71 For Marx's account and critique of the basic ideas of Lagrange's theory of analytical functions, see : PV, 90-92 .

72 The reference here is to those separate rough drafts, part of which has been published in the present volume under the title " The First Drafts" (see, PV,67-76).

73 In the manuscript devoted to the history of differential calculus,there are two places, arranged almost immediately one after the other, in which Marx proposed to introduce : 1) the investigations on the theorems of Taylor and MacLaurin, 2) a discussion of Lagrange's theory of analytical functions (see, p. 83). These intentions of Marx remained unrealised, though on these themes he had a vast amount of material at his disposal. He collected them from the sources at his command, even before he arrived at his own point of view on the nature of differential calculus, which he enunciated in his works sent to Engels. These materials are in the main in the nature of a conspectus, however, they do contain resumes or Marx's critical comments. From among these the most significant comments are contained in the

manuscripts: 1) "Taylor's theorem, MacLaurin's theorem and the Lagrangian theory of derived functions" (for the details of which, see : PV, 231) and 2) "Taylor's theorem" (this remains incomplete). Here the corresponding extracts from these are being reproduced (as some kind of a realisation of Marx's aforementioned intentions). For the extracts from the other notes on the same theme, see: PV,132, 214.

74 In the manuals of differential calculus at Marx's disposal,the derivatives of all the elementary functions, save the trigonometric ones, were in fact deduced with the help of the binomial theorem. On this Marx himself wrote in the manuscript "Theorems of Taylor and MacLaurin, first systematisation of the material "(see, PV, 219-220). Later on Marx formulated another method of differentiating this class of functions, and called it "algebraic" (see, the manuscript "On the Concept of the Derived Function"). Thus it is clear, that chronologically this manuscript is prior to the manuscript "On the Concept of the Derived Function" and "On the Differential".

75 Thus in Hind's book (pp. 84-85) after an example,containing the deduction of the binomial theorem with the help of the expansion of $(x+h)^m$ into Taylor's series, theorems of Taylor and MacLaurin have been deduced from the binomial theorem.

76 Here (see also : PV, 274) Marx directly says, that by an "increase" in the value of the variable x, he means any change of this value : a positive, as well as a negative increment h.

77 Since, according to Marx, a function in x is an expression, so it is a combination of signs,which is examined in respect of the entry of the variable x into it.

In the given instance,the terms of MacLaurin's series,i.e., the products ("combinations") of the following two expressions are being referred to : 1) x^k $(k=0,1,2,3 \ldots.)$ and, 2) the corresponding "constant function" $\dfrac{f^{(k)}(0)}{K!}$.

78 Marx calls an expression a " constant function ", if it does not contain entries of the variable x. (y), $\left(\dfrac{dy}{dx}\right)$, $\left(\dfrac{d^2y}{dx^2}\right)$ etc.are expressions of the function $f(x)$ and of its successive derivatives,in which all the entries of the variable x have been substituted by a constant — zero. The result of such a substitution in y, correspondingly in the derivative $\dfrac{d^ky}{dx^k}$, is indicated in the manuscript by (y), correspondingly by $\left(\dfrac{d^ky}{dx^k}\right)$.

Marx adopted this notation from Boucharlat (see : Boucharlat, p.40). It has been retained in the present volume.

79 Here Marx did not explain what he meant by "the irrational nature of the constant function". Apparently, the issue here is the reason behind the emergence of

"exceptions" in both the cases: it is the presence in the expansions,in both the cases,of terms,not having any reasonable mathematical sense.

In the first case: directly bereft of such sense (for example,a "fraction" of the form $\frac{0}{0}$), in the second: for the definite values of the variable x (for example , $\frac{c}{x-a}$, for $x = a$).

Here "irrationality", does not indicate algebraic, "ir-ratio- nal-ity". This word has been used here,as opposed to the word "rational "(compare: "the rational differential calculus of Euler and d'Alembert" as opposed to "the mystical differential calculus of Newton and Leibnitz"). At the very end of his manuscript "Theorems of Taylor and MacLaurin,first systematisation of the material", Marx gave the general characteristics of the instances of their inapplicability,in brief(see: PV, 230).

80 Here the expression : "presented as a finite equation",evidently means presented in the following form :

$f(x+h) = P_0 + P_1 h + P_2 h^2 + \cdots + P_n h^n$, were n is a positive and integral number, $p_i (i = 0, 1, 2, \cdots, n)$ — are functions in x.

81 To understand Marx's critique of the proof of Taylor's theorem,which follows,we must take into consideration a more detailed account of this proof,according to the sources used by Marx. For such an account see: Appendix, pp. 336-339.

82 This fragment from Marx's manuscript "Taylor's theorem" has been reproduced here,as it contains,in the most concentrated form,Marx's points of view: on the shortcomings of the proof of Taylor's theorem,as it was known to Marx ; on the "algebraic"roots of this theorem in the binomial theorem; and on its essential difference from the latter. (For greater details,see: PV, 264). As it is difficult to read and understand the first paragraph in this fragmentary form,we mention that here Marx sums up his critique of Hind's account of the proof of Taylor's theorem.In it (see: Hind, § 74, pp.83-44,§§ 77-80, pp.92-96):

1) Taylor's theorem is proved by presupposing that the expression $f(x+h)$ may be expanded into a series of the form $f(x) = Ph^{\alpha} + Qh^{\beta} + Rh^{\gamma} + \cdots$, where $P, Q, R, \cdots$ are functions of the variable x, and the indices $\alpha, \beta, \gamma, \cdots$ are ascending,integral and positive numbers ;

2) the "exceptions" to Taylor's theorem are considered to be conditioned by the fact,that for some particular values of the variable x, these conditions are not fulfilled (some of the coefficients $P, Q, R, \cdots$ are not defined : and they "do not have finite values" at these points);

3) following Lagrange, an attempt has been made to show, that, generally speaking,i.e., excluding the separate particular values of the variable x, the presuppositions under which Taylor's theorem is proved, hold good for every function $f(x)$ (i.e., the indices

α, β, γ, $\cdots$ cannot have negative or fractional values,the functions P, Q, R, $\cdots$ do not turn "into infinity").

The comments of Marx,which follow,are devoted to a criticism of this sort of attempt, they show its unprovability.

83 The words "for example, $x = a$", are related to the following example considered by Hind: $f(x+h)$ is to be expanded into Taylor's series, where $f(x) = x^2 + \sqrt{x-a}$. When $x = a$, $f(x+h)$ has the sensible value $(a+h)^2 + \sqrt{h}$, while its representation in Taylor's series gives,according to Hind, only " $a^2 + 2ah + h^2 + 0 + \infty - \infty + \infty - \infty$ etc., which does not determine anything " (see: Hind,p.93.).

84 What is meant here is the function $y = x^m$, where m is an integral and positive number.

85 A literal translation of this clause would be : " which can not give a result along the path of differentiation".

86 Literally : "in the possible historical part of this manuscript".

87 In the manuscript "On the History of Differential Calculus" Marx observed,that from a simple difference in the form of representing the change of values of a function,flows the fundamental differences in the treatment of differential calculus(see:pp. 83-84). Here he refers to some "separate sheets", wherin "while analysing the method of d'Alembert" he has disclosed this idea (ibid). There are two groups of such sheets : one group is marked with capital Latin letters from A to H (see, PV, 248), the other — with small Latin letters from a to n (see, PV, 246).

As d'Alembert defined the derivative through the concept of limit, Marx naturally began the analysis of his method with a critique of the concept of limit. From the information provided in the Appendix (see : "On the concept of "limit" in the sources consulted by Marx", p.303), it will be quite clear that this concept is unsatisfactory.This part of this manuscript consists of the sheets A-D (in accordance with its content,it is being published here under the title : On the non-univocality of the terms "limit" and "limiting value").The sheets E-H also have a direct relation with the above mentioned place of the manuscript on the history of differential calculus. These are being published here under the title: " Comparison of d'Alembert's method with the algebraic method ". Sheets a–g of the other group,are in essence devoted to the same question.They are being published here under the title: "Analysis of d'Alembert's method in the light of yet another example ". For the contents of the remaining pages of this group (h to n) see : PV, 246-247.

In consonance with Marx's reference to these attached separate sheets,as being devoted to an analysis of d'Alembert's method,here these have been joined together under the general title : *Appendix to the manuscript "On the History of Differential Calculus". Analysis of d'Alembert 's Method.*

88 In other words,here the proposal is to examine the expression $3x^2 + 3xh + h^2$ for the non-negative values of x and h, assuming that, h indefinitely tends to zero (remaining

different from zero). Let us recall, that the sources used by Marx did not contain the concept of absolute value ; it was not needed while considering the sum of non-negative summands.

89 Here Marx proceeds to substantiate his conclusion, that "perhaps the concept of limiting value has been incorrectly interpreted, and is constantly so interpreted " (see, p.98), and consequently it is appropriate to substitute it by some other new term,understood univocally.For this purpose he proposed the term "absolutely minimal expression", which means the limit, in a sense which is now common (see,p. 97 and Appendix, pp. 303-304)

Marx's criticism of the definitions of "limiting value" and of the methods of using this concept in the books of Hind and Boucharlat,is related first of all to the fact,that in these books "limit" is interpreted actually,i.e., it is viewed as the "last" value of the function for the "last" value of the argument ; he said " it is infantile ; its emergence should be sought in the first mystical and mystificatory method of calculus" (see, p. 98). In the present paragraph,evidently, "limiting value" is understood according to Hind's definition of it (see, Appendix, pp. 303-304) .In practice he treated it as coinciding with the one sided limits of the function,for the argument tending to some number to the right or to the left,in the given case the function $3x^2 + 3xh + h^2$ with the one sided limit to the right,is considered as a function of h, for $h \to +0$. However, as distinct from Hind, Marx stressed that this "limiting value" has a meaning, only if it is not understood actually: computed under the assumption, that $h \neq 0$ (here $h > 0$), i.e., he treated it the way we now do. At the same time, in application to the function $3x^2 + 3xh + h^2$ considered, hereby the demand contained in the definition of "limits"(as the exact upper and lower bounds of the values of the variable), with which Hind's book begins,is not transgressed.In fact, as Marx observes,firstly,this function constantly tends to its limit (here,of course, the lower), as h tends to zero and,secondly, never having attained it, what is more, hence, never having crossed it, i.e., it wittingly fulfils both the demands of Hind's definition (Hind himself usually did not verify the observance of these demands; see: Appendix,pp.303-304).

90 If the (one sided) limit of the function $3x^2 + 3xh + h^2$, as h tends to zero (to the right hand side, i.e.,as h diminishes) is interpreted actually, i.e.,the argument h is assumed to be attaining its limiting ("last") value 0, then as the set of values of the function — in respect of which,according to Hind's definition,the limit must exactly be the lower bound — it is enough to choose a set, consisting of only one value of the function for $h = 0$ (see,Appendix, pp. 303-304), hence,in the given case, consisting of only one number $3x^2$; however,to look at it as the limiting value of the same $3x^2$, as h tends to zero,would be — as Marx says lower down — a banal tautology.In other words,it is natural to speak of $3x^2$ as the limiting value of $3x^2 + 3xh + h^2$ as h tends to zero, whereas to look at $3x^2$ as the limiting value of the same $3x^2$, also as h tends to zero, is not meaningful here — first of all, in so far as it is quite superfluous, it does not give us anything new.

91 Here the expression $\frac{0}{0}$ itself is considered to be the limit of the ratio $\frac{y_1 - y}{x_1 - x}$, as it has been done in Boucharlat's book (see,Appendix, pp. 308-309), but with a difference: here also the limiting value (again in the sense of Hind) of the functions $x_1 - x$ and $y_1 - y$ as $x_1 \to +x$, is not understood by Marx actually, i.e., it is written under the assumption that $x_1 \neq x$ (here $x_1 > x$).

92 Here again the issue is this : $\frac{0}{0}$ $\left(\text{or } \frac{dy}{dx}\right)$ should not be treated actually, i.e. as the value of the ratio $\frac{y_1 - y}{h}$, when $h = 0$; as in that case, following Hind and by obtaining the limiting expression $\frac{0}{0}$ through the simple assumption of $h = 0$, it would suit to permit the consideration of this expression — in which no trace of the ratio $\frac{y_1 - y}{h}$, containing the variable h remained — as the limiting value of the same $\frac{0}{0}$ (considered as a "constant" function of h) as $h \to +0$, which gives us nothing new. However,in respect of the expression $\frac{y_1 - y}{h}$, considered when h is different from zero (here $h > 0$), $\frac{0}{0}$, to be more precise,the derived function standing opposite it "as its real equivalent", is, as said Marx,"its absolutely minimal expression", i.e., it is the limit,in the sense now common.

93 In the original initially it was: "to the above mentioned differential equations", but Marx struck out the words "above mentioned". However, it is clear, that as before, equations in the proper sense of the term are not at issue here,but rather,the true formulae of the differential calculus,which have the form of equalities.

94 In the manuscript: "with the geometric one"; clearly, this is a slip of pen.

95 As has already been noted, in the sources used by Marx zero was not considered to be a finite magnitude.That is why,here it has been said,that the difference $x_1 - x = h$, becoming as small as you please, all the same it remains different from zero.

96 Instead of $x + \tau\dot{x}$, Marx here simply writes $x + \dot{x}$. About the reasons behind such a substitution see: PV,68 , and also note [60].

97 This fragment contains the contents of the separate sheets *a–g*. Sheets *h–n* contain only some rough calculations or unfinished notes.It is difficult to understand their meaning. These are not being published here. About them see: the description given on pp. 246-247 of the present volume. Sheets *a–g* are devoted to an analysis of d'Alembert's method, conducted in the light of the same example of a composite function, which has been examined in the manuscript "On the Differential".

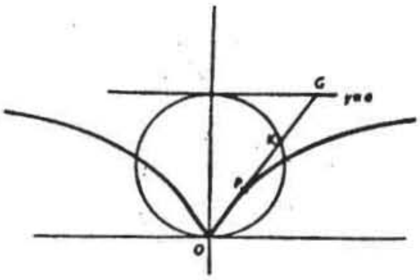
98 The symbols $f(x)$, $f(u)$ used here are short forms of the expressions : "a function in x", "some (other) function in u". While analysing this very example in the manuscript "On the Differential", written later on, Marx designated these functions by different letters.

98a The *Cissoid* is the algebraic curve with the equation

$$y^2(a-x) = x^3 ,$$

or in polar coordinates

$$r = a \sin \varphi \tan \varphi .$$



99 There is a rule in Feller and Odermann's book, which permits the trader to compute a sum x, which he must overcharge for his goods, if he wishes to get such and such a sum for it, after allowing such and such a rebate *from* hundred or *on* hundred.

100 Here it is difficult to understand, what sort of "hocus pocus"of Boucharlat Marx has in view. In any case, it may be observed, that Boucharlat defined dy only through dx (to be more precise, he defined only their ratio $\frac{dy}{dx}$). That is why, in order to know what is dy, one should be able to answer such a question for dx. But if, for every x, $y = x$, then according to Boucharlat we get, only $dy = dx$, i.e., $dx = dx$, which does not give any answer to the question, as to what one should understand by dx. If Sauri simply says that dx is the infinitesimal increment of x (here Marx has not yet faced the difficulties connected with the actual infinitesimals in the calculi of Newton and Leibnitz), then Boucharlat only pretends, as though he is in fact explaining, what is dx. It is not accidental that Marx has put the corresponding words of Boucharlat within quotation marks : dx "is itself the differential of x ". For the full text of § 9 of Boucharlat's book and comments on it,see: Appendix, p. 328.

101 For the Russian translation of Newton's work (with commentaries by D.D. Morduhai-Boltovsky) see : I.Newton, Matematechskie raboty, M., ONTI, 1937, pp. 3-24. The extracts are from the beginning ("Quadrature of simple curves, Rule I") and the end ("Demonstration of the quadrature of simple curves according to the first rule") of this work of Newton (p. 3 and pp. 22-23).

102 At the beginning of Newton's work, after the words : "The following notations are adopted : $AB = x$, $BD = y$; let a, b, c be known quantities, and m, n — be whole numbers", under the title "Quadrature of simple curves, Rule I", there is an assertion : "If. $ax^{\frac{m}{n}} = y$, then $\frac{an}{m+n} x^{\frac{m+n}{n}}$ = the area $A\,B\,D$", which is not proved there, but is only formulated, and "explained by examples". Here, namely, that is what Marx had in view. However, the demonstration adduced by Newton, at the end of the work also does not satisfy Marx (see further, the text of the manuscript and note [103]).

103 Then follows the comment, which Marx marked out by a vertical line on the left ,to indicate, that it is not a conspectus,but an insertion, which belongs to Marx himself. In this insertion Marx criticised the Newtonian proof of the theorem (which is usually called the Newton-Leibnitz theorem), on the connection between the definite integral and the original function, adduced below. Herein Marx's statement, to the effect that Newton already knew — from geometry,treated analytically — the content of the theorem demonstrated by him, is historically fully justified. As is well known, Newton's teacher Barrow did in fact observe the connection between the problems of finding the area and the construction of tangents, i.e., between quadratures, computation of definite integrals, and differentiation.The essence of the other reproofs of Marx, addressed to Newton, consists of the fact, that the latter does not introduce the concept of the definite integral as the limit of a sum,and accordingly, in this sense does not complete any process of integration, and also of the fact, that having observed in the integrand function $f(x)$ the derivative in respect of x of $\int_0^x f(t)\,dt$, from there Newton draws the conclusion, that also, conversely, the primitive of $f(x)$ is $\int_0^x f(t)\,dt$, without taking into consideration the fact that the primitive is being defined non-univocally.

104 By analytical geometry Marx here means geometrical applications of mathematical analysis, or geometry, analytically considered. It is well known, that the elements of such a geometry were shaped prior to Newton, in the works of Fermat, Pascal and Huygens.

105 Evidently, here Marx wanted to say that, from "analytical geometry" it was known to Newton, that if $z(x)$ is the area, bound (as it is usually done) by the curve $y = f(x)$,

then it is the integral of $y\,dx$, i.e., of $f(x)\,dx$, where $f(x)$, i.e., the expression borrowed from the equation of the curve, is the result of differentiating the function $z(x)$.

106 Here indeed there is a lack of clarity in Newton's account.It is connected with his understanding of the "limit" as the actually attainable "end" of a process of "infinite" (indefinite) approach to the "final"(limiting) state (see, Appendix, p. 313). The (curvilinear) trapezium $BD\,\delta\beta$ (see the diagram on p. 129) is substituted in Newton's account by a rectangle with the same base $B\beta$ and height BK, which is presumably so chosen, that the rectangle $KB\beta H$ is equivalent to the trapezium $BD\,\delta\beta$. Hence, the latter is supposed to be rectifiable. However, in this assumption it is not clear, as to what gives Newton the right to think, that in the "final" state, i.e., when by diminishing indefinitely $B\beta$ ultimately turned into 0, the height KD, which Newton designated by the letter v, must coincide with the ordinate DB or y. For then we shall have only $KD \cdot 0 = BD \cdot 0$, and this equality is true for any KD and BD.

107 Apparently, here Marx has in view that which he has already said (in points 2) and 3)) above, on Newton's demonstration, namely, that "he does not construct the curve from the equation $y =$ etc., but does it geometrically, assuming the area to be given", and "actually avoids integrating or showing, how this inverse process may be accomplished with the help of calculus", i.e., in this proof Newton does not at all use the equation of the curve $y = f(x)$: no operation of integration or differentiation is carried out with $f(x)$(what is differentiated is not the equation of the curve, but the equation of the area). The definition of the integral as the limit of the sum is hidden in Newton,behind his understanding of the differential of the area of a curvilinear trapezium, bound in the interval $[a, b]$ of the curve $y = f(x)$, as $y\,dx$. However, stated in a modern language, neither does Newton give an exact definition of the area of such a trapezium, what is more, nor does he give such a definition of the integral

$$\int_0^x f(x)\,dx,$$

which could have contained within itself the constructive method of its approximate computation. It is for this reason, that Marx reproaches Newton.

108 That is, the dependence of y upon x is revealed, also in the fact that both the increments Δx and Δy turn into zero at the same time.

109 According to Marx, the transformation of Δx into dx consists of the fact, that Δx other than zero, turns into dx equal to zero. Marx calls this transformation, the negation of Δx. We note, that the formal negation of something not equal to zero, is also equal to zero (if the double negation of A gives A). On the differences of dialectical negation of negation from such formal double negation see : PV, 19 (beginning of the manuscript "On the Concept of the Derived Function").

110 In the modern text-books of mathematical analysis, usually the differential is introduced as the principal linear part of the increment of a function, and at the same time in practice it also appears as an operational symbol of the differential calculus. Marx has made a distinction between these two functions of the differential.

The differential particles dx, dy, which are exactly equal to zero, correspond to the differential as an operational symbol, defined by the formula : $dy = f'(x)dx$; and the differential dy as the principal part of the increment of the function $y = f(x)$, is defined by the formula $dy = f'(x)h$, where h is a "finite" (other than zero) increment of the independent variable x. Euler too treats the issue analogously (see, Appendix, p. 316)

111 It is clear from the latter manuscripts, first of all from those published in the first part of the present volume, that later on Marx changed his evaluation of Lagrange's method. However, in the given case his attitude boils down to this : there is no point in raising an objection against Lagrange's method of explaining the interrelation of the coefficients of the expansion $f(x + h)$ into a series according to the powers of h and the successive derivatives of the function $f(x)$.

111a *Constructivism* is one of the trends of mathematics. Like classical mathematics, constructivist mathematics too investigates the formal and quantitative relations of objective reality; but there are some differences that characterize them.

Those who belong to the classical trend believe that they know (and sometimes they may know) the properties of the objects they investigate, that is why they do not pay attention to the methods of constructing those mathematical objects. The constructivist, on the other hand, limits his/her work to the *constructivisable objects* alone. The existence of a constructivisable object is considered to be proved, only when the possible and effective methods of its construction are indicated.

The concept of constructivisable object is foundational *to constructivist mathematics and logic,* that is why this concept is not defined — it is only explained with the help of examples. The letters that go into the construction of a word are examples of primary constructivisable objects. Words are built from these constructivisable objects — according to some agreed rules ; and from the words — the more complex constructivisable objects, such as clauses, sentences and texts. Consider the pair < 404, 55 > . Here, the use of the algorithm for column-wise subtraction successively generates the following constructivisable objects :

$$\begin{array}{r} 404 \\ \underline{55} \\ \end{array} \qquad \begin{array}{r} 404 \\ \underline{55} \\ 9 \end{array} \qquad \begin{array}{r} 404 \\ \underline{55} \\ 49 \end{array} \qquad \begin{array}{r} 404 \\ \underline{55} \\ 349 \end{array}.$$

Every constructivisable object is determined by the one immediately preceding it. If an object is not given by direct observation or by an algorithm then it is not a constructivisable object. The collection of all the natural numbers, considered as an

actually infinite set or the Series of Natural Numbers — is not a constructivisable object, since there exists no algorithm for constructing this series [*Petrov yu. A.* Logicheskie problemy abstraktsii beskonechnosti i osuschestvimosti. M., 1967.].

In order to work with the constructivisable objects, it is necessary to know about the methods of determining the identities and differences among them. For example, consider the word "constructivisable" : in it, one and the same letters "*c*", "*s*", "*t*" and "*i*" are standing in different places. Studies on the methods of determining these identities and differences have given rise to the concept of *identification.*

Similarly, investigations into those constructivisable objects that are not realizable in practice, have given rise to the concept of *potential realization*. For example, in practice we can not write a word, which is as large as possible ; but, going beyond the boundaries of this reality, we think that it is theoretically possible to do this work [*Ruzavin G.I.* O prirode matematicheskovo znaniya. M., 1968.].

There are some similarities between the constructivist and the intuitionist concept of constructivisable objects: according to the followers of both the trends, the existence of an object under consideration is proved,only when the possible effective means of constructing that object is given — not otherwise. There are mutual differences too : the intuitionists are of the opinion that all mathematical objects are produced by some "primary intuition", this "intuition" is the creator of all the numbers ; the constructivists, on the other hand, opine that in the ultimate analysis, constructivisable mathematical objects and the methods of constructivist mathematics are nothing but idealized prototypes of the objects of this world and of the regularities that govern them. Intuitionist L.E.J. Brouwer argued that the roots of most of the idealizations of classical mathematics may be traced back to the principle of excluded middle of classical logic. Constructivists like E. Bishop have declared the universality-claim of the principle of non-contradiction of classical logic to be baseless. Bishop has called it the "principle of omniscience", which states that : an arbitrary set *A* either has an element with a given property *P* or it does not. "In case *A* is an infinite set this principle is not constructively valid, because the examination of each element of *A* to see whether one of them has the property *P* is not something that can necessarily be done by a finite routine process" [*Bishop E.* The Constructivisation of Abstract Mathematical Analysis // Proceedings of International Congress of Mathematicians (Moscow, 1966). Introduction. Izd. "MIR". M., 1968. p. 308.].

In contrast to the classical mathematical concept of *actual infinity*, constructivists have proposed the concept of *potential realization of infinity.* According to the concept of actual infinity, any collection of an infinite number of objects is an actually existing set ; and according to the concept of potential realization of infinity, the construction of an infinite set is a process without any last step. In classical mathematics : to exist

is to obey the principle of non-contradiction ; and in constructivist mathematics : to exist is to be constructivisable. In the final analysis, the content of the constructivist critique of classical analysis is as follows : to admit the concept of actual infinity is to extend over infinite sets, those principles of classical logic which are applicable in the case of finite sets (e.g., the principles of excluded middle and non-contradiction and, the rules of formal deduction based on them) ; such an extension is not permissible.

In the 30s of the present century, when the concept of algorithm became more exact through the investigations conducted by Church, Kleene, Turing and Post, this led to significant developments in constructivist mathematics. An important result of these developments : Markov's concept of natural algorithm [*Markov A. A.* Teoriya algorifmov.M., 1954]. Martin-Leof's investigations into the interrelationship of algorithm and meta-algorithm too broke new grounds [*Martin-Leof P.* Ocherki po konstruktivnoi matematike. M., 1975]. Investigations into the logic of the principles of constructivist mathematics have led to the emergence of constructivist logic.

The logically valid positions of intuitionist logic are critically reconstructed in constructivist logic. Constructivist logic is based upon the following system of axioms[here, A, B and C are arbitrary propositions and,the logical connectives $\cdot$ (and), $\vee$ (or), $\rightarrow$ (if ... then) and $\overline{}$ (the negation symbol) have been used in their usual sense ; but unlike in the classical logic, here they are not inter-translatable, since the principle of non-contradiction etc., which are foundational to these transformations — are not valid in constructivist logic] :

$$A \rightarrow (B \rightarrow A);$$
$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C));$$
$$(A \cdot B) \rightarrow A;$$
$$(A \cdot B) \rightarrow B;$$
$$A \rightarrow (A \vee B);$$
$$B \rightarrow (A \vee B);$$
$$A \rightarrow (B \rightarrow (A \cdot B));$$
$$(A \rightarrow C) \rightarrow ((B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C));$$
$$(A \rightarrow B) \rightarrow ((A \rightarrow \overline{B}) \rightarrow \overline{A});$$
$$A \rightarrow (\overline{A} \rightarrow B).$$

The foundations of constructivist logic were laid by Kronecker, Brouwer, Weyl, Heyting (all intuitionists) and, by the constructivists Kolmogorov, Shatunovsky, Vasiliev, Glivenko and, especially through the investigations conducted by the school of Markov.

[See also : *Nepeivoda N.N.* Emergence and Development of the Concept of Constructivisability in Mathematics // Present Volume, Special Supplement : *Marx and Mathematics*, Part Three, last article.]

*** *** ***

The editor of K.Marks, *Matematicheski Rukopisi* (M., 1968) did not provide any note on constructivism. She did not think that it was necessary for the readers of that edition, perhaps, in view of the fact that some of the major centres of constructivist mathematics were (and are) situated on the territory of [erstwhile] USSR. But our situation is different. Here the classical trend is overwhelmingly dominant. The terms constructivist mathematics or constructivism are absent in the glossary of mathematical terminology published by the West Bengal State Book Board, in the glossaries of mathematical and scientific terminologies published by the Bāṅglā Academy of Dhākā and, in the glossary of scientific and technical terms published by the Government of India. That is why our readers are in need of a note on this term.

*** *** ***

Through his investigations into the characteristic concepts and symbols of differential calculus (like the derivative and the differential) Marx arrived at the conclusion that they are operational by nature. Forty-four years after his death, in 1927, intuitionist Hadamard arrived at similar conclusions about the general nature of the differential calculus [see : *Glivenko V. I.* Marx and Hadamard on the Concept of Differential // Present Volume, Special Supplement : *Marx and Mathematics*, Part Two, first article]. Intuitionist L.E.J Brouwer was the first to propose the theory that the standard mathematical quantifiers and connectives are by nature operational. The constructivists accept this position of Brouwer, though they reject the philosophical idealism connected with mathematical intuitionism.

Thus the development of the constructivist trend has created material grounds for connecting Marx's mathematical investigations with the mainstream of history of mathematics, by revealing their inner linkages. This is the historic significance of the constructivist trend for the modern reader of Marx's *Mathematical Manuscripts* and, conversely, that of Marx's mathematical investigations — for the mathematicians of to-day and to-morrow. — Tr.

112 These pages of Hind's book are devoted to an estimation (from above and from below) of the remainder term of Taylor's series — to an estimation "of the quantity, which diminishes, as we stop the series at any of its indicated term" (Hind, p. 87). According to Hind, such an estimation must be carried out, when the "analytical transformation of a function" — wherein divergent serieses may also be used — is not at issue, but when some particular value of it is to be computed.

113 Later on Marx cited an example, in which the turning into zero of the numerator of the fraction, for some value of the variable x, must entail that its denominator too turns into zero. However, here he made a mistake in calculation, which cannot be rectified in his example. So this portion has been omitted. For the examples explaining this idea of Marx, see : note [114].

114 After this Marx again turns to that example which we omitted (see note [113]), but this time without the previous mistake. However, even now there is a mistake in the calculation, but it is of the nature of a slip of pen. Though the example chosen here, by Marx, is not a happy one, but the meaning is clear. Namely, it has been observed, that in differential calculus the numerator $f(x+h)-f(x)$ of the fractional expression turns into zero, as a result of the turning into zero of the denominator, whereas, generally speaking, in "algebra" such cases are also possible, but only when the equality of the denominator with zero, is a consequence of the fact that the numerator has turned into zero. For example, such would be the case of the fraction $\frac{x-a}{x^2-a^2}$, had we put in it, the numerator equal to zero, i.e., had we imparted the value a to the variable x. Or, the case of the fraction $\frac{(x-a)(x-b)}{(x-a)(x-b)(x-c)}$ (a, b, c — are pair wise different numbers), where, from the equality of the numerator with zero, quickly follows the equality of the denominator with zero, but not conversely. In this connection see also : the manuscript "On the Differential" (see : PV, 28-29) and note [18].

115 Here Marx's concept of limit is still very closely connected with the concept of the "boundaries of change" (see : Appendix, p. 303).

116 Marx later on developed this critique of Boucharlat, in his manuscript "On the Differential" (see : PV, 33-34).

117 In Boucharlat's book (5th ed.,p. 34, § 55) demonstration of Taylor's theorem began with the lemma : "If in the function y of x, the variable x is changed into $x+h$, then we get one and the same differential in both the cases : when x is a variable and h is a constant, as well as in the case when h is a variable and x is a constant". (see also : Appendix, pp. 337-338).

118 Since the basic manuals at Marx's disposal not only gave one and the same formulation of the theorems of Taylor and MacLaurin, but also gave exactly identical demonstrations of them, it was natural to assume that these belonged to the authors of the theorems themselves (and that was indeed the case with MacLaurin's theorem ; see : Appendix, p. 335).

119 Here the reference is to the proof of Taylor's theorem in Boucharlat's book (see : Boucharlat, pp. 36-37 or Appendix, p. 336 ff.).

120 Here Marx used some abbreviated (stenographic) mode of writing, substituting for the words "function y", the expression "$f(y)$". Thus, this place should be read as : "... Taylor starts not from the function y, or $y=f(x)$, but ...".

121 In this paragraph, as well as in a number of other manuscripts (see : for example, p. 96), apparently written later on, Marx compared the method of actual infinitesimals of Newton and Leibnitz, with the method of limits, interpreted actually, as attainable at the end of an infinite process of approximation of the variable to its limiting

value; besides, without any substantiation, such correlations are extended to this "end"; these are of any sense only for the pre-limit conditions (for example,when considered as given "triangles", the sides of which are equal to zero). Here Marx arrives at the conclusion that, in essence such a "method of limits" is no better than the method of actual infinitesimals (see : Appendix, pp. 303-304, 313).

122 Here the reference is to Lacroix's account of the analysis of the special case of the problem of two couriers, in his "Elements of Algebra" (§ 64, pp. 94-96). The case here is this : two couriers proceed from the two towns, separated by the distance a ; they proceed in one and the same direction ; the speed of the second (b) is greater than the speed of the first (c), besides b is constant ($b = 6$ Km/hour), and c increases : it successively attains the values 5·8 , 5·9 , 5·99 etc. ; the distances x and y, traversed by each courier, upto the point where the second overtakes the first, are to be found out. It is not difficult to notice that here the solution has the form

$$x = \frac{ab}{b-c}\ , \quad y = \frac{ac}{b-c}\ ,$$

when c is indefinitely tending to b.

123 Here Marx has the following situation in view : as h tends to 0, the term $f'(x)$ in the expression

$$f'(x) + \frac{1}{2}f''(x)h + \cdots$$

remains unchanged, since in the course of this operation x is assumed to be a constant.

124 Here Marx refers to the following two places of "Eléments d'algèbre par L. Euler" (Lyon, 1795).

a) "Here it is still necessary to dispel the quite widespread mistake of those, who think that the infinitely large can not be enlarged. This view if incompatible with the desired principles, which we have just established [by introducing $\frac{1}{\infty}$ as 0 and therefore $\frac{1}{0}$ as ∞ — Ed.]; for if $\frac{1}{0}$ designates an infinitely large number, then, since $\frac{2}{0}$ is, undoubtedly, twice $\frac{1}{0}$; it is clear,that a number even if it is infinitely large, can still become two or several limes larger"(§ 84, p. 60).

b) In another place (§ 293, p. 227) Euler represents an algebraic fraction by the series $1 + a + a^2 \cdots$ and observes : "Let us assume first of all that $a = 1$; then, our series turns into $1 + 1 + 1 +$ etc. upto infinity. The fraction $\frac{1}{1-a}$, to which it must be equal, turns into $\frac{1}{0}$, but we noted above that $\frac{1}{0}$ is an infinitely large number; hence here it gets an elegant confirmation".

125 Here "determinability of a problem" is understood as follows : the problem permits such a formulation (with the help of equations), which is univocally determined by the values of the unknown sought for.

126 Of course, here as well as lower down, by the expression "number of equations" what is meant is : the number of independent equations, hence, not one of them is a logical consequence of the rest.

127 Here the expression "proper fraction" means : a "fraction in the proper sense of the word", as wrote Lacroix, see p.3 of his "Treatise", i.e., an expression of the form $\frac{P}{Q}$, where P and Q are polynomials, expanded according to the powers of some variable — for example of x — such that the "angular" division of P by Q gives the remainder R, the power of which is less than the power of Q. According to Lacroix, the expression of a "proper fraction" in the form of an infinite series is not the expression of a function, definable by this fraction in its "proper" form, since "algebraic functions always contain only a limited number of terms" (ibid).

128 Approximate values of logarithms may be found with the help of the process of square root extraction, i.e., with the help of a few "algebraic operations". But for the exact computation of the logarithm of a real number, in the general case, a finite number of these operations are not enough. (We read in Lacroix : "Such for example, are the logarithms, which may be obtained only approximately and which depend upon the extraction of an infinite number of roots"(ibid, p.4).

129 That is to say here a definite arc of a definite circle is not at issue : an arc is not considered as such, but only in respect of the entire circumference — as a part of it, i.e., as an angle (herein, two different arcs, constituting identical parts of two different circles, are identified). This observation of Marx is explained by the fact, that in Lacroix $\sin x$ was not considered as a number, defined by another number x, but was considered as a segment, whose length depends not only on the arc, but also upon the radius of the circle, with its centre at the vertex of the angle. In fact, Lacroix wrote : "Here we digressed from the radius, though the magnitude of sin also depends upon this element, since we have in view only one circle "(ibid, p.3).

130 Here "variation of the function x" means : changes in the values of this function. Let us recall, that Marx called an "algebraic" expression, considered in respect of the entries of the variable x or its values into it — "the function x" (or "the function in x").

131 Here Marx assumes this basic theorem of algebra to be already proved.

132 Here the reference is not to an equation, but only to its left hand side ; below Marx himself said, that while speaking about the general equation $f(x) = 0$, he had in view its left hand side, i.e., $f(x)$.

133 The reference is to MacLaurin's own proof of the theorem, which bears his name. For its account in Boucharlat's book see : Boucharlat, pp. 20-21 and PV, Appendix, p. 336.

134 A considerable part of the text that follows has been struck out by pencil.

135. The reference is to the search for the multiple roots of the "proposed" equation $f(x) = 0$ (and their ratios). (As is well known, the roots of multiplicity k of the equations $f(x) = 0$ are the general roots of the equations $f(x) = 0$, $f'(x) = 0$, $\cdots$, $f^{(k)}(x) = 0$, not satisfying the equation $f^{(k+1)}(x) = 0$.) In his text-book on algebra Lacroix no where uses the terminology of differential calculus and avoids them to such an extent, that he does not even introduce the concepts of the polynomial and the derivative of the polynomial

$$a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m,$$

which he could have called, according to the definition, the polynomial

$$ma_0x^{m-1} + (m-1)a_1x^{m-2} + \cdots + a_{m-1}.$$

This is why in Lacroix, the left hand side of the "proposed" equation is also not designated by $f(x)$. It has been introduced here by Marx. The polynomials which we would now designate by $f(a)$, $f'(a)$, $f''(a)$, $\cdots$, Lacroix correspondingly designated by the letters V, A, B, ... However, the absence of an appropriate symbolism forces him sometimes to use the same letters, V, A, B, ... to designate the polynomials $f(x)$, $f'(x)$, $f''(x)$, $\cdots$. Conversely, Marx first of all draws our attention to the fact, that all these polynomials are obtained from $f(x)$, through successive differentiation.

136 It is that equation (2), about which Marx spoke at the beginning of point 3). The equation

$$x^m + Px^{m-1} + Qx^{m-2} + \cdots + Tx + U = 0$$

carries the number(1) in Marx's conspectus.

137 The result of substituting $y + a$ for x in the polynomial

$$f(x) \equiv x^m + Px^{m-1} + Qx^{m-2} + \cdots + Tx + U$$

was presented by Lacroix in the form

$$V + Ay + \frac{B}{2}y^2 + \frac{C}{2\cdot 3}y^3 + \cdots + y^m,$$

hence

$$V = f(a),\ A = f'(a),\ B = f''(a),\ C = f'''(a),\ \cdots$$

If a is a multiple root (of multiplicity k) of the equation $f(x) = 0$, then

$$f(a) = f'(a) = \cdots = f^{(k)}(a) = 0,$$

i.e., $V = A = B = C \cdots = 0$. This is what has been proved in Lacroix. Marx reproduced below the technique of this proof.

138 *Hysteron Proteron* is a logical error,consisting of putting the subsequent (hysteron) before the antecedent (proteron). Example : "the cart before the horse".

139 Taking y out of the brackets in equation (II), and thus the reduction of the search for the roots of equation (II) into a search for the roots of equation (IIa), has been depicted by Lacroix as the "division" of both the parts of equation (II) by y. See : Lacroix's book, p. 281. Marx used the same terminology here and afterwards.

140 Of course here it has been assumed that a is a root of equation (1) of multiplicity k, where $k > 1$.

141 In other words it was possible to formulate the rule simply as :

if a is a multiple root of equation (1), then it is also a root of the equation $A = 0$, where A is a derivative of the left hand side of equation (1). The calculations which follow this in Marx's manuscript, are not there in Lacroix.

142 Of course under the assumption that a is a root of multiplicity k, where $k > 2$.

143 See note[133].

144 In the original : $x = 0$; but to all appearance, it is a slip of pen.

For an understanding of the text that follows, it is enough to note, that apparently, here Marx has in view an analogy with the method of differentiation, enunciated after Lagrange by Boucharlat. It consists of the following : $f(x+h)$ is expanded into a series according to the powers of h :

$$f(x+h) = f(x) + Ph + Qh^2 + Rh^3 + \cdots,$$

after which the ratio $\frac{f(x+h) - f(x)}{h}$ turns out to be equal to

$$P + Qh + Rh^2 + \cdots \tag{1}$$

Thus, Boucharlat reduced the search for the limit of the ratio $\frac{f(x+h) - f(x)}{h}$, i.e., for the derivative $f'(x)$, into assuming $h = 0$ in the expression (1). Here for Marx a plays the role of h.

145 Apparently here Marx says that while obtaining the expression A from the expression V algebraically, in essence we do what Boucharlat would have done by applying the method of differentiation enunciated in the previous note, to the search for the derived function A of V ; here in the algebraic deduction of A from V, the unknown y plays the role of h.

146 The reference is to the method of eliminating one unknown from two equations with two unknowns, proposed by Euler in his "Introduction to the analysis of infinities" (ch. XIX, §§ 483-485, in the Russian edition of 1961, pp. 253-255). For eliminating the unknown x from Marx's equations 1) and 2), whose coefficients (P, Q , ..., P_1, Q_1, $\cdots$) contain the variable y, the first equation is multiplied by the polynomial

$$x^{n-1} + px^{n-2} + qx^{n-3} + \cdots,$$

containing the indeterminate coefficients $p, q, \ldots$, and the second is multiplied by the polynomial

$$x^{n-1} + p_1x^{n-2} + q_1x^{n-3} + \cdots,$$

containing the indeterminate coefficients p_1, q_1, $\cdots$ By equating the coefficients of the same powers of x in the products thus obtained, the problem is reduced to the solution of a system of $m+n-1$ equations, linear in respect of $m+n-2$ unknowns $p, q, \cdots, p_1, q_1, \cdots$, having the coefficients $P, Q, \cdots, P_1, Q_1, \cdots$, dependent only on x. The last of these equations, found from the first $m+n-2$ equations, by substituting the expressions for $p, q, \cdots, p_1, q_1, \cdots$ and $P, Q, \cdots, P_1, Q_1, \cdots$, in it also gives us the "final" equation (the resultant), which no more contains the variable x. In his "Elements of Algebra" Lacroix gives an account of this method of Euler, but only in the light of examples (see : §§ 192-195, pp. 264-270). (For a modern account of Euler's method see, for example, van der Waerden, B.L. Modern Algebra, M-L., 1947, § 27, p. 115-117.) [A newer English edition of it was published by "Ungar", NewYork, 1953 — Tr.]

147 Here Marx's comment comes to an end, but the conspectus of the above mentioned chapter of Lacroix's "Elements of Algebra" continues. However, what, namely, Marx wanted to say here, remains unclear. Only this much can be said, that he had in view an analogy between Euler's method for eliminating the unknown (see note [146]) and the method of demonstrating Taylor's theorem as enunciated in the books of Hind and Boucharlat. This demonstration was based upon the assumption that, if $y = f(x)$, then

$$f(x+h) = y + Ah + Bh^2 + Ch^3 + \cdots,$$

where $A, B, C, \ldots$ are "unknown functions of x, which are to be determined" (Boucharlat, p. 36).

In other words, as in the case of the indeterminate coefficients $p, q, \cdots, p_1, q_1, \cdots$, in Euler's method, here also the numerical "indeterminate coefficients" are not being referred to. The latter are considered as functions of the "second"of the two unknowns or variables (h and x in the demonstration of Taylor's theorem; correspondingly x and y in Euler's method), to be determined.

148 Like the majority of mathematicians of his time, Lacroix called a series convergent, if "the terms forming it diminished as they moved away from the first term" (Lacroix, p. 328). In his conspectus Marx too adduced this definition.

149 In other words, here the following circumstance is being utilised : the exponential function $f(t) = a^t$ satisfies the functional equation $f(x) \cdot f(z) = f(x+z)$. Namely, the following serieses are being written :

$$a^x = 1 + Ax + Bx^2 + Cx^3 + \cdots,$$

$$a^z = 1 + Az + Bz^2 + Cz^3 + \cdots,$$

$$a^{x+z} = 1 + A(x+z) + B(x+z)^2 + C(x+z)^3 + \cdots,$$

the first two are multiplied, and ; the indeterminate coefficients $A, B, C, \ldots$ are determined by equating the coefficients of the like terms of the form $x^i z^k$ in the serieses for $(a^x \cdot a^z)$ and a^{x+z}.

150 This comment is not there in the 11th (French) edition of Lacroix's "Elements of Algebra " (1815). It may be held, that if not the whole of it, then at least its last para belongs to Marx. To understand the text that follows it is enough to note, that in §§ 260-261 (pp. 355-357) of Lacroix's book the issue is about the sum of money A, lent for n years at interest (r units per year), to be repaid by the debtor in course of these n years, in yearly instalments of one and the same sum of a. Considering capital A to be turning at the end of the n-th year into $A(1+r)^n$, and the sum of money a which is to be paid at the end of the first, second etc. years, upto the end of the n-th year, to be equal to the sums $a(1+r)^{n-1}$, $a(1+r)^{n-2}, \cdots, a$, Lacroix obtained the corresponding equality

$$A(1+r)^n = a(1+r)^{n-1} + a(1+r)^{n-2} + \cdots + a,$$

whence, summing up the progression in the right hand side he obtained

$$A(1+r)^n = \frac{a[(1+r)^n - 1]}{r}. \qquad (1)$$

The equalities

$$A = \frac{a}{r}\left\{1 - \frac{1}{(1+r)^n}\right\} = \frac{a}{r} - \frac{a}{r(1+r)^n},$$

given below by Marx, are obtained by dividing both the parts of equality (1) by $(1+r)^n$. Since A is the sum to be paid at the given moment to finally settle the account, A is called the "present value" of the debt.

151 This calculation is based upon the following :

When $n = 99$ and $r = \frac{1}{20}$ we have $(1+r)^n = \left(\frac{21}{20}\right)^{99}$.

We know that $\log\left(\frac{21}{20}\right)^{99} = 99(\log 21 - \log 20) = 2{\cdot}0977$, hence $\left(\frac{21}{20}\right)^{99} \approx 125$.

152 Marx could have found such an account of the question of linear dependence (of "variation") in many text books by the English authors of his time. Thus, in R. Potts' "Elementary Algebra", section IX, p. 16 (1880), we read : "The sign $\propto$ is used instead of the words "varies as" and the proportionality [of a variable x to another variable y] written in the form of $x \propto y$ is read as : "x varies as y". Potts' proposition 23 reads : "If

x varies as y, and y varies as z, then x varies as z "(p. 17). Potts also adduced other propositions on the properties of variation.

153 This critical observation of Marx is directed against "widening" the concept of permutation, discussed above (see, PV, 190), which unavoidably entails a confusion between the concepts of "permutation" and "variation" (their merger). The main point is this : though a permutation of n elements may be viewed as a variation of n elements taken n at a time, but in so doing the difference between the permutation of n elements and the variation of n elements taken m at a time (when $m < n$) is retained, whereas in the "widening" of the concept of permutation, about which Marx wrote, this distinction loses meaning.

154 Apparently what is being referred to here, is the method of obtaining all variations of n elements taken m at a time by : a) constituting all permutations of n elements, b) separating in each of them, the first m elements from the rest and c) finally, equating all such permutations mutually, whose beginnings ("words " made of the first m letters) coincide.In so far as the number of such variations $= (n - m)\,!$, it is clear that the number of variations of n elements taken m at a time $= \dfrac{n\,!}{(n-m)\,!}$. It is also clear, that it does not require any "widening" of the concept of permutation.

155 It is clear, that by "division by the factor y", of an expression of the form $Ay + By$ Marx here means a formal transformation, consisting of the following : y is taken out of the brackets and the second factor $(A + B)$ is separated,which is viewed as the result of such a "division", independently of, whether y is equal to zero or not.

156 If the equation has a root $y = 0$, then its left hand side may be represented in the form of the product $(Ay + B)\,y$. "Having divided" — in the sense expressed in note 155 — the left hand side of this equation by y, we shall obtain the equation $Ay + B = 0$, which is a multiple root of the initial equation, when $y = 0$, and in its turn it has $y = 0$ as a root, owing to which B must be equal to zero independently of y. And that is what Marx has said.

157 MacLaurin transformed the equation by substituting

$$x = y + k$$

and, wrote the terms with the same powers of y in a column one under the other. The vertical series, referred to by Marx, corresponds to the free term of the equation, obtained as a result of this transformation.

158 Apparently, here Marx wishes to explain, why the search for the bounds of the roots should not be confined to a consideration of the free term of the transformed equation, and why one should successively turn to the coefficients of the terms prior to it, beginning with the last term (i.e., to the derivatives of the last term).

It seems that in the last clause there is a slip of pen : instead of "roots" Marx wrote "terms", evidently having in view the fact, that the positive roots are contained between

zero and their upper bounds and, the negative roots — between their lower bounds and zero.

159 Here Marx has turned his attention to the difference in the forms of expression of the powered series for $f(x+h)$. If this series is written in the form

$$f(x) + ph + q\frac{h^2}{1\cdot 2} + r\frac{h^3}{1\cdot 2\cdot 3} + \text{etc.},$$

then the coefficients $p, q, r, \ldots$ are the successive derivatives $f'(x)$, $f''(x)$, $f'''(x)$, ... However, it will be incorrect, if the series is written in the form

$$f(x) + ph + qh^2 + rh^2 + \ldots$$

160 That is, a composite function expressible through elementary functions. These may be defined, for example, as :

1) all elementary functions like x^m, a^x, $\log x$, $\sin x$ and others which are "expressible through elementary functions", 2) if a function $f(x_1, x_2, \cdots, x_n)$ $(n = 1, 2, \ldots)$ is "expressible through elementary functions", then a function obtained from it by substituting the functions "expressible through elementary functions" in place of the arguments $x_1, x_2, \cdots, x_n$, is also "expressible through elementary functions".

Evidently then, composite functions are such functions, as are "expressible through elementary functions", but are themselves not elementary, and that is why in the general case they contain a few elementary functions, like : $\log \sin x$, $\sin^m x$ etc.

161 That is, polynomials.

162 In Hind, as well as in the other sources used by Marx, expressions containing $\sqrt{-1}$ were considered indicative of the unsolvability of the problem, "impossible" expressions. Conversely, expressions not containing the imaginaries, were often called "possible". The fact that Marx has put the words "possible expression" within quotation marks, is expressive of his irony directed at this sort of attitude towards the emergence of the imaginaries.

163 In a number of places, in his manuscripts Marx adduced bibliographical informations about the memoir of Taylor, "Method of increments", which contains Taylor's theorem. A sheet, attached to the manuscript "On the History of Differential calculus" also contains a reference to this memoir. This shows that Marx intended to get acquainted with this work in the original. Such an assumption is also supported by Marx's special mention of the fact that he borrowed the informations on MacLaurin's theorem directly from MacLaurin. Though, here too, his source remains the text-books at his disposal. In Hind's book (pp. 84-85) the binomial theorem has been deduced (for a positive and integral m) from Taylor's theorem, and the result has been written in the form

$$(x+h)^m = x^m + mx^{m-1}h + \frac{m(m-1)}{1\cdot 2}x^{m-2}h^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^{m-2}h^3 + \text{ etc.}$$

The fact that the last terms of the expansion were not written, apparently provided Marx with the ground to ascribe to the author, an urge to treat this particular instance

from a more general point of view. It is natural that here Taylor himself appeared to Marx to be the "author".

164 These two sheets, are in the nature of rough drafts and they do contain slips of pen, which we have corrected, as the meaning of the text was quite clear. Here the source of point 1) is : Sauri's book (vol. III p. 3). The source of point 2) could not be established. Giving an account of Lagrange's method, Lacroix referred to the fact that this method was further improved upon by Poisson (Lacroix, Treatise, p. 160). In his list of references Lacroix mentioned a paper by Poisson in issue No. 3 of "La correspondance sur L'École Polytechnique". In the bibliographical index attached to his historical essay (see : PV, 66) Marx has mentioned Poisson without mentioning his works.

165 Here Marx stresses the difference between functions as analytical expressions (functions "in x") and functions as dependence (of one variable upon another : functions "of x") and draws our attention to the confusion caused by the fact,that these concepts are not distinguished. In modern literature, especially in the work on mathematical logic, carried out by the school of A.A. Markov, this confusion is eliminated by using two different equality signs : "$=$" and "$\doteq$ ". "$y = f(x)$" in essence signifies, that y is a function "of x ", in Marx's sense ; " $y \doteq f(x)$" signifies that y is an analytical expression — a word in the corresponding alphabet, having the form "$f(x)$" (in the example considered by Marx, $5x^4$ is a function " in x", in Marx's sense). For greater details see : note[6].

166 In accordance with what has been said in the previous note, Marx understood an expression like $f(x) = 5x^4$, in a sense, according to which we would now write $f(x) \doteq 5x^4$, i.e., a statement to the effect, that $f(x)$ has the form $5x^4$ (that $f(x)$ is a "word" of the form $5x^4$). Such expressions, as well as equations, have two sides (right and left), which Marx here counterposes, one against the other : the general indeterminate expression (the left) as opposed to the particular expression (the right).

167 From the paragraphs which follow immediately, it is clear, that Marx by no means considered such algebraic (in his sense of the word) induction of Taylor's theorem to be its actual proof.

By "differentiation" Marx sometimes understood simply : formation of the difference $f(x+h) - f(x)$; and in that case, "differentiating $f(x)$, when x increases by a positive or negative increment h", is naturally understood as : an approximate (upto this or that degree of exactitude) calculation (of the finite) increment of the function (at a given point x_0) for a given increment h of the independent variable x. On the use of Taylor's theorem for the successive differentiation of the function $f(x)$, with the help of a serial expansion of $f(x+h)$ according to the powers of h, see below.

About the word "every" (function), it is evident from the paragraph that follows (point a)), that Marx never thought that every function could be expanded into Taylor's

series. That is why it is clear, that by "every function" here he meant every function $f(x)$, for which $f(x+h)$ can be expanded into Taylor's series.

168 Evidently here the reference is to the fact,that from the expansion of $f(x+h)$ into Taylor's series, it is possible to extract the chain of differences

$$y_1 - y,\ (y')_1 - y',\ (y'')_1 - y'' \text{ etc.,}$$

about which Marx wrote in detail in the paragraph that followed. Under the influence of the sources used (see : first of all the text-book by Boucharlat), Marx used the notation y' in the sense of the augmented value of y ; i.e.:, y_1, as well as in the sense of the derivative of y. To remove this double meaning, in the next paragraph we have everywhere substituted Marx's

$$y' - y,\ y'' - y',\ y''' - y'' \text{ etc,}$$

respectively by

$$y_1 - y,\ (y')_1 - y',\ (y'')_1 - y'' \text{ etc.}$$

169 The second part of this paragraph, beginning with the words "but, conversely", contained a number of cancellations and slips of pen. But as it was not quite clear, what, namely, Marx wanted to say here, we did not take the liberty of correcting even the explicit slips of pen. Apparently here Marx had in view some generalisation (for any function, expressible through a powered series) of that directly "algebraic" differentiation of the equality

y_1 or $(x+h)^{m+1} =$

$$= x^{m+1} + (m+1)x^m \frac{h}{1} + (m+1)mx^{m-1} \frac{h^2}{1 \cdot 2} + (m+1)m(m-1)x^{m-2} \frac{h^3}{1 \cdot 2 \cdot 3} + \text{ etc.,}$$

which he studied on pages 4-5 of his manuscript. It seems, that in the words "the difference between $f(x_1)$ and $f(x)$ the letter f is simply a stenographic notation for the word "function", i.e., this place should have been translated as "the difference between the values of some function in x_1 and in x". If such a function is designated by the letter φ , then the formula

$$y_1 = A\ (x_1{}^m - x^m)$$

(which contains an explicit slip of pen) may , perhaps, be read as :

$$\varphi(x_1) - \varphi(x) = A\ (x_1{}^m - x^m).$$

By the words "an arbitrary constant A ", evidently, here he meant: "some" ("some not closely defined ") constant.

170 Here Marx has in view the method of formal differentiation of a function, represented through a powered series, the first prescription of which reads : if

$$f(x) = A + Bx + Cx^2 + Dx^3 + \text{etc.,}$$

then

$$f(x+h) = A + B\,(x+h) + C\,(x+h)^2 + D\,(x+h)^3 + \text{etc.}$$

$$= A + B\,(x + h) + C\,(x^2 + 2xh + h^2) + D\,(x^3 + 3x^2h + 3xh^2 + h^3) + \text{ etc.}$$

This in fact suits the use of the binomial theorem, beginning with the second power of the binomial $(x + h)$.

171 The reference, apparently, is to the manuscript "On the Concept of the Derived Function" (see : PV, 19).

172 John Landen obtained the binomial theorem of Newton, and besides he obtained its generalisation for any real index of power, with the help of his "residual analysis" — which is somewhat analogous to Marx's "algebraic method of differentiation" (see : Appendix, p. 320).

Marx did not want to finally formulate his own manuscript on the history of the methods of differential calculus and the theorems of Taylor and MacLaurin, before getting acquainted with the works of John Landen. But unfortunately he could not realise this intention of his. It should also be mentioned, that, as will be clear from the further description of manuscript 4302, Landen's proof would not have fully satisfied Marx, as Landen proceeded from the presupposition about the expansibility, and besides univocal expansibility of $(a + x)^p$ into a series in ascending and integral powers, whereas Marx thought that it is essential to substantiate such an expansibility. And he did it for an integral and positive p, referring to the distributivity of multiplication in respect of addition, when with this aim in view Marx presented $(x + h)^6$ in the form of

$$(x + h)\,(x + h)\,(x + h)\,(x + h)\,(x + h)\,(x + h).$$

The English translation of the 1828 edition of Boucharlat's book, used by Marx, did not contain the "Appendix I " to its 5-th edition, devoted to a "proof of Newton's formula with the help of differential calculus ".

173 It is clear from what follows, that here Marx has in view only the method of searching the first derivative, consisting of the following : if

$$f(x + h) = A + Bh + Ch^2 + Dh^3 + \cdots,$$

where A, B, C ..... are functions only of x, then after assuming $h = 0$ we notice that $A = f(x)$, and thus (for $h \neq 0$) we go over to the equality

$$\frac{f(x + h) - f(x)}{h} = B + Ch + Dh^2 + \cdots,$$

and in the right hand side we obtain the "preliminary" derivative, i.e., an expression, in which it is enough to assume $h = 0$, for obtaining the derivative of $f(x)$. Here the question of the methods of "freeing" the further derivatives is yet to be considered.

174 Here Marx wants to say, that in equation 2a) the coefficients of the powers of h are themselves not the successive derivatives of $(x + h)^m$, but are some fractional parts of them.

175 Since it is not clear why in place of the functions $f'(x)$, $f''(x)$ etc., their symbolic equivalents $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc., may not be simply substituted, it is evident that here Marx

wanted to say something else. Basing ourselves upon what he wrote afterwards, it is natural to assume, that here we have another case of a slip of pen, and that what he intended to say consisted of the following:

If we have only the expansion

$$f(x+h) = f(x) + Ah + Bh^2 + Ch^3 + \cdots \tag{1}$$

with the indeterminate coefficients A, B, C, ... , then we must still "with the help of differentiation"— here : formation of the ratio

$$\frac{f(x+h)-f(x)}{h} = A + Bh + Ch^2 + \cdots$$

and then the assumption of $h = 0$ — at first establish that $A = f'(x)$, and then obtain from the expansion of $f(x+h)$, the expansion of $f'(x+h)$:

$$f'(x+h) = f'(x) + 2Bh + 3Ch^2 + \cdots, \tag{2}$$

whence

$$\frac{f'(x+h)-f'(x)}{h} = 2B + 3Ch + \cdots$$

and, further, for $h = 0$

$$f''(x) = 2B, \text{ i.e., } B = \frac{1}{1\cdot 2}f''(x), \text{ etc.}$$

Only after this shall we be able to substitute for the indeterminate coefficients A, B, C, ... their "symbolic equivalents", i.e., the expressions involving the symbols $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\cdots$ (and the numerical coefficients).

176 Let us recall, that as in Euler's calculus of zeros (see : Appendix, p. 316), here the "removed" (through the assumption of $x_1 = x$) differences $(u_1 - u)$ and $(v_1 - v)$ (in Marx's notation) are mutually "equal", if their ratio is equal to one, i.e., the limit of the ratio $\frac{u_1 - u}{v_1 - v}$ is equal to 1, as $x_1 \to x$.

177 See : Appendix, "Theorems of Taylor and MacLaurin and Lagrange 's Theory of Analytical Functions in the sources consulted by Marx", p. 333.

178 In the original, this entire paragraph contains a great many cancellations. It is difficult to say, how it should be read. For example, may it not be interpreted as : "It is never viewed as taking only one [determinate] particular value a" ? Apparently, such an interpretation must correspond to Marx's understanding of the words "generally speaking" in Lagrange's demonstration.

INDEX OF QUOTED AND MENTIONED LITERATURE

Alembert J. d', Traité de l'équilibre et du mouvement des fluides. Paris. 1754. — *66, 67, 80.*

Becker O., Hofmann J.E., Geschichte der Mathematik, Bonn, 1951. — 333.

Boetius Anicius Manlius Severnius, Anicii Manlii Torquati Severini Boetii De institutione arithmeticae libri duo. Ed. G.Friedlein. Leipzig, 1867. — 112.

Boucharlat J.-L., Élémens de calcul différential et de calcul intégral, 5-me ed., Paris, 1838. English translation of the 3rd French ed.: An elementary treatise on the differential and integral calculus, Cambridge-London, 1828. —*4, 33, 60, 119-121*, 123, 124,133, 134, 135, 136, 145, 148, 151, 154-158, 160, 161, 163, 166, 214, 218, 219, *225*, 227-228, 229, 230, 233, 236, 237, 239, 241-242, *243*, 244, 260-262, *263*, 265, 287, 288, 293, 303, 308-309, 326-332, 335-339.

Cantor M., Vorlesungen über Geschichte der Mathematik, Bd.3, 2.Aufl., Leipzig,1901. — 333.

Devely (I.E.L.), Algèbre d'Emile, 2 vol., Genéve, 1805. — 167.

Engels F., Anti-Dühring, Progress, Moscow, 1978. — 1, 3, 4.

Engels F., Dialectics of Nature, Progress, Moscow, 1976. — 8.

Euclid, Euclid's Elements, in 3 Volumes, M.-L., 1949-1950. — *189, 202.*

Euler L., Élémens d'algèbre, Lyon, 1795. — 168, *170*, 172, *200*, 201.

Euler L., Institutiones calculi differentialis cum ejus usu in analysi finitorum ac doctrina serierum, Berlin 1755. Russian translation: Differentsialnoe ischeslienie, M.-L., 1949. — *66*, 147, 316-319, 333, 335.

Euler L., Introductio in anaysin infinitorum, Laussane, 1748. —*66.*

Feller F.E., Odermann C.G., Das Ganze der kaufmännischen Arithmetik, 7 aufl., Leipzig, 1859. — 2, 116, *117*, 122, *151.*

Foster J.L., An essay on the principle of commercial exchanges, London, 1804. — *117.*

Franklin, F.,Matematicheskii analiz, M., 1950. — 147.

Goodwin H., An elementary course of mathematics, 4th ed., Cambridge, 1853. — 189, 197.

Goschen G. L.,The theory of the foreign exchanges,8th ed., London, 1875. — 116.

Hall Th. G., A treatise on plane trigonometry, London,1833. — 241.

Hall Th. G., A treatise on the differential and integral calculus, and the calculus of variations, 5th ed., London, 1852. — 134, 151, 157, 158, 166, 214, 219, 229, 230, 237, 242, 261, *262*, 263, 265.

Hall Th. G., The elements of algebra, 3rd ed., Cambridge, 1850. —189, 194, 196, 199, 210.

Halley E., Methodus nova, accurata et facilis inveniendi radices aequationum quarumcunque generaliter, sine praevia reductione, "Philosophical Transactions of the Royal society of London", London, 1694. — 200.

Hausner O., Vergleichende Statistik von Europa, 2 Bde., Lemberg, 1865. — *117.*

Hemming G.W., An elementary treatise on the differential and integral calculus, Second ed., Cambridge, 1852. — *213*, 214, 228, 242, 243, 272.

Hind J., The principles of the differential calculus; with its applications to curves and curved surfaces, 2nd ed., Cambridge, 1831. — *81*, 123, 133, *135*, 148, 151, 155, 156, 157, 214, 217, 225, 227, 228, 230, 233, 237, 239, 240, 241, 242, 243, 245, 246, 259, 261, 265, 289, 290, 293, 303-308, 320, 335.

Hind J., The elements of algebra, 4th ed., Cambridge, 1839. — 187, 199.

Hind J., The elements of plane and spherical trigonometry, 3rd ed., Cambridge, 1837. — 186-188, 212, 239, 240, 241.

Hymers J., A treatise on conic sections and the application of algebra to geometry, 3rd ed., Cambridge, 1845. — 119.

Juschkewitsch A.P., Euler und Lagrange über die Grundlagen der Analysis. In : "Sammelband zu Ehren des 250 Geburtstages Leonard Eulers", Berlin, 1959. — 317-318.

Knopp K., Theorie und Anwendung der unendlichen Reihen. 2 Aufl., Berlin, 1924. — 301.

Lacroix S.F., Traité du calcul différentiel et du calcul intégral. 3 vol., seconde éd., Paris, 1810-1819. — 9, 146, *152*, 153, 171, 262, 305, 309, 310, 320, 325.

Lacroix S.F., Traité élémentaire de calcul différentiel et de calcul intégral, 2-e ed., Paris, 1806. Eng. tr. : An elementary treatise on the differential and integral calculus, Cambridge, 1816. — 133, 146.

Lacroix S.F., Élémens d'algèbre, 11-mé ed., Paris, 1815. — 167-168, 171, 177, 179, *180*, 184, 185, 186, 187, 214, 228, 229, 235.

Lacroix S.F., Complément des éléments d' algèbre, 4- e éd. Paris, 1817. 7- me éd., Paris, 1863. — 198, 199, 325.

Lagrange J. L., Théorie des fonctions analytiques, Paris, 1813. Or in : Oeuvres de Lagrange, Vol. IX, Paris, 1881. — *66, 67, 80*, 311-312, 325.

Lagrange J. L., Nouvelle méthode pour résondre les équations, litterales par le moyen des séries, " Mémoires de l'Académie royale de Sciences et Belles Lettres de Berlin", Vol. XXIV. 1770. — 134.

Landen J., The residual analysis, London, 1764. — *90*, 311, 320-325.

Landen J., A discourse of the residual analysis, London, 1758. — 325.

Lhuilier S., Principiorum calculi differentialis et integralis expositio elementaries, Tübingen, 1795. — 333.

MacLaurin C., A treatise of algebra in 3 parts, 1st ed., 1748; 6th ed., London, 1796. — 178, 185, *192, 193*, 194, 197, 201, 202, 203, *205, 206*, 207, *208, 209*, 210, *211, 227, 235*, 336.

MacLaurin C., Geometria organica, sive descriptio linearum curvarum universalis, London, 1720. — *229*.

Marks K., Matematicheskie rukopisi, in : " Pod Znamenem marksizma", No. 1, 1933 and, in : the collection " Marksizm i estestvoznania", M., 1933. — 1, 137.

Marks K., O poniyatii funktsii, in : "Voprosy filosofii", No. 11, 1958. — 171ff.

Moigno F., Leçons de calcul différentiel et de calcul intégral, rédigées d' aprés les méthodes et les ouvrages publiés ou inédits de Mr. L. A. Cauchy, 2 Vol., Paris, 1840 et 1844. — *66*.

Natanson I.P., Proizvodnye, integraly i riady, v kn. "Entsiklopedia elementarnoi matematiki", T.III, Gostekhizdat, M.-L., 1952. —301.

Newton I., Philosophiae naturalis principia mathematica, London, 1687. Russian ed.: Newton I., Matematicheskie nachala naturalnoi filosofii, perevod s lat. s poiyasn.i primech. A.N. Krylova, Izvestia Nikolaevskoi morskoi akademii, Spb., 1915-1916. — 4, *66, 67*, 313-315.

Newton I., Arithmetica universalis, sive de compositione et resolutione arithmetica liber, Cambridge, 1707. — *90, 192, 210, 228, 232, 234.*

Newton I., Analysis per quantitatum series, fluxiones et differentias, cum enumeratione linearum tertii ordinis, 1711. — *66, 67.*

chnymm chislom chlenov. V sb. : Isaak Newton, Matematicheskie raboty, M.-L., 1937. — 4, 126, 146.

Poppe J. H.M., Geschichte der Mathematik seit der ältesten bis auf die neuste Zeit, Tübingen, 1828. — 109-112.

Potts R., Elementary algebra, 1880. — 190.

Sadler M.Th., Ireland; its evils and their remedies, 2nd ed. London, 1829. — *118.*

Sauri, Cours complet de mathématiques. 5 Vols. Paris, 1778. — 4, 113, 115, 119, 121, 122, 123, 124, 125, 126, 132, 158, 166, 229, 230, 237, 242.

Spehr Fr. W., Vollständiger Lehrbegriff der reinen Kombinationslehre etc. Braunschweig, 1824. — 192, 212.

Stern M.A., Lehrbuch der algebraischen Analysis, 1860. — 212.

Struik D.J., Kratkii ocherk istorii matematiki, M., 1964. — 333.

Thibaut B., Grundriss der allgemeinen Arithmetik oder Analysis zum Gebrauch bei akademischen Vorlesungen, Götingen, 1809. — 191, 192, 195, 212.

Tseiten G.G., Istoria matematiki v XVI i XVII vekakh, M.-L.,1938. — 333.

Vileitner G., Istoria matematiki ot Dekarta do serediny XIX stoletia, M., 1960. —333.

Wood J., Elements of algebra, 3rd ed., Cambridge, 1810. — 213.

NAME INDEX

SPECIAL SUPPLEMENT

MARX AND MATHEMATICS

INTRODUCTION

This special supplement to Marx's *Mathematical Manuscripts* has grown out of an attempt to write a new preface to it. The supplement has three parts.

PART ONE : HISTORY, contains materials pertaining to the history of evolution of Marx's mathematical investigations, to that of the work leading to the publication of their results and, a bibliography, listing the majority of existing publications the field. The bibliography is not exhaustive.

PART TWO : INVESTIGATIONS, contains four articles inspired by Marx's mathematical manuscripts. Three of them are translations from Russian. They relate Marx's mathematical investigations respectively to : the history of analysis, the study of history of mathematics in the erstwhile USSR and, the logic of Marx's *Capital.* The fourth article provides an outline of the problem of situating Marx's mathematical manuscripts in the history of ideas as a whole.

Parts one and two reflect the past — the work that has already been done. The question arises : where do we go from here ? Marx conducted his investigations in the 19th century, basing himself mainly upon the developments in analysis upto the end of 18th-beginning of 19th centuries (e.g., upto the time of Lagrange). We are living in the last decade of the 20th century. In Marx's lifetime, and even a couple of decades after his death — right upto the first decade of this century — practically, there existed *only one* (now called the classical) trend in mathematical analysis, but now there are *many* (classical, intuitionist, constructivist and non-standard — some of them overlap and, branch out into multiple sub-trends). Indeed, the fact that even to-day we speak of mathematics in the singular, reflects a particular view of it. We must also train ourselves to speak, in the plural, of the mathematics*es*. [This graaphic barbarism has been intoroduced to jolt the reader out of the prevalent singular use of the word *mathematics*, which is morphologically both plural and singular in English. In Bengali and Russian we have greater graphic clarity : *gonit* > *gonitsamuha* ; *matematika* > *matematiki.*] The ontological and epistemological consequences of these more recent developments in the history of mathematics are profound : that which was considered to be *one* has also revealed itself as *many.* The entire problematique of truth and certainty has entered into an era of radical reconstruction all over again. At long last, the strongest bastion of theoretical dogmatism — the monopoly of the classical mathematical paradigm — is crumbling before our own eyes. It is clear, that is why, that there is no point in beginning our investigations from just where Marx left his work unfinished — when he died in 1883. To-day, even a survey of the post-Lagrange developments must base itself upon the contemporary attainments, the frontiers of which are being extended daily, hourly.

PART THREE : MATHEMATICSES, has been planned to provide the reader of this volume with a perspectival update on the relevant developments. Owing to reasons beyond the control of the present author here the cut year is 1987 — when all the five articles included in this part were published in Russian, as part of the proceedings of a symposium on "The Regularities and Modern Tendencies of the Development of Mathematics", held in September 1985, in Obninsk.

All the articles included in this supplement are, to my knowledge, being published in English for the first time. All translations from Russian are mine. It will be noted that all the articles

herein included — save my own — are translations from Russian. A number of factors determined this choice. A vast amount of relevant work exists in Russian, and our readers are generally unaware of them — this situation requires correction. Where a collective effort is needed, I was constrained to work single-handedly, without any kind of institutional support. I have deliberately excluded the materials already available in English, since our readers have greater access to them, thanks to our historical contacts with the English-using world. Thus, a very relevant article by *John Kadvany*, A Mathematical Bildungsroman // *History and Theory*, 1, 1989, pp. 25-42, which should otherwise have been reprinted in part three of this supplement, remains excluded. I strongly recommend it for any reader of Marx and Mathematics. I have heard that the Italian publications on Marx's mathematical manuscripts are really very good; but I do not read Italian ; in fact, apart from Russian I do not read any other European language. This personal limitation has also contributed to the inadequacies of this supplement. Let us hope that in future some one will make the other relevant materials accessible to us.

The limitations of the present volume and of this supplement will be overcome with the publication of a better and complete edition of all the mathematical manuscripts of Marx (some 400 pages of them still remain unpublished), as well as with the publication of newer and newer studies on them, executed with ever greater competence ; but more importantly, we must join hands in opening up new frontiers in mathematical theory and practice and, in the cognate disciplines (e.g., in informatics) and technologies, and thus carry forward the critical and transformative spirit embodied in the mathematical manuscripts of Karl Marx .

Calcutta, June 15, 1993. Pradip Baksi.

PART ONE : HISTORY

ON THE HISTORY OF KARL MARX'S MATHEMATICAL MANUSCRIPTS

Letters (excerpts)

Marx's mathematical investigations have been discussed in a number of letters of Marx and Engels. English translations of the relevant sections of some of the hitherto published relevant letters are being reproduced here.

Marx to Engles

In Manchester

[London,] 11 January [1858]

Dear Frederick,

*   *   *

In elaborating the PRINCIPLES of economics[1]* I have been so damnably held up by errors in calculation that in DESPAIR I have applied myself to a rapid revision of algebra. I have never felt at home with arithmetic. But by making a detour via algebra, I shall quickly get back into the way of things.

*   *   *

your

K.M.

[MECW(E), 40, 244]

Marx to Engels

In Manchester

London, 6 July 1863

Dear Engels,

*   *   *

... My spare time is now devoted to differential and integral calculus. Apropos, I have a superfluity of works on the subject and will send you one, should you wish to tackle it. I should consider it to be almost essential to your military studies. Moreover, it is a much easier branch of mathematics (so far as mere technicalities are concerned) than, say, the more advanced aspects of algebra. Save for a knowledge of the more ordinary kind of algebra and trigonometry, no preliminary study is required except a general familiarity with conic sections .

*   *   *

your

K.M.

[MECW(E), 41, 484]

*For the notes see pp.402-403 of this supplement. — Ed.

Engels to Friedrich Albert Lange[2]

In Duisburg

Manchester, 29 March 1865
7 Southgate

Dear Sir,

* * *

There is a remark about old Hegel which I cannot let pass without comment : you deny him any deeper knowledge of the mathematical sciences . Hegel knew so much mathematics that none of his disciples was capable of editing the numerous mathematical manuscripts that he left behind[3]. The only man who, to my knowledge, has enough understanding of mathematics and philosophy to be able to do so is Marx.

* * *

Yours very respectfully
Friedrich Engels

[MECW(E), 42, 138]

Marx to Engels

In London

[Manchester,] 31 May 1873
25 Dover Street

Dear Fred,

* * *

I have been telling Moore[4] about a problem with which I have been racking my brains for some time now. However, he thinks it is insoluble, at least *pro tempore*, because of the many factors involved, factors which for the most part have yet to be discovered . The problem is this : you know about those graphs in which the movements of prices, discount rates, etc., etc., over the year etc., are shown in rising and falling zigzags. I have variously attempted to analyse crises by calculating these UPS AND DOWNS as irregular curves and I believed (and still believe it would be possible if the material were sufficiently studied) that I might be able to determine mathematically the principal laws governing crises[5]. As I said, Moore thinks it cannot be done at present and I have resolved to give it up FOR THE TIME BEING.

* * *

your
K.M.

[MECW(E), 44, 504.]

Engels to Marx

In London

August 18, 1881

Dear Moor[6],

. . . yesterday I found the courage at last to study your mathematical manuscripts[7] even without reference books, and I was pleased to find that I did not need them. I compliment you on your work. The thing is as clear as daylight, so that we can wonder enough at the way the mathematicians insist on mystifying it. But this comes from the one-sided way these gentlemen think . To put $\frac{dy}{dx}=\frac{0}{0}$, firmly and point blank, does not enter their skulls. And yet it is clear that $\frac{dy}{dx}$ can only be the pure expression of a completed process if the last trace of the *quanta* x and y has disappeared, leaving the expression of the preceding process of their change without any quantity.

You need not fear that any mathematician has preceded you here. This kind of differentiation is indeed much simpler than all others, so that just now I applied it myself to derive a formula I had suddenly lost, confirming it afterwards in the usual way. The procedure must have made the greatest sensation, especially, as is clearly proved, since the usual method of neglecting *dxdy* etc. is *positively false*. And that is the special beauty of it ; only if $\frac{dy}{dx}=\frac{0}{0}$, *is* the mathematical operation absolutely correct.

So old Hegel guessed quite correctly when he said that differentiation had for its basic condition, that the variables must be raised to different powers, and at least one of them to at least the second, or $1/2$, power[8]. Now we also know why.

If we say that in $y=f(x)$ the x and y are variables, then this claim has no further consequences, as long as we do not move on, and x and y are still *pro tempore*, in fact constants. Only when they really change, i.e., *inside the function*, do they indeed become variables, and only then can the relation still hidden in the original equation reveal itself — not the relation of the two magnitudes but of their variability. The first derivative $\frac{\Delta y}{\Delta x}$ shows this relation as it happens in the course of a real change , i.e., in each *given* change ; the completed derivative $\frac{dy}{dx}$ shows it in its generality, pure, and hence we can come from $\frac{dy}{dx}$ to each $\frac{\Delta y}{\Delta x}$, while the latter itself only covers the special case. However, to pass from the special case to the general relationship , the special case must be abolished (*aufgehoben*) as such.

Hence, after the function has passed through the process from x to x' [in the notation now in use : to x_1 — Ed.] with all its consequences, x' can be allowed calmly to become x again ; it is no longer the old x, which was variable in name only; it has passed through *actual change*, and the *result* of the change remains, even if we again abolish (*aufheben*) it.

At last we see clearly, what mathematicians have claimed for a long time, without being able to present rational grounds, that the (derivative or) *differential quotient* is the original, the differentials dx and dy are derived : the derivation of the formulae demands that both so-called irrational [here "irrational" means "non-rational" — Ed.] factors stand at the same time on one side of the equation, and only if you put the equation back into its first form $\frac{dy}{dx} = f'(x)$, as you can, are you free of the irrationals [i.e. "non-rationals" — Ed.] and instead have their rational expression.

The thing has taken such a hold of me that it not only goes round my head all day, but last week in a dream I gave a chap my shirt-buttons to differentiate, and he ran off with them.

Yours

F.E.

[Mathematical Manuscripts of Karl Marx, London, 1983, XXVII-XXVIII.]

Engels to Marx

In Ventnor

London, November 21, 1882

Dear Moor,

... enclosed a mathematical essay by [Samuel] Moore. The conclusion that "the algebraic method is only the differential method disguised" refers of course only to his own method of geometrical construction and is pretty correct there too. I have written to him that you place no value on the way the thing is represented in geometrical construction, the application to the equation of curves being quite enough. Further, the fundamental difference between your method and the old one is that you make x change to x' [in the notation now in use : to x_1 — Ed.], thus making them really vary, while the other way starts from $x + h$, which is always only the sum of two magnitudes, but never the variation of a magnitude. Your x therefore, even when it has passed through x' and again became the first x, is still other than it was; while x remains fixed the whole time, if h is first added to it and then taken away again. However, every graphical representation of the variation is necessarily the representation of the *completed* process, of the *result*, hence of a quantity which became constant, the line x; its supplement is represented as $x + h$, two pieces of a line. From this it already follows that a graphical representation of how x' again becomes x, is impossible ...

Your

F.A.

[Mathematical Manuscripts of Karl Marx, London, 1983, p. XXIX.]

Marx to Engels

In London

November 22, 1882
1, St. Boniface Gardens,
Ventnor.

Dear Fred,

. . . Sam, as you saw immediately, criticises the analytical method applied by me by just pushing it aside, and instead busies himself with the geometrical application, about which I said not one word. In the same way, I could get rid of the development of the proper so-called differential method — beginning with the mystical method of Newton and Leibnitz, and then going on to the rationalistic method of d'Alembert and Euler, and finishing with the strictly algebraic method of Lagrange (Which, however, always begins from the same original basic outlook [as that of] Newton-Leibnitz) — I could get rid of this whole historical development of analysis by saying that *practically* nothing essential has changed in the geometrical application of the differential calculus, i.e., in the geometrical representation.

The Sun is now shining, so the moment for going for a walk has come, so no more *pro nunc* of mathematics, but I'll come back later to the differential methods occasionally in detail . . .

Yours

K.M.

[Mathematical Manuscripts of Karl Marx, London, 1983, p. XXX.]

Reminiscences (excerpts)

Marx's friends and comrades have mentioned his mathematical studies in their reminiscences or elsewhere. English translations of the relevant portions of some such writings are being published here.

Engels' Speech At Marx's Funeral (Excerpts)

*** *** ***

. . . in every single field, wherever Marx has conducted investigations and, it goes without saying that the fields of his investigations were many and variagated, and no where did he tackle the task of investigation superficially— in every field, even in mathematics , he has obtained independent results.

*** *** ***

17 March 1883

[Marx- Engels Smriti, Pragati, Moscow, 1976, p.6 (in Bengali)]

From The Preface To The Second Edition Of *Anti-Dühring*

*** *** ***

Marx and I were pretty well the only people to rescue conscious dialectics from German idealist philosophy and apply it in the materialist conception of nature and history. But a knowledge of mathematics and natural science is essential to a conception of nature which is dialectical and at the same time materialist . Marx was well versed in mathematics, but we could keep up with the natural sciences only piecemeal, intermittently and sporadically.

*** *** ***

. . . to me there could be no question of building the laws of dialectics into nature, but of discovering them in it and evolving them from it.

But to do this systematically and in each separate department,is a gigantic task. Not only is the domain to be mastered almost boundless; natural science in this entire domain is itself undergoing such a mighty process of being revolutionised that even people who can devote the whole of their spare time to it can hardly keep pace. Since Karl Marx's death, however, my time has been requisitioned for more urgent duties, and I have therefore been compelled to lay aside my work. For the present, I must content myself with the indications given in this book, and must wait to find some later opportunity to put together and publish the results which I have arrived at, perhaps in conjunction with the extremely important mathematical manuscripts left by Marx[9].

*** *** ***

London, September 23, 1885

[From : Engels, F. *Anti-Dühring* (Eng. Ed.), Progress, M., 1978, pp. 15-16, 18.]

From Paul Lafargue's Reminiscences Of Karl Marx

*** *** ***

Apart from studying the works of poets and novelists, the other astonishing means that Marx discovered for giving rest to his brain was studying mathematics. He had a special kind of inclination to mathematics. Algebra even provided him with mental solace, he used to study this branch of mathematics during the most painful moments of his eventful life. He could not concentrate upon his usual scientific activities during the last days of his wife's illness, in those days he could forget the pain that seared his mind owing to her illness, only by studying mathematics. During these days of mental agony he wrote a treatise on infinitesimal calculus. Experts are of the opinion that this work of him is of immense scientific value. In higher mathematics Marx saw the most consistent and at once the most simple expression of dialectical movements. He was of the opinion, that so long as a science does not get used to the use of mathematics, it can not be called a truly mature science.

*** *** ***

[First published in 1890.]

[Marx-Engels Smriti, Pragati, Moscow, 1976, pp. 31-32 (in Bengali).]

A NOTE
ON THE HISTORY OF COLLECTING, DECIPHERING, EDITING AND PUBLICATION OF MARX'S MATHEMATICAL MANUSCRIPTS

PRADIP BAKSI

Karl Marx died in 1883. A partial edition of his *Mathematical Manuscripts* came out in print 85 years after his death, in 1968. During the first years of this intervening period, these manuscripts arrived in Germany, together with the other unpublished manuscripts of Marx and Engels, from England. These manuscripts became the property of the Social Democratic Party of Germany[10]. We know that Engels considered Marx's mathematical manuscripts to be important. He expressed his desire to publish them, in his preface to the second (1885) edition of *Anti-Dühring*. This wish remained unfulfilled during his life time. Frederick Engels died in 1895. After his death, it was the Social Democratic Party of Germany that became primarily responsible for publishing the said manuscripts. This party failed to fulfill this responsibility. What is more, one of the important leaders and theoreticians, first of the Social Democratic Party and subsequently of the Communist Party of Germany, Franz Mehring (1846-1919) declared, at the behest of some mathematicians (of whom exactly, we do not know), that Marx's mathematical manuscripts are of no importance [19, p. 14 ; all references in this article are to the entries in the Bibliography appended at the end of Part One of this Supplement, pp.404-408]. Before the Russian revolution of 1917, a Russian revolutionary emigrant David Borisovich Ryazanov (Goldendakh) (1870-1938) worked for some time in the archives of the Social Democratic Party, in Berlin. At that time he noticed that a part of Marx's mathematical manuscripts was not there in the archives. He located them at the residence of an important leader of the Social Democratic Party, Eduard Bernstein (1850-1932). Subsequently Ryazanov approached a leader of the Austrian Social Democratic Party Frederick Adler (1879-1960) and, requested him to take the initiative for publishing the mathematical manuscripts of Marx. Ryazanov's attempt too failed to bear any fruit, but in the process ten mathematical note-books of Marx went into the personal custody of Adler.

After the revolution of 1917 a Marx-Engels Institute was established in Moscow and, Ryzanov was appointed its first director. This institute of Moscow acquired a contractual right to photo-copy the manuscripts of Marx and Engels, from the archives of the Social Democratic Party of Germany. In persuance of this contract Ryazanov and his colleagues demanded the mathematical manuscripts of Marx for photo-copying purposes. It was only then that the authorities of the archives of the Social Democratic Party of Germany could recover the aforementioned ten note-books of Marx from Adler. Ryazanov's efforts in this direction were reported in the July 1924 issue of "*Inprekor*" published from Vienna [64]. At long last in 1925 the Marx-Engles Institute of Moscow succeded in obtaining the photocopies of 865 pages of Marx's mathematical manuscripts [16,p.56.]. A German mathematician E. Gumbeil was already acquainted with these manuscripts. He was brought to Moscow and given the task of editing them. R. Mateika and R.S. Bogdan helped him in the task of deciphering the texts. In 1927 Gumbeil declared that the press-copy of Marx's mathematical manuscripts was ready [ibid]. However, some other associates of the Institute (for instance,

E. Kolman) had a different opinion in this regard [see : 9, p.185 and 29, p.101]. Ryazanov, the first director of the Institute, was expelled from the C.P.S.U in 1931. He was killed in 1938. Vladimir Viktorovich Adoratsky (1878-1945) became the next director of the Institute. Gumbeil was removed and, in his place the task of editing the mathematical manuscripts of Marx was given to Sofya Aleksandrovna Yanovskaya (1896-1966). Initially she was assisted by D.A. Rykov and A.E. Nahimouskaya. In 1933, on the occasion of the 50th death anniversary of Karl Marx, two of his articles on the nature of differential calculus and, an editorial article of Sm. Yanovskaya were published in the journal "*Pod Znamenem Marksizma*" and in a collection of essays entitled "*Marksizm i estestvoznanie*". Upto 1968, those who were interested in the said mss had to remain contented mainly with these publications. After 1932, a Swedish mathematician Wildhaber remained associated, for sometime, with the Moscow-team working on the mathematical manuscripts of Marx. A member of the Soviet delegation to the Tenth International Congress of Mathematicians (1932) declared in one of the sessions of the congress that the entirety of Marx's mathematical writings are going to be published soon [27]. This promise too remained unfulfilled. Arrived the Second world war. During the War the archives and the library of the Institute were shifted to some place in the Soviet Far East. Work of the Institute slowed down. After the War the pace of work picked up slowly. However, Sm. Yanovskaya — the editor of Marx's mathematical manuscripts — was also required to cope with the teaching of mathematical logic in the Moscow University and, with the task of translating and editing text-books of mathematical logic. Her health began to deteriorate. A Congress of Mathematicians was organised in 1950 at Budapest. Here the delegates from all the other socialist countries repeatedly asked the members of the Soviet delegation : when, at long last, are they going to publish the mathematical manuscripts of Marx ? The members of the Soviet delegation had no definite answer to this question [19, pp.205-206]. After the return of this delegation to the USSR, the responsible authorities began to take more vigorous steps. Now Sm. Yanovskaya was given a new assistant: Konstantin Alekseievich Rybnikov. An important event of the 1950s was the publication of a note of Marx entitled "*On The Concept Of Function*" in the journal "*Voprosy Filosofii* " No. 11, 1958)[11]. As the work entered into the 1960s, it was noticed that the manuscripts are opaque in quite a few places (similar problems were being encountered by the editors of many other manuscripts of Marx and Engels). That is why the question of verifying the text once more from the originals was posed with some urgency. Meanwhile — in fact before the Second World War — on the 19th of May 1938, the entire archives of the Social Democratic Party of Germany, inclusive of the manuscripts and letters of Marx and Engels, has become the property of the International Institute of Social History, Amsterdam (see : note [10]). That is why, a Soviet delegation was sent to Amsterdam in August 1964, in search of the necessary papers [19, p. 207]. A member of this delegation was Sm. Olga Konstantinovna Senekina. She noticed, that when — during 1924-1930 — the team of Soviet workers under the guidence of D.B. Ryazanov, was busy in photo-copying some 55,000 pages of the writings of Marx and Engels, then quite a few pages of Marx's mathematical manuscripts remained un-photocopied, due to inadvertence. Now the photo-copies of those pages were obtained. This pushed up the volume of the hitherto known mathematical manuscripts of Marx to nearly 1000 pages (see : the Preface to the 1968 edition of these mss). The editor of the 1968 edition

of Marx's *Mathematical Manuscripts,* Sm. S.A. Yanovskaya died in 1966. Two years after her death, in 1968, a 639 page edition of Marx's mathematical manuscripts was brought out under the joint supervision of K.A. Rybnikov, O.K.Senekina and A.Rivkin. In the Preface to this edition, the work that went on around these mss prior to 1931, remains unreported. However, the work of the 1933-1968 period has been described in detail [PV,14-15]. An even more important "silence" of this Preface involves the fact that it is an incomplete edition. This becomes evident to anyone who cares to go through this edition. Why were certain pages of Marx's mathematical manuscripts dropped ? Apparently, the persons responsible for the publication considered these pages to be mathematically insignificant. It goes without saying, that such an approach towards the manuscripts of dead authors is — to say the least — unhistorical.

It may be mentioned here that the work of the Institute of Marxism-Leninism of the CPSU (which evolved out of the Marx-Engels Institute of Lenin's time) has always been affected by the power-struggle within the top leadership of that party. Beginning with Ryazanov, many competent workers were removed and killed. Those who were spared the hospitality of labour camps or death, left or were forced to leave the country. The noble work of editing and publishing the works of Marx and Engels became the subject-matter of self-promotion and intrigues of mean-minded and incompetent persons. In an atmosphere of all-round decadence of Soviet barrack socialism, quite a few generations of that society lost all interest in Marxism, thanks to the criminal activities of the party and state leaders. Those who retained some honest interest failed to remain in the good books of the powers that be. In these circumstances, the 1968 partial edition of Marx's mathematical manuscripts was left just like that for about two decades. Even more truncated editions were brought out in some other countries (see : the bibliography). Words went around that the 1968 edition is by and large satisfactory and, that it would be included in that form in the contemplated complete works of Marx and Engels*. This "consensus" was broken in the wake of Perestroika. A decision was taken towards the end of 1987, to prepare a new and complete edition of Marx's mathematical manuscripts on the basis of all the hitherto available papers. At this stage, the task of editing was given to Sm. Irina Konstantinovna Antonova of the Institute of Marxism-Leninism, Moscow. Towards the middle of 1988 she told the present author that she hopes to complete her part of the job by the end of 1990. Then it would require the approval of the experts of the Institutes of Marxism-Leninism of Berlin and Moscow, before publication. These plans have been overtaken by larger movements of history. Tumultuous changes have taken place by the end of 1991. First the G.D.R. and then the U.S.S.R. has ceased to exist. The ruling communist parties of these two countries have been abolished and the Institutes of Marxism-Leninism controlled by them have been wound up. The publication of a complete edition of Marx's mathematical manuscripts as part of the projected complete works of Marx and Engels has now become a matter of uncertainty all over again. When, where and how they would be published in future or whether they would be at all published ever — who knows ?

---

* For a different opinion on this question see : *Malodshii* (PV, 423-424). — Ed.

Notes

1. In the summer of 1857 Marx began to write a series of economic manuscripts in order to sum up and systematize the results of his extensive economic research started in the 1840s and continued most intensively in the 1850s. (In the first half of 1850s he filled 24 paginated and several unpaginated notebooks with excerpts from the works of other economists, books of statistics, documents and periodicals.) These manuscripts were the preliminary versions of an extensive economic work in which he intended to investigate the laws governing the development of capitalist production and to criticise bourgeois political economy. Marx outlined the main points of this treatise in an unfinished draft of the 'Introduction' (one of the first manuscripts of the series) and in his letters to Engels, Lassalle and Weydemeyer. Further economic study prompted Marx to specify and change his original plan. The central place in the series is occupied by the extensive manuscript, *Critique of Political Economy* (widely known as *Grundrisse*), on which Marx worked from October 1857 to May 1858. In this preliminary draft of his future *Capital* Marx expounded his theory of surplus value. After the first instalment had been prepared for publication in 1859 under the title *A Contribution to the Critique of Political Economy*, Marx added several more manuscripts to the series in 1861.

   The manuscripts of 1857-61 were first published in German by the Institute of Marxism-Leninism of the C C C P S U in 1939 under the editorial heading *Grundrisse der Kritik der politischen Ökonomie (Rohentwurf)*. These manuscripts and *A Contribution to the Critique of Political Economy. Part One* are included in vols. 29 and 30 of the English Edition of the Collected Works of Karl Marx and Frederick Engels (Progress, Moscow, 1975-).

2. *Lange, Friedrich Albert* (1828-1875) : German philosopher, economist, neo-Kantian ; member of the Standing Committee of the General Association of German Workers (1864-66), member of the International, delegate to the Lausanne Congress (1867).

3. I began my inquiries about Hegel's mathematical manuscripts in 1980. In this connection, I received a letter from Dr. Helmut Schneider of the Hegel-Archiv, Ruhr-Universität Bochum. In this letter dated the 7th of February 1982, he stated that the mathematical manuscripts of Hegel are neither there in their archives, nor have these mss been published so far. — P. B.

4. *Moore, Samuel* (1838-1911) : English lawyer, member of the International, translated into English Volume One of *Capital* (in collaboration with Edward Aveling) and the *Manifesto of the Communist Party* ; friend of Marx and Engels.

5. In this connection see :

   a) *Zoltan, K.* Marx és a válságok törtvényeinek matematikai tanulmányozása [Marx and the mathematical investigations into the laws of crisis], *Közgazdasági Szemle*, Budapest, 1962, 12, pp. 1464-1483 ;

   b) *Dunayaeva, V.* K voprosu O matematicheskom metode v "Kapitale" K. Marksa [On the question of mathematical method in Karl Marx's "Capital"], *Voprosy Ekonomiki*, 1967, 8, pp. 18-30.

6. "Moor" was Marx's nickname in Marx family. His close friends also called him by that name [see : Marx-Engels Reminiscences. Bengali ed. Progress. M., 1976].
7. See : PV, 19 and 26 : "On the Concept of the Derived Function" and "On the Differential".
8. See : *Hegel, G.W.F.*, Science of Logic (Tr. : W.H. Jhonston and L.G. Struthers), G.Allen and Unwin, London, 1929 ; Volume One, Book One, Section Two, IIC — The Quantitative Infinity, pp. 241-332 [especially: IIC (c) — The Purpose Of The Differential Calculus Deduced From Its Application, pp. 291-320].
9. Here Engels expressed his desire to publish his *Dialectics of Nature* and Marx's *Mathematical Manuscripts* together. This wish of his remained unfulfilled. The *Dialectics Of Nature* was first published in 1925 and a part of Marx's *Mathematical Manuscripts* in 1933, an enlarged but nevertheless incomplete edition of the same came out in 1968.
10. For a general overview of the history of preservation, change of ownership and publication of the manuscripts, letters etc. of Marx and Engels, see : Saha, Dr. Panchanan — *Marx Engelser Pandulipi Kibhabe Raksha Pelo ?* Indo-GDR Friendship Society, Calcutta, 1983.
11. See : O Ponyatii Funktsii, *Voprosy Filosofii,* 1958, 11, pp. 89-95 & PV, 171-177.

BIBLIOGRAPHY

Different Editions Of Marx's Mathematical Manuscripts

1. *Matematicheskie Rukopisi,* Izd. "Nauka", Moscow, 1968 Ed. Sofya Aleksandrovna Yanovskaya; pp. 639 + illustrations. Language : original and Russian.
2. *Mathematische Manuskripte,* "Scripter Verlag", Kronberg Taunus (FRG), 1974 ; Ed. Wolfgang Endemann. Photocopy of the German portions of Part I of (1.) above (pp. 19-106 of the present volume).
3. *Manoscritti Matematici,* "Dedalo Libri", Bari. 1975 ; Ed. Francesco Matarrese and Agusto Ponzio. Italian translation of Part I of (1.) above, plus two editiorial essays.
4. *Mathematical Manuscripts Of Karl Marx,* New Park Publications Ltd., London, 1983; Ed. Cyril Smith. English translation of Part I of (1.) above, plus the English translations: of the Preface, Appendix, and the relevant Notes of (1.); of three relevant letters of Marx-Engels and, E. Kolman's review of (1.); of the essay "Gegel i matematika" ("Hegel and mathematics") (1931) by E. Kolman and S. A. Yanovskaya; and the article "Hegel, Marx and Calculus" by C. Smith.
5. *Les manuscrits mathématiques,* "Economica", Paris, 1985; Ed. A. Alcouffe. French translation of Part I of (1.) above, plus the editorial essay : Marx, Hegel, et le "calcul".

Books And Articles On Marx's Mathematical Manuscripts

1. *Alcouffe A.*, Marx, Hegel, et le "Calcul" // *Marx, K,* Les manuscrits mathématiques, Paris : Economica, 1985. pp. 9-109.
2. *Baksi P.*, Markser Gonit Vishayak Pandulipi Prasange (On Marx's mathematical manuscripts) // *Gonit,* Calcutta, 1983, 2(2), pp. 51-58.
3. *Baksi P.*, Karl Markser Gonit Vishayak Pandulipi O Tar Tatparya (The mathematical manuscripts of Karl Marx and their significance) // *Marksbad, Gonit O Tarkashastra* — a collection of essays, Jnananveshan, Calcutta, 1985, pp.30-65.
4. *Baksi P.* On the Problem of Situating Marx's Mathematical Manuscripts in the History of Ideas // Present Volume, Special Supplement, Part Two, last article.
5. *Basheleishvili B.,* K Voprosu o zakone edinstva protivopolozhnostiei v matematicheskikh rukopisyakh K. Marksa (On the question of the law of unity of opposites in the mathematical manuscripts of K. Marx) // Studencheskaya Nauchnaya Konferentsia, Tibilisi, 17-23 aprelia, 1952. Izd. Tibilisskovo gos universiteta im I. V. Stalina, 1952.
6. *Bottazzini U.,* Processi algebrici e processi dialettici nei Manoscritti Matematici di Marx, 1975. Unpublished. A later French version of it was published in : *Dijalektika,* Beograd, 1980, 3/4, pp. 69- 85.

7. *Budach L.*, Karl Marx und die Matematikeinmal anders gesehen // Karl Marx, die Berliner Universität und die Verantwortung für die Festigung des Friedens und des Sozialismus, Berlin 1983, s. 34-40.
8. *Burkhardt F.*, Marx und die Mathematik // Wiss. Zeitschr. der Humbold.-Univ. zu Berlin, Ges.- u. Sprachwiss, Reihe, 1968, Hft. 6, s. 863-866.
9. *Burkhardt F.*, Karl Marx und die Mathematik// Karl Marx "Das Kapital". Erbe und Verpflichtung. Leipzig, Karl Marx-Universität, 1968, s. 695-701.
10. *Casanova G.*, Karl Marx et les mathématiques// *La Pensée*, n. s. Paris, 1948, N. 20. pp. 68-72.
11. *Endemann W.*, Einleitung // *Marx, K.*, Mathematische Manuscripte, Kronberg ts. Scripter Verlag, 1974. s, 15-49.
12. *Gerdes P.*, Marx demystifies calculus // Studies in Marxism, vol. 16, 1985; MEP Publs. Minneapolis, USA. Reviewed in : *Science and Nature*, 7/8. pp. 119-123.
13. *Glivenko V. I.*, Poniatie Differentsiala u Marksa i Adamara (Marx and Hadamard on the concept of differential) // *Pod Znamenem Marksizma*, 1934, 5, str. 79-85; & PV,411-419.
14. *Gokieli L.P.*, Matematicheskie Rukopisi Karla Marksa i voprosy obosnovanya Matematiki (Mathematical Masnuscripts of Karl Marx and the Questions of the Foundations of Mathematics). Izd AN Gruzinskoi SSR Tibilisi, 1947. (A Georgian edition of the book was published in the same year.)
15. *Guerraggio A ., Vidoni F.*, Nel laboratorio di Marx : Scienze naturali e matematika. Milano : Angeli, 1982.
16. *Gumbeil E.*, O matematicheskikh rukopisyakh K. Marksa (On the mathematical manuscripts of K. Marx)// *Letopis Marksizma*, M.-L., 1927, 3, str. 56-60.
17. *Herzmann J., Tomek I.*, Marksova kritika metafyzickébo pojeti Základnich pojmů diferenciálniho poctu // *Filoz. Cas.*, Praha, 1981, 1, s. 78-93.
18. *Ibragimov S.*, Matematicheskie proizvedenya K. Marksa (Mathematical works of K. Marks)// *Azerbaijan Kommunist*, 1968, 1, str. 9-14.
19. *Katolin L.*, "Mee byli togda derzhkimi parnyami..." ("We were then the impudent twin.."). M. Izd. "Znanie", 1973 (2-oe izd. 1979).
20. *Kennedy H. C.*, Review of Karl Marx's *Matematische Manuscripte* (Scripter Verlag, F. R. G., 1974) and of *Manoscritti matematici di K. Marx* (Bari : Dedalo Libri, 1975)// *Historia Mathematica*, 1976, 3, pp. 490-494.
21. *Kennedy H. C.*, Karl Marx and the foundations of differential calculus// *Historia Mathematica*, 1977, 4, pp. 303-318; correction in : *Historia Mathematica*, 1977, 5, p.92.
22. *Kennedy H. C.*, Marx, Peano and the Differentials. Unpublished. Revised copy of a paper read at the 15th International Congress of the History of Science, Edinburg, August 1977.
23. *Kennedy H. C.*, Marx's Mathematical Manuscripts// *Science and Nature*, 1978, 1, pp. 59-62.

24. *Kholshevnikov A.*, O matematicheskikh rukopisyakh Marksa (On Marx's mathematical manuscripts)// *Front nauki i tekhniki*, 1933, 2, str. 100-106.

25. *Kisileva N. A.*, Karl Marks i Matematika// *Matematika v shkole*, 1969, 1, str. 5-10.

26. *Kolman, A.*, Karl Marks i matematika// *Dijalektika*, Beograd, 1968, 3, s. 27-41.

27. *Kolman E.*, Eine neue Grundlegung der Differentialrechnung durch Karl Marx// Verhandlungen des Internationalen Mathematiker-Kongress, Zurich, 1932; II Band/ Sektions-Vortrage, 1932, s. 349-351.

28. *Kolman E.*, Eine neue Grundlegung der Differentialrechnung durch Karl Marx // *Archeion. Archivio di storia della scienza*, 1933, 15, pp. 379-384.

29. *Kolman E.*, K. Marks i matematika (O 'Matematicheskikh Rukopisyakh' K. Marksa)// *Voprosy istorii estestvoznania i tekhniki*, 1968, 25, str. 101-102. For an English translation of the same see : *Mathematical Manuscripts of Karl Marx*, London, 1983.

30. *Kondakov N I.* (Ed.), Matematicheskie rukopisi Karla Marksa// Logicheskii Slovar-Spravochnik. M., Nauka. 1975, str. 341.

31. *Kuzicheva Z. A.*, *Rybnikov K. A.*, Matematika v nauchnykh issledovanyakh K. Marksa (Mathematics in the scientific investigations of K. Marx)// Istoriya i metodologiya estestvennykh nauk, Vyp. XXXII, Matematika, Mekhanika. Izd. Moskovskovo Un-ta, Avgust, 1986, str. 3-13.

32. *Loi M.*, Review of : Marx, K., *Les manuscrits mathématiques* (Paris : Economica, 1985)// *La Pensée*, Paris, 1987. 256, pp. 129- 131.

33. *Lombardo R. L.*, Dai 'Manoscritti matematici' di K. Marx// *Critica Marxista-Quaderni*, 1972, 6. pp. 273-277.

34. *Magani L.*, Marx matematico : Il fondamenti del calcolo differenziale// Conoscere Marx. Milano : Angeli, 1983, pp. 55- 89.

35. *Maiburova D.*, O matematicheskikh rukopisyakh K. Marksa// *Obshestvennye Nauki v Uzbekistane*, 1968,5, str, 26-30.

36. *Miller M.*, Karl Marx' Begrungung der differntialrechnung// Wiss. Z. Hochsch Verkehrswesen "Friedrich List". Dresden 16, DDR.,1969, s. 649-659.

37. *Molodshii V. N.*, O Matematicheskikh rukopisyakh K. Marksa// *Matematika v shkole*, 1969, 1, str. 10-23.

38. *Molodshii V. N.*, "Matematicheskie Rukopisi" K. Marksa i razvitie istorii matematiki v SSSR (Marx's "Mathematical Manuscripts" and the development of History of Mathematics in the USSR) // *Voprosy Istorii Estestvoznania i Tekhniki*, 1983, 2, str. 29-34; PV, 420-426.

39. *Mueller J.*, Karl Marx und die mathematik// *Tägliche Rundschau*, jg. 9, Nr. 72 (1953-54).

40. *Nahimouskaya A.*, Ab matematychnykh pratsakh K. Marksa// *Zapiski Bellaruskai Akademii Nauk*, kn. 1. 1933, str. 1-30.

41. *Novak I.P.*, O matematicheskikh rukopisyakh K. Marksa// Problemy Filosofii. Alma-Ata, 1968, str. 311-320.

42. *Panfilov V. A.*, Vliyanie materialisticheskoi dialektiki konechnovo i beskonechnovo na sposob differentsirovaniya K. Marksa (Influence of the materialist dialectics of the finite and the infinite on K. Marx's mode of differentiation). Dissertation. Dnepropetrovsk Gosudarstvenny Universitet, 1978. A copy of the mss exists in the INION Library, Moscow.

43. *Przelaskowski W.*, Prace matematyczne Karloa Marksa // *Economista*, Warszawa, 1960, 1, s. 159-165.

44. *Przelaskowski W.*, Pojecie funkcji w pracach Karloa Marksa// *Studia ekonomiczne*, Warszawa, 1960, 4, s. 187-203.

45. *Przhesmitsky V. I.*, Operativnyi logicheskii apparat rabotayuschii v "Kapitale" i "Matematicheskikh Rukopisyakh" K. Marksa (On the Operational Logical Apparatus operative in Karl Marx's "Capital" and "Mathematical Manuscripts") // Voprosy dialekticheskoi logiki, printsipi i formy myshleniya. M. 1985. s. 70-81; PV,427-434.

46. *Psheliaskovsky V.*, Matematicheskie metody v ekonomii v svete " Matematicheskikh Rukopisiei " Karla Marksa (Mathematical Methods in Economics in the light of the "Mathematical Manuscripts " of Karl Marx). Doklad na mezhdunar. ekon. konf. 5-8 Sent., 1972, Varshva.

47. *Piaskovskii B. V., Konvai B I.*, K. Marks i matematika// Filosofski problemi suchasnovo prirodoznavstva. Mitvid. Naukovi zbirnik. Vip. 11, Kiiv, Vig. Kiivskovo Un-tu, 1968, str. 47-60.

48. *Rabinovich I. M.*, O poliarnom svoistve znaka ravenstva v "Matematicheskikh Rukopisyakh" K. Marksa (On the polar properties of the equality sign in K. Marx's "Mathematical Manuscripts") // Nauka i tekhnika, voprosy istorii i teorii . M.-L., 1977.

49. *Reiske G., Sehenk G.*, Marx und die mathematik// *Deutsche Zeitschrift fur Philosophie*, Berlin, 1972, 4, s. 475-483.

50. *Ruzavin G. I.*, "Matematicheskie Rukopisi" K. Marksa i nekotorye problemy metodiki matematiki ("Mathematical Manuscripts" of K. Marx and some problems of the methodology of mathematics)// *Voprosy Filosofii*, 1968, 12, str, 59-70.

51. *Rybnikov K. A.*, O rabotakh K. Marksa po matematike (On K. Marx's works on Mathematics). Dissertation. 1954. Unpublished.

52. *Rybnikov K A.*, Matematicheskie Rukopisi Marksa// *Uspekhi Matematicheskikh Nauk*, 1955, 10(1), str. 197-199.

53. *Rybnikov K. A.*, K voprosu o poniatii funktsii (On the question of the concept of function)// *Voprosy Filosofii*, 1958, 11, str. 89-92.

54. *Ryvkin A.*, Marx y las matemáticas // *Documentos Politicos*, Bogota, 1968, N. 75. Mayo-Jun, pp. 34-38.

55. *Shugain O. V.*, Mathematichni rukopisi K. Marksa i "Kapital"//*Visnik Kiivskovo Universitetu*, 1968, Seria Filosofii, 2, str. 92-98.

56. *Slavkov S.*, Samostoiatelnite raboti na Karl Marks v oblasta na matematikata// *Spisanie na Bulgarsk. Akad. na Naukite*, Sofia, 1961, 4, str. 43-61.

57. *Slavkov S.*, Karl Marks i niakoi problemi na matematikata. Sofia, "Nauka i Izkustvo", 1963, 122s.

58. *Slavkov S.*, Karl Marks i niakoi voprosi na dialektikata v matematikata// Karl Marks i Filosofskovo Znanie. Sofia . Izd na BAN, 1984, s. 7-42.

59. *Smith C.*, Hegel Marx and calculus // Mathematical Manuscripts of Karl Marx. London, 1983, pp. 256-270.

60. *Snij Mo-fu*, On the mathematical manuscripts of Marx (in Chinese). Sin kiesjuz, Peking, 1952.

61. *Struik D. J.*, Marx and Mathematics // *Science and Society*, 1948, 12, pp, 181-196.

62. *Toms M.*, Nad matematických rykopisy Karla Marxe// *Politekonomie*, Praha, 1980, 6, s. 569-576.

63. *Treder H.-J.*, Die Beziehungen von Marx und Engels zu Mathematik und Naturwissenschaft// *Marx-Engels-Jahrbuch*. Berlin: Dietz, 1986, 9, s, 34-58.

64. *Varjas A.*, Marx als mathematiker. Ein Beitrag zur Entwiklung der Marx'schen dialektik// *Internationale Presse-korrespondenz*, Wien, 1924, 18. VII. No. 92, s. 1165-1167.

65. *Yanovskaya S. A.*, O matematicheskikh rukopisyakh K. Marksa// *Pod Znamenem Marksizma*, 1933, 1, str, 74-115.

66. *Yanovskaya S. A.*, Predislovie (Preface), *Matematicheskie Rukopisi* K. Marksa. M., Izd. Nauka, 1968 (pp. 1-15 of the present volume).

67. *Zeman J.*, *Zapletal I.*, Matematicke rukopisy K. Marxe a jejich Vztach k dnesku // *Filoz. Cas.*, Praha, 1983, 4, s. 536- 547.

68. *Zoltan K.*, Marx és a válságok törtvényeinek matematikai tanulmányozása (Marx and the mathematical investigations into the laws of crisis)// *Közgazdasági Szemle*, Budapest, 1962, 12, pp. 1464-1483.

PART TWO : INVESTIGATIONS

INVESTIGATIONS INSPIRED BY MARX'S MATHEMATICAL MANUSCRIPTS : A SELECTION

MARX AND HADAMARD ON THE CONCEPT OF DIFFERENTIAL

VASILI IVANOVICH GLIVENKO

1. Two Points Of View on the concept of Differential.

In the history of differential calculus one comes across two basic points of view on the concept of differential.

According to the first, the concept of differential *immediately* reflects some external reality; for the sake of brevity we shall call it the o b j e c t i v e point of view. This was the point of view of the inventors of differential calculus. For them — the differential was an infinitesimal increment of the variable. The external reality reflected in the words " infinitesimal increment ", was somehow thought to be self-evident.

The concept of derivative was not there, its role was fulfilled by the quotient of two differentials. Thus, unlike our objective point of view of the derivative, in this conception the derivative was not viewed as an immediate reflection of some external reality.

Modern analysis too has retained the objective point of view of the differential, though here it has acquired a different meaning. Here , first of all, we separate from among the totality of available variables, those which we consider to be independent, and the differential of such a variable dx is simply considered to be its arbitrary finite increment Δx. For the remaining variables, which are functions of the separated independent variables, some other definitions of the differential are well known ; these definitions are different in form . For the sake of simplicity, we shall limit ourselves to a single function of a single variable x :

$$y = f(x).$$

Then according to Stolz's definition, used in the better text books of modern analysis (those of de la Vallée Poussin and Courant, for example) if $\Delta y = A \cdot \Delta x + \alpha \cdot \Delta x$, wherein A is not depèndent on Δx, and α together with Δx tends to zero, then $dy = A \cdot \Delta x$.

Briefly, the differential dy is the principal linear part of the finite increment Δy. Other definitions of the differential are also accepted : according to the most widespread among them,

$$dy = f'(x) \ : \ \Delta x \, ;$$

according to Cauchy's definition,

$$dy = \frac{\lim f(x + h \cdot \Delta x) - f(x)}{h} \qquad h \to 0 \, .$$

But when one intends to explain the most general meaning of these definitions, then one has recourse to Stolz's definition : it is not difficult to establish their equivalence with Stolz's definition.

There are deep going differences among the conceptions of the differential as an infinitesimal increment (in the meaning that was bestowed upon the words "infinitesimal

increment" by the inventors of differential calculus), as an arbitrary finite increment or as its principal linear part ; but all the same in all these cases, we deal with the objective point of view about the concept of differential.

In both the cases the differential immediately reflects some external reality, every time, just like the variables x and y themselves.

According to the second point of view, the *derivative*

$$f'(x) = \lim \frac{\Delta y}{\Delta x} \qquad \Delta x \to 0$$

immediately reflects some external reality ; for the sake of brevity we shall call it the o p e r a t i o n a l point of view .

Here the concept of differential reflects the well known aspects of those mathematical operations, from which the definition of the derivative and the computations with the derivatives follow.

From this point of view, the differentials are *introduced* in the form of ratios of differentials, ratios — that are symbolized in the *derivatives* :

$$f'(x) = \frac{dy}{dx} .$$

After this, it is not difficult to understand, that the operations with the symbolic ratios $\frac{dy}{dx}$, according to the very rules that are applicable to the algebraic fractions, will not lead to any contradiction. But neither is it mandatory, that we seek an *immediate* interpretation of each and every result that follows from these operations. In particular, nothing obstructs us from viewing the formula

$$dy = f'(x)\, dx$$

(obtained by freeing the aforementioned formula of the derivative of the denominator), as only another expression of the formula

$$\frac{dy}{dx} = f'(x).$$

Substantiation of the operational point of view had to wait for a considerably longer period of time, than what was required for the substantiation of the objective point of view about the differential. The difficulty here was not with establishing the very *possibility* of thus, and not otherwise, interpreting the differential, but rather with the discovery of the *meaning* of such interpretation.

In this direction, the first methodologically exhaustive work was done by K. Marx; this work was written about fifty years ago, but was published only last year [K. Marks, *Matematicheskie Rukopisi* (Mathematical Manuscripts) // *Pod Znamenem Marksizma*, 1933, 1, str. 15-73]. J. Hadamard has treated the differential along the same lines, but differently, in his modern text book [J. Hadamard, *Cours d'Analyse*, Paris, Hermann, 1927, pp. 2-10.].

2. *The Differential of Marx and of Hadamard.*

The well known theorem about the differentiation of a function will play the principal role in what follows : let, as before,

$$y = f(x) ;$$

let x be the function of some variable t ; then

$$\frac{dy}{dx} = f'(x) \cdot \frac{dx}{dt} . \tag{1}$$

Marx's idea is as follows. When the starting point of the differential calculus, the equality

$$\frac{dy}{dx} = f'(x) ,$$

is taken isolatedly, then, at first, it has only a descriptive character. The real result of the operations, through which the derivative is determined, stands on the right hand side of this equality; the left hand side serves only as a symbol of those very operations,which lead to this result. But after the meaning of this symbol has been defined by such an equality, in consonance with this definition, it appears on both the sides of formula (1). [Hadamard proceeds from a formula , which is analogous to formula (1), but involves two variables : if

$$z = f(x , y) ,$$

then ,

$$\frac{dz}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt} ;$$

Marx has used a particular instance of this last formula : if $z = x \cdot y$, then,

$$\frac{dz}{dt} = y \cdot \frac{dx}{dt} + x \cdot \frac{dy}{dt} .$$

However, we lose nothing by illustrating their deduction by the simpler formula (1).] But there we are already dealing with operations under similar symbols : they themselves become the object of a calculus. "Thereby the differential calculus appears as a specific type of calculus, already independently operating on its own ground ". Naturally, the formulae of this calculus have the meaning of *operational* formulae : thus, formula (1), upon establishing the connection between $\frac{dx}{dt}$ and $\frac{dy}{dt}$, thereby shows, what one must do,so that having $\frac{dx}{dt}$ one may obtain $\frac{dy}{dt}$. And from this very point of view this formula appears to be even more excessively complex. Since the nature of the variable t has no significance here, reference to this variable — to the differential dt — may be dropped altogether, just as one drops the common factor. According to Marx, this is not only p o s s i b l e, but also n e c e s s a r y : for the removal of the illusion, as though the formula (1) is true only in respect of some really independent variable. Thus we get a new formula

(2) $dy = f'(x) \cdot dx$.

This is an operational *symbol* : once we obtain the formula (2) as a result of computations, it is enough to divide both the sides of this formula by dx, for obtaining the derivative ; it is enough to divide both the sides by dt — for obtaining formula (1) etc. Here the differential calculus as such finds its own natural fulfilment , and the " operational equation " (1), " as a preparatory equation, becomes superfluous, after it fulfils its task of supplying the general symbolic formula for differentiation " (2), " which directly leads us to our goal ". [we may recall , that for Marx , strictly speaking, the equations (1) and (2) are not at issue; he was concerned with the equations

$$\frac{d(xy)}{dt} = y \cdot \frac{dx}{dt} + x \cdot \frac{dy}{dt}$$

and

$$d(xy) = y \cdot dx + x \cdot dy .$$

But clearly, that does not change the affairs.]

Hadamard approaches the differential from the same operational point of view, but differently. Having established formula (1), he proposes to write out formula (2), by simply indicating this convention, and this alone, that whatever be the functional dependence of x and y on the parameter t , the equality (1) does hold good.

Hadamard insists on this definition of the differential, since it permits the use of the formulae containing operational symbols, for obtaining the derivatives. Let us assume, that computations with the differentials — in the sense that follows from Hadamard's definition — did indeed produce the formula,

$$dy = A \cdot dx .$$

Then one can assert that

$$f'(x) = A .$$

Actually, as per our assumption , for any t , we have :

$$\frac{dy}{dt} = A \cdot \frac{dx}{dt} ;$$

on the other hand, we know, that for any t :

$$\frac{dy}{dt} = f'(x) \cdot \frac{dx}{dt} .$$

From a comparison of these equalities, we get what is to be proved.

Hadamard defined the *second* differentials analogously. Proceeding from the formula (1), Hadamard states the dependence between the *second* derivatives to be

$$\frac{d^2y}{dt^2} = f'(x) \cdot \frac{d^2x}{dt^2} + f''(x) \cdot \left(\frac{dx}{dt}\right)^2 \tag{3}$$

and proposes to write the formula

$$d^2y = f'(x) \cdot d^2x + f''(x) \cdot dx^2 , \tag{4}$$

as indicating this and this alone, that the equality (3) holds good, whatever be the functional dependence of the variables x and y on the parameter t. Formula (4) may again be used as the operational symbol for obtaining derivatives.

Let us assume, that computations with the differentials did indeed give us the formula

$$d^2 y = A \cdot d^2 x + B \cdot dx^2 ;$$

then we can assert that

$$f'(x) = A \, , \; f''(x) = B \, .$$

Indeed, as per our assumptions, we have for any t:

$$\frac{d^2 y}{dt^2} = f'(x) \cdot \frac{d^2 x}{dt^2} + f''(x) \cdot \left(\frac{dx}{dt}\right)^2 .$$

From a comparison of these equalities, taking into consideration the fact that the derivatives of t can have any value whatsoever, we get what is required to be proved.

But is it necessary to abandon the definition of the differential as the principal linear part of an increment, in order to be able to thus use the folmulae containing the differentials as operational symbols ? This is not a very simple question and it requires to be discussed separately.

3. *The Operational Point Of View and the Differential as the Principal Linear Part of an Increment.*

We have seen, that the definition of the differential as the principal linear part of an increment demands, first of all, that we apportion one of the variables as independent. In reality, however , there are no absolutely independent variables. Even in the process of solving one and the same problem of geometry, mechanics etc., it is often impossible to consider one and the same variable independent, from the beginning to the end . It is clear, that the formulae of differential calculus, when applied to such problems, will really become full-fledged operational formulae, only if it is not required of us, that having once made a choice of an independent variable we should retain it in that capacity for the entire course of the computations; in other words, if the formulae, containing the differentials, and written with the assumption that x is the independent variable, remain unchanged, even when x turns out to be a function of another independent variable. In this sense, the invariance of the formulae, containing the differentials, thus happens to be an *essential condition* for the concordance of the objective and the operational points of view about the definition of the differential.

The definition of the *first* differential, as the principal linear part of an increment , satisfies this essential condition. Let us recall the proof of this well known fact. At issue here is formula (2) : $dy = f'(x) \cdot dx$.

When x is an independent variable, this formula is obtained as under. First of all

$$\Delta y = A \cdot \Delta x + \alpha \cdot \Delta x \, ,$$

$$\frac{\Delta y}{\Delta x} = A + \alpha,$$

$$\lim \frac{\Delta y}{\Delta x} = A \qquad \Delta x \to 0\,,$$

$$f'(x) = A\,,$$

$$dy = A \cdot \Delta x\,,$$

$$(2^*) \qquad dy = f'(x) \cdot \Delta x\,.$$

Let us note, that in this argument the value of x is fixed, and Δx arbitrarily tends to zero : consequently, Δx is viewed as a variable, *not dependent on* x. This remark will play a vital role in what follows. As of now, let us return to what we have obtained :

$$dy = f'(x) \cdot \Delta x\,;$$

this apart, since $dx = \Delta x$, formula (2) quickly follows from it.

Now, suppose that x is not an independent variable, and $x = \Phi(t)$, such that it is also the case that $y = \psi(t)$. Then one may get convinced about the validity of formula (2) as under : Owing to (2*) :

$$dy = \psi'(t) \cdot \Delta t\,;$$

according to formula (1) :

$$\psi' t \cdot \Delta t = f'(x) \cdot \Phi'(t) \cdot \Delta t$$

and, finally, owing to (2*) :

$f'(x) \cdot \Phi' \Delta t = f'(x) \cdot dx$ [from (2*) · we have $\Phi'(t) \cdot \Delta t = dx$] .

By comparing the last three equalities, we get formula (2). Clearly, (2) may serve as an operational formula, quite equivalent to Hadamard's definition.

Difficulties arise , when we try to define the *second* differential. The definition of the second differential is already included in the definition of the differential as the principal linear part of an increment, and there is no room for any additional "arbitrary " understanding . In fact, according to this definition, the differential dy is itself a function of x, and that is why the second differential d^2y, the differential of the differential, is thereby defined as $d(dy)$; it only remains to be computed. For this, let us note, that from the formula (2*) it follows that

$$d^2y = d\,[f'(x) \cdot \Delta x]\,.$$

Further, as we have already noted, if we do not wish to make all our proofs of invariance of formula (2) — infinite, then we must view Δx as a variable independent of x. Hence, it easily follows, that the factor Δx stands after the differential sign, while differentiating from x, and we get :

$$d^2y = [\,df'(x)\,] \cdot \Delta x\,,$$

whence, having (2*) in view :

$$(5^*) \qquad d^2y = f''(x) \cdot \Delta x^2\,.$$

We get the final formula for the second differential, by substituting $dx = \Delta x$:

$$(5) \qquad d^2y = f''(x) \cdot dx^2\,.$$

This formula, now found in the assumption, that x is an independent variable, turns out to be *non-invariant.* In fact, let ,

$$x = \Phi(t)\,,\ y = \psi(t)\,,$$

then, owing to (5*)

$$d^2y = \psi''(t) \cdot \Delta t^2\,;$$

according to formula (3)

$$\psi''(t) \cdot \Delta t^2 = f'(x) \cdot \Phi''(t) \cdot \Delta t^2 + f''(x) \cdot [\Phi'(t)]^2 \cdot \Delta t^2$$

and, finally owing to (5*)

$$f'(x) \cdot \Phi''(t) \cdot \Delta t^2 = f'(x) \cdot d^2x,$$

and owing to (2*)

$$f'(x) \cdot [\Phi'(t)]^2 \cdot \Delta t^2 = f''(x) \cdot dx^2\,.$$

Comparing the last four equalities we get :

$$d^2y = f'(x) \cdot d^2x + f''(x) \cdot dx^2\,.$$

This result does not coincide with formula (5).

The conclusion is clear. If we really want the differential calculus to be a full-fledged *calculus*, if we wish to have the right to use its formulae, as we use the algebra, without examining at every step, how they were obtained, then we shall be satisfied with the operational definition of the differential, as its basic definition. The concept of the differential as the principal linear part of an increment , turns out to be only an interpretation, suitable only for definite particular instances. When , in pursuit of an immediate and objective interpretation of each and every symbol, one accepts the principal linear part of an increment as the *definition* of the differential, i.e. when attempt is made to reduce the concept of differential as a whole to it, then one gets a defective result, since, by so doing, one fails to arrive at the differential calculus *as such*.

4. *The Operational Point of View and the Objective Understanding of the Differential in General.*

Our conclusion may be explained only by stating, that, n a m e l y , i t i s t h e o p e r a t i o n a l u n d e r s t a n d i n g o f t h e d i f f e r e n t i a l c a l c u l u s , w h i c h r e f l e c t s t h e r e a l i t y c o r r e c t l y a n d f u l l y [even if, owing to the fact that in reality, as has already been mentioned, there are no absolutely independent variables].

Does this exclude any objective understanding, whatsoever, of the differential symbols ?

Let us put the question more exactly : given a system of differential symbols dx, dy, d^2x, d^2y etc., mutually related — just as the derivatives of any variable t,

$$\frac{dx}{dt},\ \frac{dy}{dt},\ \frac{d^2x}{dt^2},\ \frac{d^2y}{dt^2}\ \text{etc. are —}$$

is any interpretation of these symbols, independent of the variable t possible ?

We saw, that the interpretation as the principal linear part of an increment is possible only for the symbols of the first order dx, dy ; from this , however, it does not follow that other interpretations, which would be suitable for the symbols of any order , are not possible.

Such interpretations do really exist. For example, geometrical interpretations are possible.

Let us take a curve defined by the equation :

$$y = f(x) .$$

Let us assume, that on this curve

$$x = x(s) ,$$
$$y = y(s) ,$$

where s is the length of a segment of the curve (from a definite point and, it is positve or negative — depending upon the direction chosen). Now the differential symbols dx , dy , d^2x , d^2y may be interpreted as follows. Let us introduce some arbitrary *constants* not equal to zero, indicated through ds and d^2s and put :

$$dx = x'(s) \cdot ds ,$$
$$dy = y'(s) \cdot ds ,$$
$$d^2x = x'(s) \cdot d^2s + x''(s) \cdot ds^2 ,$$
$$d^2y = y'(s) \cdot d^2s + y''(s) \cdot ds^2 .$$

It is easily seen, that this interpretation satisfies the equalities (2) and (4). The length s of the curve-segment , is a property of the curve — not dependent upon its analytical presentation. That is why the said interpretation too is not dependent on it. Naturally, whoever uses the differentials in this or that part of mathematics, bases his notion of the differential upon those interpretations, to which he is accustomed. Thus, interpretations like the one indicated are constantly followed in investigations pertaining to differential geometry, however, these are not created for the differentials themselves of the co-ordinates x and y, but rather for these or those expressions formed with them, expressions — that are of geometric interest.

It is clear however, that the presence of this sort of interpretation does not, in essence, solve the problem of the differential. Marx solved this problem through a dialectical investigation of how the transition from algebra to the differential calculus was accomplished in mathematics. As a result of his investigations there arose the understanding of the differential calculus, as an algebra of its own kind, constructed over the ordinary algebra — which includes the differential symbols, besides the numbers. The definition of the differential provided in Hadamard's text book shows that mathematicians are also arriving at that understanding of the general character of the differential calculus, where dialectics had arrived in the hands of a materialist philosopher — some half a century ago.

Source : *Pod Znamenem Marksizma*, 1934, 5, str. 79- 85.

About the Author : Vasili Ivanovich Glivenko (1897-1940), mathematician and logician ; graduated from Moscow University in 1925, obtained his doctoral degree in 1928 ; taught in Karl Libknekht Teacher-Training Institute Moscow, from, 1928 to 1940 ; made great contributions to the instuitionist and constructivist logics.

Other Publications :

1. Sur la logique de M. Brouwer // *Bull. Acad. Sci. de Belgique* (5), 14(1928).
2. Logika Protivorechii. 1929.
3. Osnovy Obshiei Teorii Struktur. 1937.

MARX'S "MATHEMATICAL MANUSCRIPTS" AND THE DEVELOPMENT OF HISTORY OF MATHEMATICS IN THE USSR

VLADIMIR NIKOLAVICH MOLODSHII

The "Mathematical Manuscripts" of Karl Marx were published in 1968 [1], in connection with the 150-th anniversary of his birth. "All those manuscripts of Marx, which were more or less complete, or those which contained his own comments on this or that mathematical question", were included in this publication — in the original language and in Russian translation [1, s. 3]. A part of Marx's mathematical manuscripts, containing the results of his reflections on the nature of differential calculus, were published in 1933, in Russian translation [2].

For the Soviet historians of mathematics, the mathematical manuscripts of K. Marx, were important supplements to the fundamental works of the classics of Marxism-Leninism, upon which they constantly based their investigations. Here I have in view : Marx's "Capital", Engels' "Anti- Dühring" and "Dialectics of Nature" and, Lenin's "Materialism and Empirio-Criticism" and "Philosophical Notebooks". Marx's "Mathematical Manuscripts" helped the Soviet scholars to better orient themselves on philosophico-methodological questions — questions, that are important for the history of mathematics, which to some extent determined the concrete themes of their investigations, especially on the history of mathematical analysis and of its substantiation in the 17th-19th centuries.

Marx undertook a deeper study of mathematics in connection with his economic investigations [1, s. 4-6]. His mathematical manuscripts show, that subsequently he became interested in purely mathematical problems — in questions pertaining to the problem of substantiation of the differential calculus, and in its history. Marx noticed the deficiencies of the basic conceptions of differential calculus of the end of the 17th-beginning of the 19th centuries and, he began to elaborate his own conception of "algebraic differentiation" and of the philosophico-methodological and historical questions intimatley connected with it [1, s. 6-22]. Marx's conception is essentially different from that of Lagrange.

Marx treated the differential of a function as an operational symbol. In this connection he investigated questions related to the nature of mathematical abstractions and to its symbols, pertaining to making the definitions of the variable and the function more exact and, questions related to the mathematical means of describing movement. Marx discussed the question of regularity of the developments of mathematical conceptions in a "historical essay", in the light of the course of development of the differential calculus and the results of the attempts to substantiate it in the 17th-18th centuries.

The question of the nature of mathematical abstractions plays no small role even in the elaboration of the problems of the foundations of modern mathematics and of mathematical logic. It is enough to recall the struggle between the supporters and opponents of the concept of actual infinity. The dialectico-materialist elaboration of the problem of formation and of types of abstractions constantly drew the attention of some philosophers and mathematicians of our country — like S.A. Yanovskaya [3]. Facts from the history of calculi, axiomatic method and questions of mathematical logic were analysed in their investigations in a new light [4]. Comparisons with Marx's "Capital" were also undertaken. Marx himself had, on more than one

occasion, indicated that the logic of mathematics and the logic of "Capital" resemble each other [5] ; and this was not a chance remark. Yanovskaya used the results obtained in this field, while elaborating the question of the epistemological foundations of the criterion of truth in mathematics (the practice of non-contradictority) [6] and, those of the concept of mathematical rigor [7]. The creative output of S.A. Yanovskaya include the papers entitled : "On the so-called Definitions through Abstractions" (1935) and "The Problem of Introducing and Excluding the Asbtractions of Orders Higher than One" (1965) [7].

Modern investigators — especially philosophers and logicians — are drawn towards the question of operational strength of mathematical symbols. V.I. Glivenko was the first to offer an extended and purely mathematical evaluation of Marx's treatment of the signs *dx* and *dy* as operational symbols [8.1], (somewhat later, M. Fréchet too expressed analogous ideas) [8.2]. Marx's ideas about the operational strength of these symbols are true even in respect of the wider range of materials provided by modern mathematics. As soon as the discussion turns to the scientific interpretations of the reasons behind the emergence and development of effective mathematical conceptions and methods — especially those of the 19th and 20th centuries, the aforementioned fact turns out to be important for history of mathematics.

Before the publication of Marx's mathematical manuscripts, historians of mathematics studied the ideas of the classics of Marxism-Leninism, on the inner-regularities of the development of the sceinces, from the other sources available. But in Marx's "historical essay" on the development of the foundations of differential calculus, they came across Marx's own analysis of the inner regularities of the development of mathematics.

Marx showed that when the necessary conditions are created within the existing mathematical theories, then a new mathematical theory may arise and develop. This new theory "stands on its own legs", when its basic concepts and methods assume the specificities characteristic of it alone; the embryonic forms of these concepts contained in the initial mathematical theories, do not have these specificities. Marx stressed , that a new theory is not perfected and does not get recognition at once ; that happens only through the struggle between its adherents and the followers of the old ideas.

Having compared three conceptions — those of Newton, d'Alembert and Lagrange — Marx observed, that in the period under consideration, the elaboration of the means of substantiating the differential calculus proceeded along the lines of perfecting and making things more exact.

The first investigations inspired by these ideas include : S.A. Yanovskaya's "Misheil Roll kak kritik analiza beskonechno malaykh" ("Michel Rolle as a critic of infinitesimal analysis") [first published in 1947, reprinted in : 3, s. 76-106]; and K.A. Rybnikov's "Ob algebraicheskikh korniyakh differensialnovo ischeslieniya" ("On the algebraic roots of the differential calculus") [9], and "O roli algorifmov v istorii obosnovaniya matematicheskovo analiza " (" On the role of algorithms in the history of substantiation of mathematical analysis") [10].

Subsequently, not only the historians of mathematics, but also mathematicians and philosophers, began to take an interest in the question of inner regularities of the development of mathematics. Elaboration of this question was found to be essential for analysing the nature and mechanism of the scientific revolutions in mathematics.

Marx's "historical essay" helped to reveal the philosophico-methodological foundations of the mistakes and insufficiently substantiated conclusions of some of the leading mathematicians

of the centuries gone by. Yanovskaya discussed the foundations of the mistakes of Saccheri's proof of the parallel lines axiom in : [11]. Rybnikov revealed the inexactitude in the arguments of I. Bernoulli and Ya. Bernoulli, in their analysis of one of the questions of variational analysis [12]. E. Ya. Bakahmutskaya compared the critical remarks of K. Marx and T.P. Osipovsky on Lagrarnge's conception of algebraic differentiation [13].

A study of the text-books of mathematics, published mainly in the 17th-18th centuries showed, that the process of making them more perfect at times gave rise to ideas, that went out of the framework of the then prevalent scientific ideas, but subsequently they became components of new conceptions together with a new methodology [14].

A.P. Yuskhevich noted [15], and then S.A. Yanovskaya and N.I. Likholetov showed [16], that from 1804 to 1834, the teaching of differential calculus in Moscow University went through three successive stages, reproducing its "mystical", "rationalistic" and "purely algebraic" forms respectively. Ideas of Cauchy replaced them by the middle of the 19th century. Roughly the same can be said about the teaching of mathematical analysis in the other universities and institutions of higher learning of Russia, during the first half of the 19th century [17 and 18].

Investigations of the Soviet historians of mathematics confirmed the correctness of Marx's critique of the conceptions of differential calculus held by Newton-Leibnitz, d'Alemert and Lagrange, as well as of his position, that nevertheless, the elaboration of the questions of substantiation of differential calculus, as per these conceptions, went along a line of ascent (and that is why it produced concrete results). [The Leibnitz -Newton apparatus of the differential calculus, together with its "principle of getting rid of " the infinitesimals of higher orders, has been provided with a new scientific substantiation in the non-standard mathematical analysis (see : "Matematicheskaya Entsiklopedia", T. III, M., 1982, s. 1019-1020). However, this does not reduce the merit of Marx's critique, since the Leibnitz-Newton apparatus has been based upon essentially different presuppositions in the theories of its inventors on the one hand, and in the non-standard analysis on the other]. Marx's "historical essay" served as a starting point for those sections of some of the text books of history of mathematics, which contained a discussion of the development of mathematical analysis [19]. The second edition of the Great Soviet Encyclopaedia [20] and the "Filosofskaya Entsiklopedia" [21] contain a description of the contents and of the methodological significance of the mathematical manuscripts of K. Marx. Articles were devoted to these manuscripts in many journals, in pariticular in the "Uspekhi Matematicheskikh Nauk" [22], "Pod Znamenem marksizma" [23], "Voprosy Filosofii" [24] and in "Matematika v Shkole" [25]. "Istoria Otchestvennoi Matematiki" [26] contains a brief description of the role of Marx's "Mathematical Manuscripts" in the development of history of mathematics in the USSR, during the last fifty years.

The questions about the stimuli, regularity and forms of scientific revolutions in the mathematics of 19th-20th centuries have principled significance for a dialectical materialist elaboration of the history of mathematics of the same period. Fruitful investigations of these questions are inseparable from the analysis of the law-governed development of the mathematical conceptions and theories as a single whole together with their basic concepts, principles, methods of proof and norms of mathematical rigor. Namely thus did Marx pose and investigate the question of development of the means of stubstantiating the differential calculus from the period of Newton and Leibnitz to that of Lagrange. When the problem of scientific

revolutions will gain its proper place in the investigations of our historians of mathematics, then they will be convinced about the fact that Marx's ideas about the regularities of development of mathematics, about the nature of mathematical abstractions and operational symbols,and about the struggle between the new and the old, are capable of helping them more than they did earlier. The present author became convinced about this while investigating the scienitific revolutions in the theory of numbers of the 18th-beginning of the 19th centuries, and in the mathematical analysis and geometry of the first half of the 19th century. Other authors have also recognised and correspondingly used this fact in their investigations [27].

The ideas expressed by V.I.Lenin in his "Materialism and Empirio-Criticism" and in his "Philosophical Notebooks" are also important for those historians of mathematics, who are engaged in the analysis of the scientific revolutions in the mathematics — especially, of the period beginning with the end of the 19th centutry.

The last — 50th — volume of the second edition of the collected works of Marx and Engels has been published recently. This edition includes 39 main and 11 supplementary volumes. This second edition contains nearly 800 new works and 700 letters more than those in the first [see : Sokrovishnitsa revoltsionnoi mysli (k zavershenia 2-ovo izdaniya sochinenii K. Marksa i F. Engelsa), *Pravda*, 1983, 28 January]. In the 11 supplementary volumes, the historians of mathematics will find some new statements of Marx and Engels related to mathematics, to its role in the elaboration of the questions of the social sciences. This is of great significance for the study of the history of mathematizatiön of knowledge in the 19th century.

The entirety of Marx's manuscripts pertaining to the philosophico-methodological questions of mathematics and of its history, has not yet been published. These will be included in the multi-volume academic edition of the works of Marx and Engels, being prepared by the Institute of Marxism-Leninism of the CC of CPSU together with the Institute of Marxism-Leninism of the CC of SUPG [28]. However, a preliminary description of a part of this heritage has been published [29] ; this publication may be used with profit.

Marx read (during 1878-79) Du Bois-Reymond's "Leibnitzian Ideas in Modern Natural Sciences" [30] and noted his statement to the effect that: ".... Aristotle's and Locke's view, that the soul is a *tabula rasa,* is supported by the investigations of Reimann, Helmholtz and others about the axioms of geometry" [29, s. 87]. It would be interesting to reproduce the rest of Marx's conspectus of this speech of Du Bois-Reymond and to compare it with the statements of Lobachevsky, Bolyai, Reimann and Helmholtz about the nature of the presuppositions in geometry. A comparison of these notes of Marx with V.I. Lenin's conspectus of A.Rey's "Modern Philosophy" may also be of use. The following statement of A.Rey, noted by Lenin, in fact develops the aforementioned statement of Du Bois-Reymond, noted by Marx : "By constantly moving further from the space accessible to sense perception and by moving up to the geometrical space, mathematics, however, does not move away from real space, i.e. from the *true relations among things. But rather, comes closer to them*" [31]. At issue here are the mathematical abstractions characteristic of the mathematics of the second half of 19th century and, how the naturalists approached the dialectico-materialist interpretation of their nature. This is an important issue for those who are investigating the history of mathematics of the 19th-20th centuries.

Marx also took notes form the works of Leibnitz and Descartes : from what they wrote about motion, and from the Leibnitz-Clarke correspondence [32], and from some posthumous publication of Descartes [33]. An analysis of these notes would certainly help the historians of mathematics to understand the "Mathematical Manuscripts" of Marx better.

One should not forget, that as of now, the scientific writings, especially those remaining in the shape of manuscripts of Marx and Engels have not been collected in full. One cannot exclude the possibility of discovery of such new materials as may be of interest to the historians of mathematics [see : 28 and 29].

Mathematicians and historians of mathematics of other countries have also taken note of Marx's mathematical manuscripts. Noteworthy in this connection is an essay of D. J. Struik [34], where he has compared Marx's conception of "algebraic differntiation" with the conceptions of Cauchy and his successors. Svyatoslav Slavkov's monograph on Marx's mathematical manuscripts [35] was published in 1963. The German and Italian editions of [the first part of] Marx's "Mathematical Manuscripts"(1968), were published in 1974 and 1975 respectively. H.C. Kennedy read his paper on Marx's mathematical manuscripts in the 15th International Mathematics Congress[36]. Soviet scholars should analyse these materials and ascertain the nature of the influence exerted by the works of Marx upon the development of philosophico-historico-mathematical investigations in the world.

[This is a re-written and updated version of the paper read by the present author at the Second School of History of Mathematics, Liepai, 3-10, VII, 1978. The first version of this paper was published in : *Istoriko-Matematicheskie Issledovania*, Vyp. XXVI, s. 9-17.]

Literature

1. *Marks K.*, Matematicheskie Rukopisi, M., Nauka, 1968 (pred., prim. i kommentarii S.A. Yanovskoi). 639 str. + risunki.
2. *Marks K.*, Matematicheskie Rukopisi// *Pod Znamenem Marksizma*, 1933, 1, s. 15-73; v sb.: Marksizm i Estestvoznanie, M. Politizdat. 1933, s. 5-61.
3. *Yanovskaya S.A.*, Metododologicheskie Problemy Nauki. M., Mysl, 1972.
4. Sbornik Statei po Filosofii Matematiki. M., Uchpedgiz, 1936.
5. *Dunaeva V.*, K voprosu o matematicheskom metode v "Kapitale " K. Marksa// *Voprosy Ekonomiki*, 1967, 8, s. 18-30.
6. *Yanovskaya S.A.*, Soderzhateilnaya Istinnost i Formailno-Logicheskaya Dokazuemost v Matematike// Praktika i Poznanie. M., Nauka, 1973, s. 247-272.
7. See in [3] : O roli matematicheskoi strogosti v istorii tvorcheskovo razvitiya matematiki i spetsiailno o "Geometrii" Dekarta// *Istoriko-Matematicheskie Issledovaniya* (dalee: *IMI*), vyp. XVII, 1966, s. 151-184.

8.1. Glivenko V. I., Poniatie Differntsiala u Marksa i Adamara//*Pod Znamenem Marksizma*, 1934, 5, s. 79-85.

8.2. Fréchet M., Sur la notion de différentielle// J. Mathémat., t. 16., 1937, p. 233-250.

9. *IMI*, vyp. XI, 1958, s. 583-592.

10. Tr. In-ta istorii estestvozn. i tekh., 1957, 17, s 287-299.

11. *Yanovskaya S. A.*, O mirovozzrenii N. I. Lobachevskovo// *IMI*, vyp. III, 1950, s. 60-64; see : [3], s, 107-149.

12. *Rybinkov K. A.*, Pervye etapi razvitiya variatsionnovo ischisleniya// *IMI*, vyp, II, 1949, s. 404-429.

13. *Bakhmutskaya E. Ya.*, Timofei Feodorovich Osipovsky i evo "Kurs Mathematiki"// *IMI*. vyp. V, 1952, s, 28-74.

14. See, for example : *V. N. Molodshii*, Osnovy ycheniya o chisle v XVIII-nachale XIX v. M., Prosveshenie, 1963; *A.P. Yushkevich*, da Kuinya'Zh. A. i problemy obosnovaniya matematicheskovo analiza // *IMI*, vyp, XVIII, 1973, s: 157-175.

15. *Yuskhevich A.P.*, Matematika v MGU za pervye 100 let sushestovaniya // *IMI*, vyp. I, 1948, s. 43-140.

16. *Likholetov N.I. i Yanoskaya S. A.*, Iz istorii prepodavaniya matematiki v Moskovskom Universitete (1804-1860gg.)// *IMI*, vyp. VII., 1955, s. 129-130.

17. *Lobachevsky N.I.*, Nauchno-pedagogicheskoe nasledie. M., Nauka, 1976.

18. *Galchenkova R.I.*, Matematika v Leningradskom (Peterburgskom) Universitete v XIX v. // *IMI*, vyp. XIV, 1961, s. 355-392.

19. *Rybnikov K.A.*, Istoriya Matematiki. 2-e izd, Izd. MGU, 1974.

20. BSE, izd. 2-e, T.26, 1954, s. 496-498.

21. Filosofskaya Entsiklopedia, T.3, 1964, s, 342-343.

22. *Rybnikov K. A.* , Matematicheskie Rukopisi K. Marksa// *Uspekhi matem. nauk*, 10 :1 (63), 1955, s. 197-199. *Rozov N. Kh.*, Matematicheskie Rukopsi K. Marksa// *Uspekhi matem. nauk*, T. 23, vyp. 5. (143), 1968, s. 205-210.

23. *Yanovskaya S. A.*, O matematichesikh rukopisyakh Marksa// *Pod Znamenem Marksizma*, 1933, 1, s. 74-115.

24. *Ruzavin G. I.*, Matematicheskie rukopisi K. Marksa i nekotorye problemy metodologii matematiki// *Voprosy Filosofii*, 1968, 12, s. 59-70.

25. *Kiseleva N. I.*, Karl Marks i matematika// *Matematika v shkole*, 1969, 1, s. 5-10

Molodshii V. N., O matematicheskikh rukopisyakh K. Marksa// *tam zhe*, s. 10-23.

26. Istoriya otchestvennoi matematiki, T.4. kn.2, Kiev, 1970, s. 460,461,465,496; *A.P. Yushkevich*, O razviti isorii matematiki v SSSR//*IMI*, vyp. XI, 1958; *IMI*, vyp. XXIV, 1979.

27. *Rybikov K. A.*, O tak nazyvaemykh tvorcheskikh i kriticheskikh periodakh v istorii matematicheskovo anliza // *IMI*, vyp. VII, 1954, s. 643-665.

Molodshii V.N., Osnovy ucheniya o chisle v XVIII i nachale XIX v. M., Prosveshenie, 1963 ;

Molodshii V. N., O. Koshi i revolutsiya v matematicheskom analize pervoi chetveorti XIX v.// *IMI*, vyp. XXIII, 1978, s. 32-55.

Molodshii V. N., O filosofsko-metodologicheskikh predposylkakh otkrytiya i razrabotki N. I. Lobachevskim neevklidovoi geometrii// *Filosofskie nauki*, 1980, 4, s. 75-85.

28. *Senekina O. K.*, F. Engels i istoriya nauki i tekhniki (po rukopisnym materialalm F. Engelsa v Institute Marksizma-Leninizma pri Ts K K PSS) // *Voprosy istorii estestvoznaniya i tekhniki*, vyp. 3(32), 1970, s. 14-19.

29. *Krinitskii A.M.*, Rabota K. Marksa nad voprosami estestvoznaniya (soobshenie po neopublikovannym materialam) // *Voprosy Filosofii*, 1(3), 1984, s. 72-92.

30. *Du Bois-Reymomd E.*, Zwei Festreden in öffentlishen Sitzungen der kg. Preuss. Akademie der Wissenschaften, 1871.

31. *Lenin V.I.*, Poln. sobr. soch., T. 29, s. 482 [Eng. ed., M., 1961, v. 38, p. 418].

32. *Leibnitii*, Opera philosophica (ed. Erdmann), Berlin, 1840, pars prior, p. 774-775.

33. *Descartes R.*, Opuscula postuma phisica et mathematica. Amsterdami, 1701.

34. *Struik D.J.*, Marx and Mathematics// *Science and Society*, 1948, 12, No. 1, pp. 181-196.

35. *Slavkov S.*, K.Marks i niakoi problemy na matematikata. Sofia, 1963.

36. *Kennedy H.C.*, Karl Marx and the Foundations of Differential Calculus// *Historia Mathematica*, 1977, 4, pp. 303-318; Russian tr. with additions//*IMI*, vyp. XXVI, 1982.

Source : *Voprosy Estestvoznaniya i Tekhniki*, 1983, 2, pp. 29-34.

Author : Vladimir Nikolaevich Molodshii, Doctor of Physico-mathematical Sciences, Professor.

ON THE OPERATIONAL LOGICAL APPARATUS OPERATIVE IN KARL MARX'S "CAPITAL" AND "MATHEMATICAL MANUSCRIPTS"

V. I. PRZHESMITSKY

I. *On the logic of Marx's "Capital", the logical apparatus of the concrete sciences and, the actual, real contradictions.*

The question of the operational logical apparatus operative in Marx's "Capital" is topical, not only in connection with the necessity of defending Marxism and the foundations of its philosophy, but also in connection with the requirements of the concrete sciences, in particular, those of physics and mathematics. In these sciences, specific logical problems are arising continuously; these problems demand that we turn to a logic, which is more powerful than that ordinary, traditional logic, which has become "mathematical".

Following Marx and Engels we shall use the term "Logic" to indicate the science, wherein the "laws of human thought" are investigated or, which is "a discipline about the laws of the very process of thought " [Marx K., Engels F., *Collected Works*, Russian Edition T. 20, s. 91; T. 21. s. 316] (Eng. ed., v. 25, p. 84 ; Marx-Engels, On Religion (in Bengali), Progress, M., 1981 p. 258 — Tr.). This discipline enters into all the fields of knowledge in the same way, as the method of MOVEMENT of thought from the problems to be sloved to their true solutions, as "the method for seeking new results, for the transition from the known to the unknown", as "the method of investigation and of thinking" [ibid, T. 20. s. 138; T. 21, s. 303] (Eng. ed., v. 25, p. 125 ; Marx-Engels, On Religion (in Bengali), 1981, p. 288 — Tr.). While studying the works of Marx and Engels, one cannot fail to notice, that already in the 1840s, they were using a qualitatively new (dialectical) logic which they themselves developed. After the publication of Marx's "Contribution to the Critique of Political Economy", in 1858 — they openly declared that a dialectical logic has been created, and that it is successfully operative in science. In this connection, they noted, that the scientific result obtained by them in logic is, " in its significance, hardly inferior to the fundamental materialist ideas" [ibid, T. 13, s. 497] (Eng. ed., v. 16, p. 475 — Tr.). This result was obtained on the basis of the principles of philosophical materialism : "to comprehend the specific logic of a specific subject-matter" [ibid, T. 1, s. 325]," ... an understanding of nature as she is, without any extraneous additions..." [ibid, T. 20. s. 513] (Eng. ed., v. 25, pp. 478-479 — Tr.), "one must not introduce any arbitrary sub-divisions" into the subject-matter under investigation, and the logical aspect of the subject-matter must "find its unity in itself" [ibid, T. 40. s. 10] (Eng. ed., v. 1, p. 12 — Tr.).

Marx and Engels began their work in the field of logic, while they were still young. Already in 1837, Marx wrote confidentially to his father : "... this work ... which had caused me to rack my brains endlessl, (since it was actually intended to be a new logic) ... [it is] my dearst child" [ibid, T. 40. s. 15.] (Eng. ed. v. 1, p. 18 — Tr.).

This statement indicates the indisputable beginning of the marxist [? — Tr.] dialectical revolution in logic. The new logic may be derived, with help of strict logical means, from its primary element, its primary foundation, its initial "cell". Such an element must fixate the total overcoming of that one-sidedness of scientific thought — by logic — which emanates from the Aristotelian principle of non-antinomicity of truth. A study of Marx's and Engels'

works show, that their purely logical demand is as under : to do away with the metaphysical anti-dialectical prejudice, the globalization of the priciple of non-antinomicity and TO USE "IN THE PROPER CASES BOTH" THE DISJUNCTION ($p \dot{\vee} \overline{p}$) AND THE CONJUNCTION (p & $\overline{p}$), AS PERMISSIBLE LOGICAL FORMS OF TRUTH, understandably, basing oneself upon the possible "use of the opposites" [ibid, T. 20, s. 528] (Eng. ed., v. 25, p. 493 — Tr.), where it is necessary (our stress — V.P.).

Such an attitude to the conjunction (p & $\overline{p}$) as a possible form of truth, fixated/consolidated in the Principles of logic as a science, is strictly scientifically substantiated. This (substantiation) consists of the two following facts, related to the DIALECTICS OF NATURE : (I) a contradiction in a scientific theory — that, which in the final count takes the form of the conjunction (p & $\overline{p}$), (has) by no means (its origin only in thought) in any and every case; depending upon the concrete conditions, the concrete contents of thought, the foundation and the root of a concrete contradiction lies : either really in thought, or — in the perception of reality and only then in thought, or — in the very essence of the subject-matter of thought, i.e. exclusively in the very nature of things; (II) the set of three basic types of contradictions are divided in scientific theories, into concretely defined sub-sets or classes of fully defined specificities, these are strictly differentiated in the writings of Marx and Engels.

According to Marx and Engels, the first two types of contradictions constitute the class of "trivial" and, "false" or "seeming" contradictions and, they are to be contrasted with the remaining class of "REAL or TRUE" contradictions.

Through their investigations in the "Capital" Marx and Engels made it possible for us to understand the importance of the distinction between the real contradictions characteristic of reality and the contradictions of the sophist type, of that between the REAL contradictions and the really TRIVIAL (ILLUSORY, SEEMING) contradictions, those that give rise to the PARADOXES. It must be pointed out however that ordinary logic lacks the means required for visualising and expressing these distinctions, in so far as these distinctions are not immediately given by the forms of expressions, but rather by the contents thereby reflected. Since the logical form of all types of contradictions happen to be the one and same antinomy — the conjunction (p & $\overline{p}$), the determination of whether this or that real concrete contradiction is genuine and true or an illusion or a paradox is impossible with the help of ordinary logic. For this logic even the task of distinguishing between a REAL contradiction from a sophistic-contradiction, becomes a BACK-BREAKING task in a number of instances.

Now let us take a look at the *real* contradictions of scientific theories.

We may take the following contradiction, as an example of the really false or seeming contradiction, that of the illusory contradiction or paradox, from amidst the writings of Marx and Engels : "why does the capitalist, whose sole concern is the production of exchange value, continually strive to depress the exchange value of commodities ? A riddle with which Quesnay, one of the founders of Political Economy, tormented his opponents, and to which they could give him no answer" [ibid, T. 23, s, 331; *Capital* (Eng. ed.), I, p. 303]. The following example of a real, *genuine,* i.e. true contradiction may also be considered from among the writings of Marx and Engels : "Motion itself is a contradiction — even simple mechanical change of position can only come about through a body being at one and the same moment of time both in one place and in another place, being in one and the same place and

also not in it. And the continuous origination and simultaneous solution of this contradiction is precisely what motion is" [ibid, T. 20. s 123 ; Eng. ed., v. 25. p.111]. We may consider, alternatively, yet another example :"... it is a contradiction to depict one body as constantly falling towards another, and as, at the same time, constantly flying away from it. The ellipse is a form of motion which, while allowing this contradiction to go on, at the same time reconciles it" [ibid, T. 23, s 114; *Capital* (Eng. ed.), I, p. 106]

Remarkably, all these types and classes of contradictions, which we meet within the scientific theories, have been — one and all — investigated in the writings of Marx and Engels. The task of recognising and revealing the specificities of these contradictions has been made an ordinary, routine and everyday affair, like that of solving the riddles or revealing the secrets of nature or those of the objects of thought, which have often been termed — after Hegel — as the problems of "solving" (logical) contradictions. Such solutions are carried out by seeking out the essential, natural, intermediate, logically-mediating links. Incidentally, often many such intermediate links are required to be sought, as, for instance, from the standpoint of elementary algebra many intermediate terms are required, to understand that $\frac{0}{0}$ may represent an actual magnitude [ibid, T. 23, s. 326; *Capital* (Eng. ed.), I, p. 290].

The works of Marx and Engels also help us to understand the fact that the difference of the actual real contradictions from the really seeming and false ones — the paradoxes — which are BASIC to logic (and not to any theory of development whatsoever), leads to the following.

1) The real and actual contradictions in theory are antinomical-truths, as truths they may be logically used in appropriate situations either as natural elements of true scientific thought or as logical TRANSITIONS encountered in logical movement of thought — which the developed sciences cannot do without, whereas just like the sophistic-contradictions, the paradox type contradictions cannot play that role.

2) The real and actual contradiction characteristic of an object of thought, is contained in the very ESSENCE of the object or it may even be THAT VERY OBJECT (as in particular, is the contradiction of MOTION), while the really paradox producing contradiction is merely a contradiction between the outer side (and correspondingly, the external appearence) of the object of thought and its ESSENCE (and correspondingly, its inner regularity).

3) From the standpoint of ordinary logic, the real parodox generating contradictions, just like any trivial contradiction — cease to EXIST upon their resolution and, permit non-antinomical representation of the phenomenon indicated by them. In contrast to this, no actual, real contradiction (like the contradiction of MOTION) can be depicted non-antinomically and in this sense they are INDESTRUCTIBLE.

The actual, real contradictions do not vanish even in those situations, when together with their carriers — as is characterisitic of them, they are DEVELOPED — i.e., transformed, unfolded and, are supplemented with new phenomena, with new contradictions derived out of the old ones. Since the development of an object DOES NOT ELIMINATE these real contradictions but only creates a *modus vivendi,* i. e. a form in which they can move forward side by side, a form of their movement. And such in general is the method — explained Marx — with the help of which real contradictions are resolved [see : *Capital* (Eng. ed.), I, p. 106]:

All that has been said above oblige us, following Marx and Engels, to see and to take note of the fact, in all the disciplines (and especially in logic and in philosophy as a whole), that the aninomies and the symbolic conjunctions which express them, may depict, not only the false and trivial contradictions, absurdities and paradoxes, but also the real, actual contradictions. And in that case such a conjunction can be the TRUE form of a TRUE antinomical proposition, VITALLY IMPORTANT FOR THE SCIENCE of logical transition along the path of dialectical negation together with a subsequent negation of the negation and, Marx has graphically demonstrated the justifiability and result-yielding character of such conjunctions in mathematics [see : Marks K., *Matematicheskie Rukopisi,* M., 1968, s. 29-75]. It must be clearly stated here, that now, when the works of Marx, Engels and Lenin are to a significant extent available to all, and are within the reach of every Marxist, we should not fail to notice this fact.

Here it may be indicated as a positive fact, that mathematicians in the USSR have already noted, that the formal-logical type of thinking and the very principle of non-antinomicity may, under certain circumstances lead even " to the appearance of delusions and mistakes "[see : Rybnikov K. A., *Vvedenie v metodologiyu matematiki* (Introduction to the methodology of mathematics), M., 1979, s. 50]. And that is why, this type of thinking "did not and does not occupy a central position ... in developed human consciousness. The acquisition of mathematical knowledge, and its composition, includes within itself a lot of elements that are not amenable to formal-logical analysis. Quite a few of the methods of mathematical "operation" happen to be "non-logical" [Rybnikov, *op. cit.,* s., 56-57]. And in so far as even to-day, we are required to use a "means of logical deduction which arose in the past" [*ibid,* s. 83-84], including the principle of non-contradiction as the principle of non-antionomicity, "notwithstanding the evident logical discrepancies and lack of explanations of the operational part of analysis", when "the problem of construction of models and their corresponding definite logical requirements have come to occupy the fore-court "[*ibid*, s. 94]. Here the development of mathematics (as well as that of theoretical mechanics and physics) is in ever greater need of a recognition of that DIALECTICS OF NATURE in logic, about the non-naturalness of which N.N. Luzin, D.D.Morduhai-Boltovsky, P.S. Novikoṽ and others have spoken repeatedly.

II. *The operational logical apparatus of the "Capital" and of the "Mathematical Manuscripts" of K. Marx.*

The operational logical apparatus adduced in the courses of traditional logic, called forth to ensure the logical movement of scientific thought, does not guarantee the banning of false identifications, antinomies or disjunctions and, even of logical arbitrariness in scientific thought. The logic of the "Captial" of Marx not only guarantees the banning of logical arbitrariness in scientific thought, but also ensures its successful development : reproduction of the essence of the object under investigation in all its contradictoriness, consistency of thought, its truth, broadness of the logical frame and of the genralisations adduced, the necessary strictness and, together with it, versatility of the conclusions. It is that is why, namely, that the logic of "Capital" is at least as superior to the ordinary logic "as the railways are to the medieval means of transport", to borrow an expression from Engels [Marx K., Engels F., *Collected Works*, Russian Edition, T. 13. s. 48; Eng. ed. v. 16, p. 476].

The following logical devices have been employed in the "Capital" for ensuring the reproduction of the essence of the object of investigation in the movement of scientific thought: contradictions connected with the solution of the problems under investigation have been revealed and formulated — these contradictions reflect the essence of the problems to be solved. Such a logical order is clearly visible in Marx's formulation of the question of essence of commodity, of the capitalist form of social production and, that of the distribution of capitalist profit and capital [ibid, T. 13, s, 47-48; Eng. ed., v. 16, pp. 475-476].

Further, the contradictions, which have been formulated as fragments of the reality under investigation, and which appear as riddles or as secrets of the object of thought, are solved purely logically, i.e., in thought. This does not change anything in their content, yet removes the veil of secrecy from their face, removes the veil of mystery. In so far as the realisation of such an operation, while resolving the real, and even seeming, contradictions, i.e., paradoxes, demands that not only the external aspect, the appearance of phenomena, but also their internal side, the regularity, and that is also the essence of the OBJECT OF THOUGHT be taken into account; here thought, while realising the logical operations, moves from the formulation of the questions to their true solutions, without tearing itself off, either from the essence of the object of thought or, from the essence of the scientific problems to be solved.

Truth is the most valuable property of thought and, that is why, it is first of all the duty of LOGIC to control the truth of scientific thought. Since, namely, "LOGIC = THE QUESTION OF TRUTH" [Lenin V. I., *Collected Works,* Eng. ed. v. 38, p. 175]. While performing this duty, the logic of Marx's "Capital" — as distinct from ordinary logic — proceeds from the fact that concrete truths happen to be both non-antinomical and antinomical. That is why, as already stated above, the new logic, the logic of "Capital" — as distinct from the Aristotelian and modern formal logic — proceeds from the demand that BESIDES "EITHER-OR", "BOTH THIS — AND THAT" is also TO BE EMPLOYED in the RIGHT PLACE. At a definite stage, this demand wards off that situation, when actually true propositions are declared to be false on the basis of the fact that the positions opposed to them, formulated earlier, turn out to be true — and this is not a sufficient ground for declaring the earlier ones false. Here we are, in the main, speaking of the ready-made propositions and concepts. This is what I would like to point out at first.

Marx has also pointed out: "Truth includes not only the result but also the path to it. The investigation of truth must itself be true; true investigation is developed truth, the dispersed elements of which are brought together in the result" [Marx K., and Engels F., *Collected Works*, Eng. ed., Vol. 1, p.113.]. That is to say, we must take care both of the consistency and of the starting points of the movement of scientific thought. The logic of "Capital" takes this into account. From the very beginning it directs the subject of investigation along a path which rejects all mysticism and scholasticism and, ensures "THE JOURNEY TO THE THINGS, AS THEY ARE, I. E. UPTO TRUTH" [Marx K., and Engels F., *Collected Works*, Russ. ed., T. 1, s. 29].

In view of the fact that the logic of "Capital" and of the "Two manuscripts on the differential calculus" of K. Marx permits in the right place, a recourse to truth equally with structural disjunction as well as with conjunction — the actualisation of the formal-logical kind of *consistency of the movement of scientific thought* becomes impossible. That is why, in this logic, the movement of scientific thought is actualised as a movement from the statement

of the question to be answered, to a true answer of it. Here the very process of movement of thought is actualised in the form of a consistent unity of two — in a certain sense contradictory to the each other — dialectical-logical processes, appropriately termed as : the ascent from the concrete to the abstract and, the ascent from the abstract to the concrete.

The ascent from the concerte to the abstract leads to an unification and organisation of the scientific data, from there one may proceed to a true solution of the problem under consideration. These data are embodied in the necessary premises of the ascent to the concrete strictly-scientific abstractions, which are called the initial abstractions for the ascent from the abstract to the concrete. Initial abstractions, either as a kind of INITIAL "CELL" of the object of thought or as a kind of true answer of the question under consideration, or as a kind of "LOGICAL BRIDGE" over the problems to be solved, turn out to be suitable for such an ascent. We shall discuss these two types of initial abstractions and ascents in detail in another paper.

The ascent from the abstract to the concrete is an intellectually realised theoretical transition according to the corresponding laws of reality (these laws are logical and special-scientific); it is a transition from the already prepared strictly-scientific initial abstractions corresponding to the concrete scientific content, to the concrete knowledge sought — to the true answer of the question under consideration.

In order to ensure that the unity of the initial abstractions of the ascent, with the laws of the reality connected with them, was natural and, that they be dependable theoretical Elements and be the logical path of ascent to the true answer of the question under consideration — logic must take into account not only the external aspect, the appearance, but also the essence of the object or phenomenon considered. That is why dialectical logic must seek or elaborate the neccessary concrete-scientific abstractions or laws of reality, initial to the ascents, only in indissoluble unity with the corresponding concrete sciences — and not otherwise. That is why in the "Capital" — where the object of investigation pertains to economics and history — this logic functions in an unity with political economy and history of mankind. In the "Two manuscripts on the differential calculus" — where the object of investigation, the differential, is a mathematical object — this logic functions in an unity with mathematics. It is understandable that this unity of the dialectical logic with a corresponding concrete science must take place only there and to that extent, where and to which extent it happens to be necessary.

The dialectical logic of Marx and Engels attains a broadness of logical frame and scientific generalisation, since it aims at using the unity of opposites: of the individual (the particular and the singular) with the general, of the concrete — with the abstract, of the real — with the possible, and of the determinate — with the indeterminate [Aristotle underrated the latter, but the indeterminate should not be underestimated] (see : Lenin V. I., *Collected Works,* Eng. ed., vol. 38, pp. 359-360). That is why, while rising from the concrete to the abstract, Marx reduces the general contained in the things to their most generalised logical expression [Marx K., and Engels F., *Collected Works*, Russ. ed., T. 3., s 180]. On this road one is required to raise scientific thought above the level of ordinary logic, i.e. above the level of the logical connections of identity and difference, of affirmation and negation, above the level of the Aristotelian laws of excluded middle and non-contradiction and, correspondingly of the formulae :

$$\neg (p \& \bar{p}) \text{ and } (p \dot{\vee} \bar{p}),$$

from among which neither one is for ever-true, nor for ever-false (but rather, both are only sometimes-true and sometimes-false), and thus rise to the level of connecting the opposites: AFFIRMATION and NEGATION, to express which one uses $(p \dot{\vee} \bar{p})$ together with $(p \& \bar{p})$.

This required representation of this law of unity of the opposites AFFIRMATION and NEGATION is superficially expressed more fully and deeply within the framework of its abstract universality, in the language of modern symbolic logic, by the formula :

$$\forall p \{[\neg p \& \neg \bar{p}] \dot{\vee} [p \& \neg \bar{p}] \dot{\vee} [\neg \bar{p} \& p] \dot{\vee} [p \& \bar{p}] \dot{\vee} [(p \& \bar{p}) \& (\neg p \& \neg \bar{p})]\}.$$

In logic we shall call this formula and the concrete law which it expresses, Engels' logical law and formula of disjunction-conjunction without forgetting the fact that this is only one of the possible PARTICULAR expressions of the ESSENCE and "NUCLEUS" of DIALECTICS, of the law of unity of opposites.

In view of the universality of this law and of the formula that expresses it, while solving a concrete problem — the object of thought, characterized by a certain predicate in p and the predicate itself, is chosen in an extremely generalised form. That is to say, always only an individual indeterminate example is chosen, from amongst the set of all the examples of a given genus, as the object of thought, which potentially carries any form inherent to their identity and difference. This holds good also for the predicates which figure in thought as per necessity. Thus scientific thought protects itself from being suppressed PREMATURELY within the limits of the Aristotelian frame-work of such conditions as, the "one and the same object", at "one and the same time", in "one and the same relation" etc.; this provides the possibility of retaining, in the movement of thought from the statement of the question to its true answer, all its possible potentiality, which can not be immediately realised and utilised in full.

The logical means that regulated, as an instrument and as a method, the strictness and the versatility of scientific thought in the "Capital" and in the other writings of Marx and Engels, has the following poles.

When thought moves from a question to its answer, those boundaries are not to be lost sight of, within the limits of which, Engels' law and formula

$$S = \{[p \& \neg \bar{p}] \dot{\vee} [\neg p \& \bar{p}]\}$$

is adequate for the object of thought. Within these boundaries they (this law and formula) demand extreme concreteness from scientific thought (and thereby this thought becomes extremely strict). But this formula may be used only within the limits of its actual range of applicability (and herein the dialectical logic of Marx and Engels retains within itself all that is really valuable and true in the ordinary logic). The movemnet of thought from the statement of a question to its true answer — according to the law that connects the opposites within the frame-work of a necessary abstract universality, which is regulated within the frame-work of Engels' law and formula of disjunction :

$$S = \{[p \& \neg \bar{p}] \dot{\vee} [\neg p \& \bar{p}] \dot{\vee} [\neg p \& \neg \bar{p}] \dot{\vee} [p \& \bar{p}] \dot{\vee} [(p \& \bar{p}) \& (\neg p \& \neg \bar{p})]\}$$

by way of excluding from it every time those terms, which lose their meaning as per the given conditions and upon solution of a given problem — alone can ensure the extreme strictness and fluidity of thought.

Source : *Voprosy dialekticheskoi logiki : printsipy i formy myshleniya* (materialy postoyanno deistvuyuschevo simpoziuma po dialekticheskoi logike). AN SSSR. Institut Filosofii, M., 1985. s. 70-81.

ON THE PROBLEM OF SITUATING MARX'S MATHEMATICAL MANUSCRIPTS IN THE HISTORY OF IDEAS

PRADIP BAKSI

I

Karl Marx completed his school education in 1835, with "a good knowledge of mathematics" [20, 644], which included arithmetic, alegebra, geometry, trigonometry and infinitesimal calculus [25, 157 ff]. However, he did not study mathematics in any university department. In the universities of Bonn and Berlin, he attended lectures on law, Greek and Roman mythology, Homer, history of modern art, anthropology, logic, geography, Isaiah and Euripides [20, 657-658 and 703-704]. While attempting an elaboration of a philosophy of law of his own, as a 19 year old student of Berlin University, he expressed his dissatisfaction ,on a methodological plane, with "the unscientific form of mathematical dogmatism" [20, 12]. In the same year he composed three poems in jest, and gave them the common title : *Mathematical Wisdom* [20, 545-546] . Two years later, in 1839, he drafted a *Plan of Hegel's Philosophy of Nature* [20, 510-514], three versions of which have come down to us ; they contain references to mechanics. His *Note books on Epicurean Philosophy* [20, 403-509], dating back to the same year and, his doctoral dissertation, written during 1840 and 1841 and submitted to the University of Jena in April 1841 : *On the Difference Between the Democritean and Epicurean Philosophy of Nature* [20, 25-105], contain evidences of his continuing philosophical interest in the fundamental physico-mathematical concepts. Thus, in spite of a lack of formal mathematical education at the university-level, mathematics was always present in Karl Marx's intellectual horizon, in some form or the other, even during his formative years. In the latter half of the 1840's Marx's interest in mathematics was rekindled by the requirements of his investigations in the field of political economy. But within a few years this interest began to draw sustenance from other sources too : for instance, in July 1850 we find him discussing the then emerging materialist conception of nature and human history in the light of the developments in mechanics and in the other sciences [19, 67-69] ; in April 1851 his friend Roland Daniels was imploring him to take up the study of physics in conenction with the projected preparation of an encyclopaedia of the sciences [8, 113]; and in September-October that year Marx did study a treatise on the history of mathematics and mechanics [PV, 109-112]. After that he went through the different branches of elementary mathematics all over again and, made a special sutdy of ordinary algebra and differential calculus. These studies continued for the next thirty odd years and, came to an end only with his death.

In "... mathematics Marx found the most consistent and at once the most simple expression of dialectical movements" [19, 31-32]. He was drawn to mathematics owing to the " many points of contact between mathematics, philosophy and dialectical logic" [16, 587]. In his more or less complete mathematical manuscripts he investigated the dialectic, the being and the becoming, the nature and history, of the fundamental concepts of differential calculus. That is why only in the context of the history of interaction of mathematical and philosophical thought can we hope to take our first steps towards a proper assessment of Marx's contributions in this field .

II

On the strength of the evidences available till date, it may be asserted, that mathematical activities began on our planet in the Neolithic era. It developed further in the centres of first urbanization: in ancient Mesopotamia, Egypt, China and India. This early mathematics was — to use a modern term — constructive, primarily oriented towards the construction of mathematical objects. The early texts containing arguments around mathematical objects and techniques grew somewhat later : in ancient Greece. Though some other civilizations — namely, the ancient Indian civilizations — had a formidable theoretical culture of their own, the interaction of mathematical and philosophical thought has been found to have been most prominent in ancient Greece alone.

It is commonly held that mathematics became theoretical with the geometrical investigations of Thales (624-548 B.C.). Thales was exposed to the mathematical practices of the ancient Egyptians, who in their turn inherited the mathematics of ancient Chaldea (Mesopotamia). Anaximandrus (611-545 B.C.) of the Ionic school founded by Thales introduced the concept of indefinite magnitude into mathematics. Another Ionic thinker Heraclitus of Ephesus (530-470 B.C.) was one of the pioneers in the conscious employment of dialectical reason in philosophy, in connection with the problem of conceptualization of change. These theoretical concerns gradually culminated in the first noticable disquiet with the indeterminate magnitudes : in the reactions of the school of Pythagoras (580-504 B.C.) to numerical analysis. And finally, in the paradoxes associated with the name of Zeno of Elea (b. 475 B.C.), the problem of definite description of the indeterminate assumed explosive dimensions. These paradoxes have a close parallel in the problematique of *Milinda Prashna* associated with the name of king Milinda or Menander (140-110 B.C.) [24].

The discovery of irrational proportions in the school of Pythagoras and the paradoxes formulated by Zeno led ancient Greek mathematics to the door steps of a "crisis of foundations". To meet this challenge Eudoxous of Cindus (408-355 B.C.) developed the method of exhaustion, already introduced by Hippocrates of Chios (450 B.C.). In the Eleatic school the concern with the indefinite went side by side with the development of the *reductio ad absurdum* argument. The problems thrown up by the contradictions involved in the attempts at a definite description of the indefinite, generated attempts to demonstrate something by expelling the obvious formal contradictions. And thus the grounds were created for raising the question of dialectics of mathematics more categorically. This the Socratics did .

In Plato's (428/427-348/347 B.C.) *Republic* we find a Socrates dissatisfied with the lack of conceptual clarity of the empirics and the Pythagoreans, proposing an investigation into the dialectic of the fundamental concepts of the existing mathematical disciplines [27, 510-511 and 524-533]. Plato attempted to put his dialectics and mathematics on a common foundation in his theory of idea-numbers. An analogous approach may also be found in Aristotle's (384-322 B.C.) remarks on an universal mathematics [3, 42]. However, it was Aristotle's systematization of logic in the image of the existing Greek mathematical theory, which influenced later mathematical developments most strongly.

Euclid (approx. 300 B.C.) benefited greatly from the critical movement of Plato's academy, revising the principles of geometry (Eudoxus was associated with this movement). After that

the method of exhaustion was further developed by Archimedes of Syracuse (287-212 B.C.). Study of the conic sections was introduced earlier by Menachmus and Plato, this was further developed by Apollonios (approx. 200 B.C.) and his followers. With these developments ancient Greek mathematics reached its zenith.

In the period that followed, at first the Hellenic centres of the mediterranean countries (like Alexandria) became the repository of the ancient Greek attainments in mathematics. Afterwards the centres of mathematical and philosophical investigations in the Greek tradition shifted to Western and Central Asia. Meanwhile Chinese and Indian mathematicses were not standing still. For a brief review of the mathematical attainments of the people of ancient and medieval Asia see : [31], [32], [33]. After the 6th century A.D. the Greek, Chinese and Indian mathematicses interacted with each other, in the centres of learning of Arabic and Persian speaking Asia. In consequence we witnessed those developments in arithmetic, algebra, geometry, trigonometry and mechanics, without which the subsequent emergence of the differential and the integral calculus would not have been possible. During the 10th-13th centuries medieval southern Europe became aware of these achievements through Hebrew and Latin translations of the available literature in Arabic. This period also witnessed a renewal of European interest in ancient Greek Philosophy and literature as a whole. At around the same time, however, the living contacts between the mathematical traditions of Asia and Europe started getting snapped. The crusades (11th-13th centuries) provide the background to these contacts and their subsequent demise.

These contacts were to be re-established some four or five centuries later, as one of the results of the colonial expansion of the European capitalist powers in the East. By this time Europe became the centre of scientific activities. The new achievements of European sciences of the modern age trickled down to the littoral towns of the colonies through the distortive filter of the colonial educational policy of the European powers. However, even this meager ration of new knowledge ushered in a process of renewal of learning in some of the colonies. The efforts of Tafazzul Hussein Khan and Raja Rammohan Roy in the realm of mathematics[1], indicate the beginning of this renewal in our own country. Marx was in general aware of this process, and of its limitations; see, for example, his article entitled *The Future Results of the British Rule in India* (written in 1853) [22, 29-34]. We have lived through and are still living through the consequences of this rule. Science education in the erstwhile colonies still remains a "lagging-behind-model" of the same in the advanced capitalist countries. Add to this lag of the present, the near total absence of awareness about our past attainments in the sciences. Those who study the history of ancient and medieval Indian sciences — a large part of which is occupied by mathematics — are promptly ticked out as purveyors of "soft science" and as nationalist propagandists. Of course the emergence of the study of history of science is connected with the rise of pātriotic consciousness in our society, and it is a significant phenomenon in the history of the sciences in India, for that reason alone. But its role does not get exhausted just there, it is merely the first lap of a long and interesting journery. The study of the history of ancient and medieval sciences has profound contemporary significance. We shall mention just one example.

D.H.H. Ingalls (1951), *C. Goekoop* (1967) and *V.A. Smirnov* (1974) have, among others, indicated the need for further investigations into the *logic of relations* present in *Gangesa's*

(13th century) *Tattva Chintamani* [29]. Such investigations are already being undertaken and will be undertaken in future, with greater competence; these will provide us with a more complete picture of the logico-mathematical activities conducted on our planet, and consequently, with greater insights for dealing with the outstanding problems in this field. Here a pertinent problem may be posed (sometime in 1974-75, my teacher Pabitra Kumar Roy made me aware of it, in Santiniketan) : in India we had a long tradition both in mathematics and logic; the great names of *Gangesa* and *Bhaskaracharya* (12th century; infinitesimal method) are associated with them; and yet, the subsequent developments in mathematics and logic leading to the emergence of mathematical logic took place in Europe, and not in India — why?

It is customary to attempt an externalist answer to this (or to any other similar) question, in terms of the relative stagnation or dynamism of the modes of production. Such attempts are valuable (for sociology of knowledge) in their own right. But how are these traits (stagnation and dynamism) of the historically existing modes of production (e.g. Asiatic Mode of Production, Capitalism etc.) *mediated* into the sphere of production and reproduction of knowledge, into that of mathematical and logical thought in particular ? We still do not have a definitive answer. As of now we can only propose a strategy for future investigations.

In ancient and mediaeval Indian mathematics the deductive aspect (arguments etc.) had a subsidiary position, the constructive aspect was preponderant [on this see : *Uspensky V.A.*, What is a Proof ? // Theme No. 6 in : Reflections on seven Themes of Philosophy of Mathematics, in part three of this special supplement]. This resulted in a considerable development of the algorithms of arithmetic, algebra and trigonometry. In contrast, the deductive moment was very strong in ancient Greek geometry. Classical Greek logical theory was abstracted mainly out of this geometry. On the other hand, ancient Indian logical theories were, in the main, abstracted out of the ancient Indian prescriptive grammatical tradition, which again was constructive (algorithmic), with its stress on the construction of "algebraic" sutras for the normative conduct of linguistic activities. Perhaps this grammatical tradition itself served as the model for the ancient Indian mathematical disciplines. F.J. Staal did not overstate the point when he asserted that : "Historically speaking the grammatical method of Panini has been as fundamental for the Indian thought, as is the geometrical method of Euclid for the western [European] thought" [30]. The modern European developments in classical mathematics and logic, culminating in the emergence of mathematical logic towards the end of the last century, re-created the ancient Greek unity of the deductive moments of mathematics and logic, on a newer plane. It is only in this century that the constructivist[2] trend has begun to assert itself in the world of mathematics (and in the cognate disciplines) as a whole. And perhaps it is not accidental, that the Indians are again making their presence felt — this time through the technological end of the spectrum, in the field of computer softwares. Falling in line with the dominant tradition in history of mathematics and logic, so far, we have been studying the ancient Indian texts of mathematics and logic, through the methodological filter of classical (western) mathematics and logic. We may now attempt a *constructivist reading* of them. Hopefully, this time there will be lesser paradigm-mismatch.

Now, before we get back to the developments of mathematical and philosophical thought in medieval and modern Europe, we must mention, further, that in the period spanning from the fall of ancient Greek civilization right upto the emergence of the new bourgeois civilization in

15th century Europe, mankind witnessed several attempts to construct encyclopaedias of the sciences. Well known among these are : the *Naturalis Historia* of Plinius Secundus Gaius, in 37 volumes (70 A.D.) ; the *Risala - i- Ikhwanus - Safa* edited by Jayad bin Rifaa, in 51 volumes (9th century); *Imago Mundi* of Petrus de Allienco (1410) and, *Margarita Philosophica* of Gregor Reisch (1496). Direct and / or indirect cumulative impact of such attempts to collect the totality of existing human knowledge contributed to the regeneration of an interest in dialectics in medieval Europe. Thus, the grounds were prepared for a closer interaction between dialectical reason and mathematical investigations on a more advanced level. However, theology was the principal arena for the development of dialectical reason in medieval Europe. And consequently we find a Nicholas of Cusa (1401-1464), mixing the ideas of a dialectical logic based on religious mysticism "with the emerging notions of mathematical analysis" [5, 81].

With the rise of capitalism in Europe came its individualist philosophy dominated by rationalist metaphysics. In this very period the operations with variable magnitudes and functions were highlighted in mathematics, through the investigations of Descartes (1596-1650), Leibnitz (1646-1716), Newton (1642-1727), Euler (1707-1783), d'Alembert (1717-1783), Lagrange (1736-1813) and their contemporaries and followers. Among them first Descartes and then Leibnitz toyed with the idea of an universal mathematics. Such developments further widened the scope for investigations into the dialectics of mathematics. But the practising mathematicians took a different course ; in a sense a natural course — that of exhausting the limits of formal reason. Bolzano (1781-1848), Dedekind (1831-1916) and Cantor (1845-1918) arrived at the conclusion that to deal with the fundamental concepts of the mathematics of the variable magnitude, i.e., of the differential and the integral calculus — the derivative and the integral — in a proper manner, infinite sets must be very precisely investigated. Cauchy (1789-1857), Weierstrass (1815-1897) and others developed the theory of limit. Consequently, the edifice of classical analysis was erected upon set theoretic foundations. But almost simultaneously, the well known paradoxes of the set theory arrived on thé scene. Russell (1872-1970), among others, arrived at analogous paradoxes through his studies in mathematical logic. Thus the formal developments in classicial analysis and mathematical logic prepared the grounds for a second "crisis of foundations " of mathematics (the first "crisis of foundations" was associated with Pythagorean numerical analysis and Zeno's paradoxes). In response there arose three schools : logicism, formalism and intuitionism. The arguments that followed led to considerable modifications, even abandonment, of some of their respective positions, in the wake of Kurt Gödel's (1906-1978) famous results about the incompleteness and inconsistency of even the most elementary formal systems [13]. Gradually, the emergence of the intuitionist, constructivist and non-standard analyses, brought an end to the monopoly of classical analysis, in the second half of this century. [But in the countries with a backward current mathematical culture, like our own (reflected in the terribly poor state of the investigations into the foundations, history and philosophy of mathematics in our country), classical analysis is often the only analysis "available" in the class-rooms of mathematics, till date.]

During this entire period of formal developments, spanning the whole of the 19th and nearly half of the 20th century, mathematical epistemology faced the famous antinomy posed by Kant

(1724-1804) : "the observable world is finite but we can not find its limits in space and time ; therefore the world is not finite but infinite, and there exists only the search for the limit according to the regulative requirements of reason" [5, 84].

The dualism inherent in this position was, to begin with, philosophically overcome within the idealist tradition of classical German philosophy itself, namely, in the dialectics of Hegel (1770-1831), wherein "... for the first time the whole world, natural, historical, intellectual is represented as a process, i.e., as in constant motion, change, transformation, development ; and the attempt is made to trace out the internal connection that makes a continuous whole of all this movement and development" [10, 34]. However, having remained chained to idealism, even such a grand attempt shrouded itself in obscurity and mysticism. This mysticism left its obvious mark in all of Hegel's writings on the topical problems of the sciences, mathematics included.

We have the testimony of Engels to the effect that Hegel left behind him "numerous mathematical manuscripts"[3]. But in view of their non-availability[4], as of now, we are constrained to fall back upon the relevant chapters of his *Science of Logic,* in our efforts to take a look at Hegel's excursions into mathematics [15, 129-170 and 198-344]. For Lenin's comments on the same see : [18, 116-119]. Hegel attempted an explanation of the mathematics of his time, especially of the differential calculus. But owing to his predetermined idealist point of departure, this attempted explanation remained artificial from the very outset. He merely dressed up his categories, like the "Quality", "Quantum", "Determination" etc., with mathematical trappings, especially with the frills of differential-calculus-terminology. At best it was an expression of some of the concepts of his dialectics in a mathematical language, in so far as it was possible; but it could not lay bare the objective dialectics of mathematics. Hegel himself conceded that his attempt is merely a philosophical explanation of the existing mathematical practice [15, 319].

In contrast to Hegal, Marx (1818-1883) studied the nature and history of the concepts and symbols of differential calculus and concluded that they are *operational.* Forty four years after Marx's death, in 1927, Jack Hadamard, an intuitionist mathematician, arrived at similar conclusions regarding the general nature of the fundamental concepts of differential calculus [see : *Glivenko V. I.,* Marx and Hadamard on the concept of Differential // Part Two of this special supplement, first article]. The intuitionists like L.E.J. Brouwer (1881-1966) and many early constructivists were unaware of Marx's mathematical investigations. Even to-day the overwhelming majority of the mathematicians, philosophers and historians of ideas are unaware of them. A quarter of a century has elapsed since the publication of the 1968 edition of Marx's mathematical manuscripts. Why then this lack of awareness about them, even in the late 20th century atmosphere of information boom? Why the responses to these manuscripts are so few in number ?

The reader of these lines may have noted the long gaps in, and the protracted nature of, the history of the efforts to unravel the dialectics of mathematics. Well, such is reality : ".... dialectical thought — precisely because it presupposes investigation of the nature of concepts themselves — is only possible for man, and for him only at a comparatively high stage of development (Buddhists and Greeks) and it attains its full development much later still through modern philosophy... " [11, 223]. It may further be mentioned in this connection that the stage

of theoretic thought associated with the emergence of materialist dialectics was preceeded, among other things, by the following encyclopaedic attempts to classify the existing branches of knowledge : that of the French encyclopaedists (1751- 1780), under the editorship of Diderot and d'Alembert; Saint Simon's (1760-1825) incomplete attempt, and Hegel's attempt to philosophically sum up the results of the natural sciences of the Newton-Linnaeus school. The end of the positivist attempt of Comte (1798-1857) and the beginning of the attempts of Marx, Engels, Daniels, Schorlemmer etc. are near contemporaneous.

To go back to the "silence" around Marx's mathematical manuscripts : intuitionist mathematics and Marx-studies could not (in spite of certain proximity of Marx's position in mathematics with those of the intuitionists) join hands and study the mathematical legacy left by Marx. Emergence of the constructivist movement from within the intuitionist trend has perhaps created more favourable grounds. In its current phase the constructivist movement has rejected Brouwer's "semi mysitical theory of the continuum" [4, 308], but has retained his general theory about the operational nature of the standard mathematical quantifiers and connectives.

So much by way of a contextual retrospective of the problem under consideration. Now, on to the problem itself.

III

A. To-day we are in a position to extend Marx's study of the differntial calculus to the study of the symbolic calculi in general. We may investigate the dialectics of the various alternatives in set theory, analysis, topology and, theory of categories. Through a two-way association of Marx-scholars and mathematicians engaged in the study of Marx's mathematical manuscripts, the mathematical heritage left by Marx may some day get integrated into the mainstreams of mathematics, just as inside of a hundred odd years after the publication of the first volume of Marx's *Capital,* his contributions in political economy got integrated into, and began shaping, the mainstreams of economic thought.

B. However, the relevance of Marx's mathematical manuscripts does not get exhausted there. In his fundamental articles on the nature and history of differential calculus, Marx has shown that the transition from ordinary algebra to differntial calculus proper, involves an *inversion of method,* expressed through the genetic development of the characteristic concepts and symbols of this calculus (differential, derivative, dx, dy, $\frac{dy}{dx}$ etc.) culminating in their future operational role (indicating strategies of the steps to be taken). I.S. Narsky has pointed out that the methodological scope of this discovery of Marx goes beyond the framework of mathematics — it applies to the meaning of signs in general [26, 156 ; quoted in : 7, 94]. Thus the theoretical apparatus of Marx's mathematical manuscripts is of relevance to the study of sign systems, to *Semiotics*[5]. Through semiotic investigations, its methodological relevance extends to the study of all the disciplines as sign systems, as "languages", i.e., as systems of articulation. Expectedly, such studies will reveal the characteristic "internal forms" of the different disciplines and their interrelations, and thus facilitate the ongoing process of integration of the sciences and

disciplines. In this vast world of semiotic investigations into the structures and functions of disciplines, extended from the computer technology to the art forms, the various trends of mathematics also have places of their own, along with the ordinary languages and other sign systems.

C. In contrast to Hegel, who attempted a philosophical explanation of the existing mathematics, Marx proposed a new way of doing mathematics bereft of metaphysics, idealism, mysticism, obfuscation and sleight of hand. In others words Marx attempted to *change* the existing practice of mathematics, its existing reality. This attempt to change the existing state of affairs of the sciences of this world — for example, of the classical political economy in the *Capital*, of the classical natural sciences in the *Dialectics of Nature,* and of classical analysis in the *Mathematical Manuscripts* — is what distinguishes the materialist dialectics from that of Hegel, and for that matter from all previous dialectics (in this connection let us recall Marx's celebrated eleventh thesis on Feuerbach). In so far as Marx and his friends were not contented with mere philosophical interpretations of the world and its sciences, the significance of their theoretic activitives became trans-philosophical. They attempted a radical reconstruction of the entire structure of human knowledge.

This attempt began towards the middle of the 19th century, but its fuller contours are gradually coming to light only in the second half of the 20th century. The first edition of the first volume of Marx's *Capital* was published in 1867. The first edition of its last (fourth) volume (edited by K. Kautsky) was issued in 1910 ; but that edition of the fourth volume (*Theories of Surplus Value)* had many radical defects; and ultimately, the final (third) part of this last (fourth) volume of *Capital* in the edition prersently in use, was brought out only in 1962. Engels' *Dialectics of Nature* was first issued in 1925; but a more complete version of Engels' studies on the dialectics of the classical natural sciences of his time could be brought out only in 1973[12]. The first edition of Marx's *Mathematical Manuscripts* came out in 1968. Marx's *Ethnological Notebooks* were brought out in 1972 [23]. The manuscripts of the first materialist attempt at investigating the dialectics of the interface of the biological and the social sciences, *Microcosmos : A Draft Outline of Physiological Anthropology,* prepared in 1850 by Marx's friend Roland Daniels (1819-1855), was brought out only in 1987 [9]. A large part of Carl Schorlemmer's (1834-1892) manuscripts on the history of chemistry remains unpublished. Nearly half of the notes and manuscripts of Marx and Engels[6] remain unpublished too ; these include : a part of Engels' notes and manuscripts on the history of England, Ireland and Germany, history of philopsophy and history of military science and, Marx's notes and manuscripts on agriculture, agro-chemistry, history of technology, geology, biology, physiology and, a part (nearly 400 pages) of his mathematical manuscripts. Publication of these notes and manuscripts and their translation into the various languages of the world will go a long way in developing our conception of the theoretical heritage of Marxism. As of now, we have just begun to understand that at the level of cognitive activities, Marxism constitutes an encyclopaedic endeavour at changing the (then and even now) prevalent ways of cognising the world. One of the open questions about the Marxist attempts in the realm of human knowledge, is about their relation with the dominant scientific programmes[7] of our time, namely, with the atomistic, Cartesian, Newtonian and Leibnitzian programmes.

It appears now, that the task of situating Marx's mathematical manuscripts in the history of ideas unfolds itself at many levels : at the level of completing text editing and publication, that of textual studies, that of situating these manuscripts within the structure and history of Marxism, that of situating them in the history of mathematical and theoretic thought, and that of situating the emerging conception of Marxsim in the history of scientific programmes. All these tasks are interrelated. And finally, we should not go about them in the spirit of an archivist of ideas or with that of a traditional historian of the past, but must take them up in the spirit of Marx, i.e, by simultaneously engaging ourselves in a comprehensive and systematic, case by case, synchronic and diachronic, study of all the newly emerged and emerging sciences and technologies of our time. In view of the gigantic strides of human knowledge since the days of Marx, and especially in late 20th century, these tasks have to be tackled by scientific collectives, aided by the latest attainments of the information processing technologies.

NOTES

1. Tafazzul Hussein Khan, the vakeel or ambassador of Nawab Asaf-ud-Daulah at Calcutta during the Government of Marquis Cornwalis (1738-1805) translated into Arabic Apollonios' *De rationis sectione*, Newton's *Philosophia Naturalis Principia Mathematica*, Thomas simpson's *Algebra* and William Emerson's *Mechanics*, during 1788-1792 [2, 39-40].

   Raja Rammohan Roy (1772-1833) wrote a modern treatise on *geometry* in Bengali [6, 407].

   These manuscripts remain untraced till date.

2. On *Constructivism* see : note 111a to Marx's *Mathematical Manuscripts* in the present volume and, *Nepeivoda N.N.*, Emergence and Development of the Concept of Constructivisability in Mathematics// Present Volume, Special Supplement : *Marx and Mathematics*, Part Three, last article.

3. F. Engels wrote to F. A. Lange on March 29, 1865 : "I can not leave unnoticed a remark you make about old Hegel, who you say lacked the more profound kind of mathematical and natural-scientific training. Hegel knew so much mathematics that none of his pupils was equal to the task of editing the *numerous mathematical manuscripts* he left behind. The only man I know who understands enough mathematics and philosophy to do this is Marx " (my emphasis — P. B.) [21, 173].

4. In response to a personal inquiry from the present author, Dr Helmut Schneider of the Hegel Archiv, Ruhr-Universität Bochum, has informed in his letter dated February 7, 1982, that neither does the Hegel Archives possess the originals of Hegel's mathematical manuscripts, nor were they ever published.

5. *Semiotics* is the study of signs (Greek *semeion* = sign). A sign is a sensorily perceptible material object, action or, event, which denotes or represents another object. Semiotics has its origins in some of the ideas of the American philosopher and logician Charles Sanders Peirce (1839-1914) and the Swiss linguist Ferdinand de Saussure (1857-1913). The range of semotic investigations are extended over all patterned communication systems : from the simplest signal systems, through the ordinary languages used by people, right upto the special languages of various disciplines. These are traditionally divided into three parts : syntactics, the study of structure; semantics, the study of meaning ; and pragmatics, the study of actual use. The growth of these investigations have given rise to prominent American French, Italian, Czech, Polish, Estonian and Russian schools of Semiotics. But in spite of both intensive and extensive developments the problem of constructing a synthetic conception of sign remains open. Such a conception must answer the

question of genesis of signs, as well as that of the generic functions of signs within a system. So far the *genetic* and *generic* approaches to the sign systems have been posed as "either-or" alternatives. A systhesis of these approaches within a "both-and" perspective is very much on the agenda of semiotic research. The mathematical manuscripts of Marx may provide this relevent perspective, in view of the fact that they contain such a conception of the characteristic signs of one language, that of the differential calculus.

6. For a brief outline of the contents of these notes and manuscripts see : [1], [17], [25] and, [28].

7. The concept of *scientific programme* has evolved out of the modern investigations in history of science, philosophy of science and science of science. The basic tenets of a scientific theory, its major premises, are always formulated within the framework of a scientific programme, which sets the ideal of scientific explanation and organisation of knowledge, and also formulates the conditions under which knowledge is considered to be both authentic and proved. The origins of this concept can be traced back to Imre Lakatos (1922-1974) ("research programme") and Thomas Kuhn (1922-) ("paradigm"). It has been further elaborated in subsequent research and alternative concepts of scientific programme have been proposed. A prominent worker in this field Piama Pavlovna Gaidenko (1934-) wrote on this concept :

" So what is a scientific programme, and why has the need arisen for such a concept?

Unlike a scientific theory, a scientific programme, as a rule, lays claim to cover all phenomena and to provide an exhaustive explanation of all facts, i.e., to an universal interpretation of everything existing. A principle or a system of principles formulated in a programme, is, hence, *universal in nature*. The well-known tenet of the Pythagoreans — "all is number", is a typical example of the concise formulation of a scientific programme. A scientific programme is most frequently formulated within the framework of philosophy (it is no accident that Engels speaks of the principled impossibility for the natural sciences "to free themselves from philosophy") [see: 11, 209- 210]. Its creators are scientists who, at the same time, also come forth as philosophers: after all a philosophical system, unlike a scientific theory, is not inclined to distinguish a group of *its own* facts, but lays claim to the universal significance of its principle. (It is precisely in analysing the structure of scientific programmes and the forms of their ties with the scientific theories, as well as in examining the evolution and change of programmes, that philosophy and history of philosophy can and must help history of science in solving its tasks).

Nevertheless, a scientific programme is not identical with a philosophical system or a definite philosophical trend. Not every philosophical system can produce a scientific programme. A scientific programme should contain not only the characteristics of the object under examination but also the, closely connected with these characteristics, possibility of elaborating a corresponding method of research. Thus, a scientific programme, so to say, determines a definite method of building a scientific theory by providing the means for the transition from a general world-outlook principle advanced in a philosophical system, to the revelation of the ties between the phenomena of the empiric world. Thus, three diverse scientific programmes came into being on the basis of ancient [Greek] philosophy: atomistic (which found its realisation in scientific theories only in modern times); mathematical (Pythagorean-Platonic, which already found its realisation in ancient times — in Euclid's *Elements* and in the mechanics of Archimedes); and finally, Aristotle's continualist programme, on the basis of which the first physical theory — the physics of the Peripatetic school — came into being. The major scientific programmes in modern times were created by Descartes, Newton and Leibnitz.

*** *** ***

A study of the emergence evolution and, finally, death of scientific programmes, the emergence and consolidation of new programmes, as well as changes in the types of ties between programmes and the scientific theories based on them makes it possible to reveal the internal ties between science and that cultural-historical entity within the framework of which it exists. Such an approach also makes it possible to trace the historically changing nature of these ties, i.e., to show how the *history of science* is internally linked with the *history of culture*.

The fact that within a definite historical period not one but two and even more scientific programmes can exist side by side, which, according to their initial principles, are opposed to each other, does not allow for a simplified conclusion about the contents of these programmes, relying on some "primary intuition" of the given culture, but *calls for a more thorough analysis of the "composition" of that culture, of the diverse tendencies coexisting in it.*

On the other hand, the existence of more than one programme in each period in the development of science shows that the idea that the history of science is an uninterrupted, so to say "linear", development of definite originally set principles and problems is unjustified. The very problems which are being tackled by science, change in the course of its history : each historical period sees their essentially different interpretation" [14, 134-137].

Such in brief is the meaning and significance of the concept of scientific programme.

REFERENCES

1. *Baksi P.*, Karl Markser Prakriti Bijnan Charcha O Bijnan Bhabona [Karl Marx's studies of and thoughts on the natural sciences] // *Mulyayan*, Sharadiya, 1395 (October, 1988), pp. 157-174 and, Nababarsha, 1396 (May, 1989), pp. 74-80.
2. *Bandopadhyay S.*, Bangiya Ranesance Paschatyavidyar Bhumika [The Role of Occidental Learning in Bengali Renaissance]. Calcutta. 1980.
3. *Beth E. W*, *Piaget J.*, Mathematical Epistemology and Psychology. Tr. by W. Mays. Dordrecht. 1966.
4. *Bishop E.*, The Constructivisation of Abstract Mathematical Analysis// Proceedings of International Congress of Mathematicians (1966). Introduction. Moscow. 1968 (pp. 307-313).
5. *Bogomolov A. S. et al*, Dialectical Logic // Themes in Soviet Philosophy. Selection of articles from *Filosofskaya Entsiklopediya* ; Tr. by T.J. Blakely. Dordrecht. 1975.
6. *Chattopadhyay N.*, Mahatma Raja Rommohan Roy. Allahabad. 1928.
7. *Chepikov M.G.*, The Intergration of science. Moscow. 1978.
8. Daniels-Marx Correspondence; published under the title *Novye materialy o K. Markse* [New materials on K.Marx] // *Voprosy Filosofii*, No.5, 1983, pp.100-126. For a Bengali translation of these letters see : Marksbad o Bijnan Samuher Dvandvikata [Marxism and the dialectics of the sciences]. Calcutta, 1986, pp. 17-84.
9. *Daniels R.*, Mikrokosmos: Entwurf einer physiologischen Anthropologie (1850)// Hrsg. von H. Elsner; Einger und annot. Von J. Bleker *et al.* Frankfurt a. M. etc., 1987.
10. *Engels F.*, Anti-Dühring. Moscow. 1978.
11. *Engels F.*, Dialectics of Nature. Moscow 1976.

12. Fridrikh Engles o Dialektike Estestvoznaniya [Frederik Engels on the dialectics of the natural sciences]. Ed. by B,M. Kedrov. Moscow. 1973.

13. From Frege to Gödel : A source Book in Mathematical Logic. 1879-1931. Ed. by Jean van Heijenoort. Harvard University Press, 1967.

14. *Gaidenko P. P.* Evolution of Science: The Cultural-Historical Aspect//*Social Sciences,* vol. XIII, No. 2, 1981. pp. 131-144. For greater details, see by the same author:

(1) Evolutsiya Poniyatiya Nauki : Stanovlenie i Razvitie Pervykh Nauchnykh Programm [Evolotion of the Concept of Science : Formation and Development of the First Scientific Programmes]. Moscow. 1980.

(2) Evolutsiya Poniyatiya Nauki (XVII-XVIII vv.) : Formirovaniye Naychnykh Programm Novovo Vremini [Evolution of the Concept of Science (17th-18th centuries): Formation of the Scientific Programmes of the Modern Age]. Moscow. 1987.

15. *Hegel G .W. F.*, Science of Logic. Volume One. Tr. by W. H. Johnston and L. G. Struthers. London 1929.

16. Karl Marx : A Biography. Moscow. 1973.

17. *Krinitsky A. M.*, Rabota K. Marksa nad voprosami estestvoznaniya (soobschenie po neopublikovannym materialam) [K. Marx's writings on the questions of natural science (a communication based upon unpublished materials)] // *Voprosy Filosofii,* 1(3), 1948, pp. 72-92.

18. *Lenin V. I.,* Philosophical Notebooks. Collected Works, Eng. ed., vol. 38. Moscow. 1961.

19. Marx-Engels Smriti [Marx-Engels: Reminiscences]. Moscow. 1976.

20. Marx K., Engels F., Collected Works, Eng. ed. vol. 1. Moscow. 1975.

21. *Marx K., Engels F.,* Selected Correspondence. Moscow. 1965.

22. *Marx K., Engels F.,* The First Indian War of Independence : 1857-1859. Moscow. 1978.

23. *Marx K.,* The Ethnological Note Books (studies of Morgan, Phear, Maine, Lubbock). Ed. by L. Krader. Assen. 1972. (Published under the title : *The Ethnlological Note Books of Karl Marx.*)

24. *Milinda Prashna* [Questions of Milinda]. Bengali tr. of *Milinda pañho* by Pandit Shreemat Dharmadhar Mahasthavir. Calcutta. 1977.

25. *Monz H.*, Karl Marx : Grundlagen der Entwicklung zu Leben und Werk. Trier, 1973 [quoted in : *Raiprikh K.,* O filosofsko-estestvennonauchnykh issledovaniyakh Karla Marksa (On the philosophico-natural-scientific investigations of Karl Marx)// *Voprosy Filosofii.* 1983. 12, p. 4].

26. *Narsky I. S.,* Dialectical Contradiction and the Logic of Cognition (in Russian). Moscow. 1969.

27. *Plato,* The Republic. Tr. by B. Jowett.// The Dialogues of Plato (in two volumes). New York. 1937.

28. *Senekina O. K..* F. Engels i istoriya nauyki i tekhniki (po rukopisnym materialam F. Engelsa v Institute Marksizma-Leninizma pri Ts K K P S S) [F. Engels and the history of science and technology (according to the manuscripts of F. Engels available in the Institute of Marxism-Leninism of the C C C P S U)// *Voprosy Istorii Estestvoznaniya i Tekhniki,* vyp . 3(32), M.. 1970. pp. 14-19.

29. *Smirnov V. A.,* On the Reconstruction of the Navya-Nyaya// *Indo-Soviet studies,* II. 1991, pp. 59-63 [Eng. tr. of Redaktorskie slova, *Vvedenide v Indiskuyu Logiku,* M., 1974].

30. *Staal F. J.,* Twee metodische richtÿnen voor de filosofie [quoted in : *Renou L.,* Panini// *Current Trendsw in Linguistics,* v.5. 1969, p. 497; quoted in : *Paribok A. V.,* On the Methodological Foundations of Indian Linguistics// *Indo-Soviet Studies,* I, 1990, p.19.].

31. *Volodarsky A.*, Soviet Studies on the History of Eastern Science // The History of Science : Soviet Research. Vol. II. Moscow. 1985.

32. *Yushkevich A. P.*, History of Mathematics of the Mediaeval Orient : A Survey of and Perspectives for Research// *Indo-Soviet Studies*, II, 1991, pp. 36-55.

33. *Yushkevich A. P.*, Mathematics and its History in Retrospective// Present volume, Special Supplement : *Marx and Mathematics*, Part Three, first article.

PART THREE : MATHEMATICSES

MATHEMATICSES :
PAST, PRESENT AND FUTURE

MATHEMATICS AND ITS HISTORY IN RETROSPECTIVE

ADOLF PAVLOVICH YUSHKEVICH

The present paper is a revised and supplemented version of a report read at the All-Union Symposium on "The Regularities and Modern Tendencies of the Development of Mathematics", held in September 1985, at Obninsk. The report dealt with the changes that have occured in our outlook about the development of mathematics from the ancient times to the modern period, as a result of the recent historico-scientific investigations. While preparing this paper I thought that it would be advisable to provide a prefatory retrospective, as to how the past of mathematics is viewed to-day : it would be a short retrospective review of the historico-mathematical investigations themselves. The literature used for preparing this paper is too vast, and the list that follows the text of the paper contains references to only a few books or papers, indicated against their corresponding numbers, more often only the name of the author of a work and the year of its publication has been mentioned.

Understandably, here we shall be dealing with only some of the changes that have taken place in our ideas about the development of mathematics at the different stages of its formation as a science. Nowadays, sometimes we come across the view that the mathematics of ancient Egypt, or Babylon, or China was yet to become a science and it became one only in ancient Greece.However, the historians of science are yet to agree as to which fields of learning may be called a science and which may not be.The present author is merely of the opinion that the aforementioned view is not sufficiently substantiated, and on this more later.

A Historiographic Retrospective. Historiography of mathematics dates back to antiquity. One finds its odd elements in the works of Plato and Aristotle, whose pupil Eudemus of Rhodes (incidentally, not a mathematician) was the first to author a treatise on the history of gemoetry. Afterwards, invididual scholars did turn their attention to the history of mathematics; but their work have long lost all significance. The study of the history of science was highly valued by such leaders of scientific and philosophical thought as F. Bacon and G.W. Leibnitz; and in the days of Enlightenment, its leading ideologists saw the motive force of progress in the growth and spread of knowledge, wherein mathematics and mechanics (inclusive of celestial mechanics) became the leading sciences. The first fundamental work on the history of these disciplines and of some parts physics — "The History of Mathematics" by the Parisian academician J.F.Montucla — was published in this period. Its first two-volume edition appeared in 1758 and, the second,much enlarged four-volume edition came out during the period 1792-1802, only after the death of its author. This work was carried out through to the end by the astronomer J.F. Laland and mathematician S.F.Lacroix [1]. This book was a great work of its time ; in spite of the then unavoidable gaps, inexactitude and the dated methodology, the modern reader will find interesting information in it, which however, should be used with circumspection. Almost simultaneously with the publication of the second edition of Montucla's work, came out a two-volume general history of mathematics by another academician of Paris — S. Bossu (1st ed., 1802), and a four-volume history of the physico-mathematical sciences by a professor of the Göttingen University A.G.Kestner (1796-1800).

The scope of the investigations into the history of mathematics was continuously widened during the 19th century.The study of the primay sources of the mathematics of the people

of medieval Orient began, the first translations of the works of the Arab and the Indian mathematicians appeared, critical editions of the works of the Greek authors Euclid, Archimedes, Apollonius and of the others were prepared, similar editions of a number of classics of the modern times were also begun, sometimes to be concluded only in the 20th century — these included the works of the mathematicians from R.Descartes and P.Fermat to A. Cauchy, B.Riemann and K.Weierstrass. The work in this direction has continued with greater intensity in the present century. Thus, the old plan of publishing the complete collected works of L.Euler (which began in 1911 and is nearing its completion only now) is being realised. The works of K.F.Gauss, N.I. Lobachevsky, G.Grassmann, P.L.Chebyshev, A.Poincaré, D.Hilbert and of others have either been published in full or in selections [for more detailed information see: the books by G.Loria [2] and K.O. May [3], which mention the classics published upto 1946 and 1973 respectively]. During the last decades of the 19th century M. Cantor, V.Buonkompayne, G.Enestrem and V.V.Bobynin took the initiative to start the first journals of history of mathematics[see : 2] and, the first courses on this subject were introduced in some of the universities — this is a rare phenomenon even to-day. The literature on the history of mathematics grew and, the famous four-volume history of mathematics by M.Cantor was published during the period 1880-1907 [4]; however, the fourth volume of this book was written by a group of scholars under the editorship of this great historian of mathematics. The work of Cantor covers the period upto 1799. It is still an useful reference book, though in certain parts it has become entirely outdated; what is more, in it the development of mathematics is viewed only in itself, outside the framework of general history and, often not even in connection with the mathematical natural sciences. The two excellent books by G. Zeiten (actually by I. Syuten) on the history of mathematics upto the beginning of the 18th century, at first published during the years 1893-1903, are of a different character : there the mathematical treatment of the subject-matter is much more deep in comparison to Cantor's work ; it is true though that they contain less of the details. Both of them have been translated into Russian [5]. In this period the interest in history of mathematics grew considerably among the mathematicians themselves, especially in the history of those disciplines in which they specialized. Hence the works on history of geometry by M. Shal (1837), those of A.Todd-Hunter (variational calculus, 1861; theory of probability, 1865), A. Enneper (elliptical functions, 1876), I. Yu. Timchenko(theory of analytical functions, 1899) and others. The aforementioned Zeiten was an outstanding specialist in algebraic geometry and a person of broad outlook.

Towards the end of the 19th and the beginning of the 20th centuries, the growth in the interest about history of mathematics was considerably promoted by the great German scientist and one of the initiators of the movement for the reform of mathematics teaching in the secondary schools, F.Klein. A three volume monograph on elementary mathematics, treated from the point of view of higher mathematics emerged out of his lectures read to the teachers of Göttingen University. First published in 1903, this book is saturated with historical materials. Its Russian translation saw two editions [6]. The history of elementary mathematics by the German pedagogue and scholar J.Tropfke [7], first published in a two-volume edition (1902-1903) and then extended upto seven volumes(1921- 1924), was mainly intended for teachers. After a long gap K.Vogel and his collaborators decided to

prepare a new edition of "Tropfke" and in 1980 a structurally quite excellent edition was brought out; this book on the history of arithmetic and algebra fully corresponds to the contemporary state of our knowledge on the subjects[8]. The subsequent new "Tropfke" (geometry) has not followed, however, owing to the demise of K.Vogel. Yet another widely known history of elementary mathematics was authored by the American scholar F. Cajorie; it was published in 1896. Its Russian Translation (1910) is accompanied by highly valuable supplements from the translator I. Yu. Timchenko [9].

Like many other mathematicians F. Klein too attached a great cognitive significance to the history of mathematics. He took the initiative to include innumerable historical informations into the famous German six-volume encyclopaedia of the mathematical sciences (1898-1934). Klein wrote in the preface of this practically nearly all-embracing collective work [10], that in it there should not only be a concise and generalised presentation of the modern condition of the mathematical sciences with their applications in the other sciences and in technology, but also a description of the evolution of the mathematical methods from the beginning of the 19th century should be provided with the help of carefully selected records and reference literature. There exists an incomplete French version of this encyclopaedia (1904-1914). One finds a fuller representation of the history of mathematics in the Italian encylopaedia of elementary mathematics — published in 3 volumes and 6 books under the guidence of L. Berzolari, J. Vivanti and D. Jili, in the years 1932-1950[11]. Here, apart from the informations contained in the main text on the history of elementary mathematics, there are independent sections on the main trends of modern mathematics and on the questions of didactics; and they follow the section on elementary mathematics. This "Encyclopaedia" is very rich and is within the reach of the students of the first years of any university. But unfortunately, it is not well known beyond the borders of Italy. Running ahead, let me add here : historico-mathematical essays or sections of essays occupy a prominent place in all the three editions of the Great Soviet Encyclopedia, thanks to the unfailing directions of the editors of the GSE, in particular of V.F. Kahan and A.N. Kolmogorov.

The ever growing interest in the history of mathematics and the recognition of its status as an independent and important section of the entire system of the mathematical sciences, in the 19th and 20th centuries, may be illustrated with the help of many facts; from among them I shall adduce only three. Two of them belong to the very beginning of the 20th century; they are, both related to the Second International Congress of Mathematicians held at Paris, in the summer of 1902.

The first incident is D. Hilbert's famous paper on the "Problems of Mathematics", read in this Congress on the 8th of August. In this paper 23 real problems from various areas were posed; they exerted a strong stimulating influence upon the subsequent development of mathematics. Hilbert's problems are usually viewed from the mathematical angle of vision,and this is understandable. But there is another side to this affair : Hilbert's judgements on the perspectives for the development of mathematics and the sorting out of its especially real problems, are based on a deep going analysis of its previous development. In his own words : "History teaches us that the sciences develop uniterruptedly. We know that every age has its own problems, which are either solved or are set aside as furitless and, substituted by

new ones, in the next epoch. In order to conceive the possible character of development of mathematical knowledge in the near future, we must take a look at those questions which still remain open, and survey those problems, which have been posed by modern science, whose solution we expect in the future. It seems to me that such a survey of the problems is especially contemporaneous today, at the dawn of a new century" [12]. Hilbert was an eminent mathematician of the end of the 19th and beginning of the 20th centuries, like A. Poincaré,and it is thus that he posed the question of the study óf the past of mathematics, in the interest of a creative prognostication of the perspective for mathematics.

The second noteworthy fact was the placing of M.Cantor's paper "On the Historiography of Mathematics" in the same Congress. It was one of the 4 plenary reports. Cantor began his survey with Montucla's work. There were 6 sections in the Congress. The sessions of the section on history and bibliography were held jointly with the section on teaching and methodology; these sessions were presided over by G. Cantor. The plenary report of V.Volterra on the life and work of three great Italian mathematicians — E. Belti, F. Briosky and F.Cazorati — was also historico-biographical in character. In all the subsequent congresses since then, there was always a section on the history of mathematics.

As the third example, mention may be made of the five-volume Soviet "Mathematical Encyclopaedia" (1973-1985). It contains innumerable historical informations and references as a matter of course, though there are no historico-mathematical articles proper.

Omitting the events of the historico-mathematical life till the end of the first world war, which disturbed the normal course of scientific progress, let us turn to the last half a century, marked by ever growing activisation of the historico-mathematical investigations, wherein a special mention must be made of the last 20-30 years. [One must mention, however, that, namely, in the years 1914-1919, F.Klein read his remarkable lectures on the development of mathematics in the 19th century, to a small circle of listeners, who gathered in his flat. These lectures were later on prepared for publications by R. Courant and O. Neugebauer; they were published in 1926, a year after Klein's death. A Russian translation of the first, historico-mathematical, part of these lectures was published in 1937 [13]. The second part is devoted to physics at the end of the 19th and beginning of the 20th centuries and to its mathematical apparatus; it contains short historico-scientific digressions, but they play a subordinate and insignificant role in it.] In this period many socio-historical, general cultural, ideological and scientific-organizational factors were in operation. We shall not list all of them, nor enunciate them in terms of their importance and shall be mentioning only some of them.

First of all, history of mathematics, like that of the other sciences, is organisationally constituted, with material support both at the international level and at that of the individual states. In 1929 the International Academy of History of Sciences was created and the first International Congress of the History of Sciences was held, at the initiative of a group of leading scientists from many countries. At present this Academy has nearly 230 full and corresponding members in its rolls; they are from many countries (26 of them are from the Soviet Union) (from the earstwhile USSR - Tr.). This Academy publishes its journal since 1948, and since 1968 has begun awarding a prize for outstanding scientific excellence, in the name of the great French historian of science A. Coire. After the founding of the UNESCO, an

International Union of the Historians of the Sciences was organised as its subsidiary, just like the International Union of Mathematicians, Specialists in Mechanics etc., all of which consisted of seperate national unions. The Soviet Association of the Historians of Science and Technology was established in 1957. There is no formal, juridical connection between the International Academy and the Internaional Union of History of Science, but in reality the members of the Academy occupy all the leading posts in the Union; and there are Soviet scholars among them. An important job of the Union and of its national sections is to organise international congresses (from 1929 to 1985 there have been 17 of them), national conferences, and symposiums of a general, as well as of a specific, character — dedicated to individual disciplines, problems, jubilee celebrations etc.

The most important pre-condition for speeding up the progress of investigations on the history of mathematics and of the other sciences, is the preparation of better workers with suitable specializations, in the institutions of higher learning and, the establishment of institutes of history of science. In different countries this has been done in different ways: great success has been achieved in this respect in the FRG, GDR (now Germany -Tr.), USSR (now exUSSR -Tr.), France and, of late in China; the USA has not been mentioned here, since there the preparation of workers and the organization of researches have their own specificities, and there is no scope for dwelling upon them in the present paper. As an example, here we shall briefly narrate the state of affairs in the USSR, mainly in the two centres at Moscow — one in the University, and another in the Academy.

Before 1917, (aforementioned) V.V.Bobynin taught an optional course on the history of mathematics, in the Moscow University, for quite some time. In the 30s, this course was renewed and later on made compulsory. A regular scientific seminar on the subject began to function since 1933 — it obtained all-union recognition ; research studentship on the history of mathematics was introduced, doctoral and post-doctoral work began to be defended in this specialization. After the second world war a special section was created for the histories of mathematics and mechanics, students' seminars were organised, and graduation theses were introduced in these disciplines. All this brought forth tangible results. This section is connected with the other kindered organizations in Leningrad, Kiev, Tashkent etc., as well as with the corresponding centre in the Academy of Sciences, USSR.

An interest in the history of mathematics existed in the Academy of Sciences of the USSR, since long. In the first years after the October Revolution, the Academy at first ordered the printing of A.V.Vasiliev's book on the development of mathematics in Russia from the epoch of Euler to that of Chebyshev (1921), and then of B.V.Steklov's book on mathematics and its significance for mankind (1923) — a book saturated with historical material. Like V.A.Steklov, many other members of the Academy, namely, A.N. Krylov, V.I.Smirnov, S.I.Vavilov, T.P.Kravets, P.O.Kuzmin, N.G.Chebotarev etc.— this list may be extended considerably — took an active interest in, and did study, the history of the physico-mathematical sciences. In the 30s and 40s certain measures were adopted, with the aim of imparting a more regular character to the investigations on the history of the sciences and technology, conducted in the Academy. The founding of the Institute of History of the Natural Sciences in 1945, and after its amalgamation with the Commission on the History of Technology in 1953, its transformation into the Institute of History of Natural Science and

Technology (with a branch in Leningrad) — was of decisive significance. It is the biggest institution of its kind in the world. Here a highly qualified group of historians of mathematics works in close collaboration, not only with their fellow-workers in Moscow and in the other Republics of the Union, but also with many foreign scientific centres and individual scholars, above all with those from the GDR, FRG (now united Germany — Tr.)., China, USA, France, Czechoslovakia (now the separated Czech and Slovak Republics — Tr.) and, Switzerland. In this Institute too, a permanent scientific seminar on the history of mathematics works and, specialists — research students and doctoral candidates — are helped in their work.

The total number of the more or less active historians of mathematics to-day, is not known. According to a directory published in 1978 [14], at that time their number was nearly 1500, now it must be considerably more and, probably, about 2000. Here, among other things, one must have in view the fact, that now investigations in this field are being conducted, not only in those countries, where the corresponding tradition has long since been established, but also in those, where earlier there did not or almost did not exist a national contingent of historians of the sciences; such countries include : the many Arab states, Turkey, India, China, Japan, Canada, those in Central and South America, and of course those Republics of the USSR (now CIS — Tr.), which constituted — socially and culturally speaking — the backward periphery of the Russian empire, before the October revolution. One of the consequences of the global decolonization of the earlier possessions of the imperialist states, has been a rapid growth of interest in them, in their own cultural past, and in general, in history.

Investigations in this field grew almost every month. The need to publish them, in turn, replaced the periodicals, that had been discontinued since the beginning of the 20th century. New historico-scientific journals or series of occasional thematic collections appeared; after the second world war their number grew — and continues to grow considerably.

A list of such publications, keeping it limited on the one hand to those that are more specialized, and on the other — to the most well known, in the chronological order of their appearence, is as under : "Istoriko-matematicheskie issledovaniya" ["Historico-Mathematical Investigations"] (1948-), published in the Russian language (wherein the papers of foreign authors are printed in their Russian translations; however, these publications mainly contain papers of the Soviet scholars); "Archive for History of the Exact Sciences" (1960-), publishes papers in the main European languages, save Russian (this is connected with the purely external conditions of publication — in this journal the papers of the Soviet authors are printed in other languages); and finally, the organ of the Mathematical Commission of the International Union of the Historians of Science — "Historia Mathematica" (1974-), publishes papers in 10 European and Oriental languages, as well as information on scientific activities, reviews and bibliographical surveys. There exists no data about the publications on the history of mathematics, though one may get some idea about their number from the fact that the 30 issues of the "Istoriko-matematicheskie issledovaniya"(owing to purely external reasons they were not published during the years 1967-1977) contained 600 papers. Apart from the aforementioned publications, there are the collections published by the Section of History of Mathematics and Mechanics of the Moscow University, by the history of mathematics seminars of the A.Poincaré Institute of Nantes and Toulouse and, the new journals being published in India, Japan etc. All this has not only opened new opportunities for publication, but has also stimulated further investigations.

We have stated above the approximate number of the investigators in the field of history of mathematics, and have added that their number is growing. We must mention in this connection the fact that simultaneously the mathematical preparation of these investigators has also improved — and this as a result of an improvement in the education of higher mathematics, determined by the rapid progress of the mathematical sciences themselves. The linguistic training of the historians has also improved:now original researches have become impossible or do become inferior in quality, without a solid knowledge of many languages.

The directory of historians of mathematics [14] contains the names of quite a few mathematicians — specialists in this or that branch of this science. Generally speaking, the relation of the mathematicians with the history of their own science has not been and is not uniform. It has lived through times of rise, as well as those of fall.

There are specialists, who are not at all interested in the history of science. On the other hand, one can count the names of dozens of mathematicians, among them there are those who are of the highest class, who are not only interested in the history of mathematics, but are active in this field (in particular, they add historical sections to their manuals), and they also organisationally co-operate with the progress of history of mathematics, at the level of organization of science. It is impossible to list the names of all such scholars. It is enough to name from among those of the older generation : P.S. Alexandrov, A.D.Alexandrov, B.L.van der Waerden, A.Weil, H.Weyl, J.Dieudonné, A.N.Kolmogorov, V.I.Smirnov, D.J.Struik ; and from among the younger ones — J.Dombr, H.Koch, Yu. I.Manin., A.N.Parshin, A.D.Solovev, V.M.Tikhomirov and K.Uzel. Of course this is a personal selection, and quite fortuitous at that, and the names of many a top mathematicians have been left out here — mathematicians who are systematically building bridges between mathematics and its history.

The hightening of the interest in history of mathematics among the mathematicians, especially among the scholars with a broad range, in the first half of the 20th century, had, in part, been conditioned by the crisis in the foundations of mathematics and the discussions generated by it, which drew the attention of many scholars to the historical retrospective. It was also influenced at a different level (in the first place, in our country), by the publication of the Russian translation of a part of the "Mathematical Manuscripts" of K.Marx, in 1933. [A greater part of Marx's Mathematical Manuscripts were published in 1968, though a complete edition of them remains to be published. —Tr.] One of the consequences of the aforementioned discussions has been a tempestuous progress of mathematical logic and, the follow-up action still continues. During the last few decades, the "storms" in the development of informatics and of the adjacent fields, as well as a revolution of its kind in computational mathematics, has again drawn the attention of a number of specialists to history. Clearly, to-day, in principle new paths are being outlined for the development of mathematics and, one of the means of trying to understand the paths of its further development, is to turn to its retrospective.

Cooperation among the mathematicians — the historians and the specialists, has already become an imperative necessity in our time. It is necessary for both the groups, and it is already yielding good results. Perhaps, here priority should be accorded to : 1) the publication of the classics and 2) to writing generalised works on the history of mathematics

of the modern and the recent times. It is enough, for instance, to mention the publication of the collected works of Euler, Gauss, Lobachevsky, Ostrogradsky, Riemann, Chebyshev, G.Cantor, Lyapunov and of Markov Sr. All these modern publications are accompanied by commentaries, without which it is difficult, and in any case less effective, to study the works of the said scholars. In this respect the recent editions of the classics of mathematics, are as a rule qualitatively superior to those of the 19th and early 20th centuries.

In respect of the works on the history of mathematics, one may mention the multi-volume Soviet history of mathematics from the ancient times to the beginning of the 20th century [15], the publication of which is still in progress, and the French history of mathematics of the 18th-19th centuries, wherein, in many sections, the 20th century too has been dwelt upon [16]. The books dealing with the history of mathematics in large geographical areas, as for instance, the collective works on the history of mathematics in our country, which cover the subject almost upto our time [17, 18], are of great value.

Jointly authored books, of the type just mentioned, are generally speaking preferable to the monographs produced by single authors : in our time one person can not produce a balanced and thoroughgoing work on the history of mathematics from the ancient to the modern times. To be spicific, the American mathematician M. Kline could not do it; his book [19] contains very interesting and competently written chapters, yet, in spite of its volume — containing 1248 pages — there are very substantial problems, related to important mathematical disciplines, for instance, regarding the theory of probabilities, and in respect of some regions, like China etc.

We have already mentioned two generalised works on the history of mathematics [15 and 16]. Now, a few words about the general orientation of the Soviet and the French collectives are in order. In the French case it was perhaps determined by the leader of the group. In the Soviet work, mathematics has been considered, not only at the level of its ideational development or self-development, but also as a social phenomenon, in its interconnections with the social requirements, with the other sciences, engineering, philosophy etc., briefly speaking, in the interconnections of the superstructure with the base (it is not for the present author — a member of the editorial and authorial collectives — to judge, how far this attempt has succeeded). In contrast, in the French work, attention has been concentrated, save in a few points of the introduction, upon the self-development of the ideas of the so-called "pure mathematics", which have almost exclusively been considered at the level of their immanental interconnections.[In the introduction of this work it has been said that "the most elementary concepts of modern mathematics" have been considered "in their historical contexts" and, in interconnection with their applications in the natural sciences. However, in the course of the work this declared objective has been very timidly realised.] It must be stressed, that in this work, generally speaking, one finds a very deep and substantial mathematical analysis of the historical process : almost all the authors are specialists in their respective fields of mathematics, who have painstakingly studied the essential literature on a given question, including many works of the mathematicians of our country (and this is not true of M.Kline's book). What we have said about this book [16], also holds good for an earlier work of N. Bourbaki — a remake of the historical essays contained in the various volumes of their "Elements of Mathematics", which have been published since 1939; this

re-writing was mainly done by A.Weil and J.Dieudonné [20].It may be stressed here, at the same time, that in the recent years, one more often notices a greater awareness about the social factors related to the development of mathematics, also in the foreign literature on the history of mathematics. The widely popular "Concise History of Mathematics" by the American geometer D.J.Struik, which was first published in 1948 and has since been reissued many times in English and in numerous translations, including one in Russian [21] — happens to be a notable example of this kind of book. Unfortunately, the valuable supplement by I. B.Pogrebyssky, characterizing the mathematics of the first half of the 20th century — laid aside by the author — has been dropped, in the latest Russian edition of this book.

Along with the generalized works, the last half a century has also seen the publication of many original works, including such monographs, as have substantially deepened our knowledge, as well as of translations from the Oriental languages — less known to the circle of historians of mathematics of Europe and USA — into the languages of Europe. These works encompass a very large time span and geographical territory. In the short list that follows we shall indicate only the names of the authors and the years of publication.

One must begin this part of the historiographical survey, with V.V.Struve's edition of the Moscow-Egyptian-Papyrus (1930), which significantly augmented our knowledge of the mathematics of *ancient Egypt* — earlier this knowledge was almost exclusively based upon the so-called Rhind Papyrus(1877). Next, we must mention the publication of and investigations upon the cuneiform *Sumero-Babylonian* texts by O.Neugebauer (1934-1951), F.Turo-Danjen (1938), E.M.Broins(1957), A.A.Waimann (1961) and others. In the field of *ancient Greek* mathematics one has to mention, at least the works of O.Bekker (1933-). M.Ya. Vygotsky (1941), I.G.Bashmakova (1958-), van der Waerden (1950), A.Sabo (1955-) and J.P.Vernan (1962). The study of the mathematics of the *middle ages* has been conducted in a number of regional directions. E.I.Bereozkina transalted almost the entirety of the so-called "Ten books" from *Chinese* into Russian ; she also came out with a preliminary survey of her investigations, in a book published in 1980. The Japanese scholar I. Mikami gave us the first sufficiently adequate description of the history of mathematics in *China and Japan*, in the English language (1913). His subsequent important papers are in Japanese and remain almost unknown in Europe, till date. In China proper, important investigations began later, first of all in the works of Li Yan and Tsyan Baotzun (1935-1937); their main essays still remain to be translated in the European languages. At present a large group of Chinese and European specialists are working on this problematique, and many important discoveries have been made — which have often been described only in the Chinese literature.,In the recent years, the original works of K. Sheml (of France), A.K.Volkov and those of the other young specialists have been published. Chronologically speaking, among the comprehensive works, first comes the volume devoted to mathematics in the multi-volume history of civilization in China, published in 1954 ; this joint work of J.Needham and Wang Ling contains a very rich bibliography, which is now, understandbly, somewhat dated [see :the bibliographical survey in the just mentioned book by E.I.Bereozkina, which, naturally, does not contain any reference to the publications since 1980.] Yet another direction of research had the mathematics of *India* as its subject matter. The first stage of the investigations in this field has been summed up in a two-volume book by B.Dutta and A.N. Singh (1935-1938);

subsequently, interesting investigations were conducted into the infinitesimal mathematics of India of the 15th-16th centuries by S.T.Rajagopal and T.V.Vedmurty(1949-). Afterwards other indian scholars, and to a lesser extent European scholars also conducted their investigations in this area. A.I. Volodarsky's book (in Russian) on the mathematics of medieval India came out in 1977. A great cycle of work has been conducted on the mathematics of the *Arab countries, Iran and Central Asia.* Here one must at least mention the names of P. Lukei (1938-), E.S.Kennedy (1947-), R.Rashid (1968-), Kh. Vernet (1952-), Kh. Samso (1966-) and, from among the Soviet scholars those of B.A.Rozenfeld (1951-), G.P. Matvievskaya (1961-), as well as their many colleagues and pupils. [In 1983 G. P. Matvievskaya and B.A.Rozenfeld published a bibliography of the literature on Arab mathematics and astronomy, in three voumes. It is a much more detailed bibliography, than the one published by G. Zuter in 1900-1902]. Finally, the fourth direction in the study of the history of medieval mathematics : its study within the frame-work of the *European region.* Here, the aforementioned K.Vogel — engaged in the study of the development of elementary mathematics in medieval Europe (and Byzantium) — made great contributions. However, the numerous works on the higher mathematics of medieval Europe, are of special interest. Here significant contributions have been made by P.Duhem,V.P.Zubov (1947-), A.C.Crombie (1953), G.L.Crosby Jr. (1955), M.Clagett (1959-), K.Wilson (1960), G.L.Busard (1961-), J.Murdoch (1961), V.S.Shirokov (1978-) and others, who have continued, and introduced more clarity to the investigations of the pioneers. Compared to the earlier understanding of the subject, medieval mathematics and its role in the global progress of science now stand illumined in a completely new light.

The principal works on the mathematics of antiquity have in the main been summed up in B.L.van der Waerden's well known monogaph (1950), and in the history of medieval mathematics, penned by the present author (1961); both of them have been translated in a number of languages; and in view of their years of publication, both appear somewhat outdated on a number of points, in the light of our present level of knowledge of the subject.

Neither the Arab countries, nor Europe knew of book printing with the help of moving types, in the middle ages — it began only in the middle of the 15th century — and, books were brought out in the manuscript form. That is why the historians of the sciences of this period are required to look for manuscripts and the search yields rich hauls. But on the other questions too, history of mathematics is largely indebted to archival investigations : these are related to the works of Newton, Leibnitz, Euler, Cauchy, Bolzano, Ostrogradsky, Bunyakovsky, Chebyshev, Kovalevskaya, Weierstrass, Dedekind, Luzin and others. In the instances herein mentioned and in many other instances, the obtained archival materials were of great significance, not only for the exact dating of various discoveries or for solving the questions of disputed priorites but also for the discovery of hitherto unknown aspects of the creativity of the great scholars, of the activities of large scientific collectives, of the international scientific community, of the emergence of scientific contacts among individual scholars and among the institutions, in which they worked, etc. As an example one may mention the three volumes of L.Euler's correspondences, letters that he wrote to the Peterburg Academy from Berlin, in the years 1741-1765. In this period he was a foreign member of the Peterburg Academy and, a full member of the Berlin Academy (he returned to Peterburg in 1766, where he had earlier worked from 1727 to 1741).

The complete history of mathematics, like that of any other science, consists not only of the history of its ideas, but also of the history of the people who created that science, that of their collectives. In this connection one may mention the fact that specialists and historians of mathematics (as well as those of the other sciences) took part in compiling the famous 16 volume dictionary of scientific bioghraphies, published under the editorship of Gillispi (1970-1980). Same is true of that big Italian biographical dictionary (1975), which contains less detailed biographies, as compared to the former, but is very rich in illustrations and provides a substantial survey of the development of the sciences since 1875 (more than 500 pages). Paucity of space forbids us to dwell upon the books about individual scholars, though they too are an inseparable part of the historico-scientific literature.

During the last 25 years a large number of monographs have been published on the history of individual disciplines;these are highly useful even for the specialist mathematicians. It is impossible to give a complete list of these monographs here and one is constrained to limit oneself to a few examples. All of them are of a very high scientific standard, though at times quite subjective in their evaluations of the role of individual discoveries or scholars. Such are some of the published monographs : on the history of the theory of numbers and algebra (G.Vussing, 1963; L.Novy 1973; A.Weil, 1983; B.L.van der Waerden, 1985; I.G.Bashmakova and E.I.Slavutin, 1985), on the development of set theory and theory of functions (F.A.Medvedev, 1965-1982; J. Kassina and M. Giiemo, 1983), on the history of the foundations of analysis from Euler to Riemann(A. Grattan-Guinnes, 1976; a book on R.Dedekind — P.Dugak, 1976; U. Bottazzini, 1981), on the history of the trigonometric serieses (A.B.Paplauskas, 1966), on the history of the theory of functions of the complex variable (S.E.Belozerov, 1962), on the history of differential equations and of functional analysis (K.Trusdell — on the equations, 1960; V.A.Dobrovolsky, 1974; V.S.Sologub, 1975; E.Lutsen, 1981), on computational mathematics and computing machines (G.Goldstein, 1977; I.A.Apokin and L.E.Maistrov, 1974), on the theory of probabilities (L.E.Maistrov, 1967 and 1980), on non-Euclidean geometry (B.A.Rozenfeld, 1976), topology(J.K.Pon, 1974), and logic (J.M.Bochenski — in English — 1961; N.I.Styazkin, 1964; T Kotarbinski, 1965). It is an incomplete list, but even if it is supplemented with a few more names of hitherto unmentioned monographs, even then that would not encompass all the basic mathematical disciplines. Work in this direction is of first order importance, and it is still continuing. Here, owing to insufficiency of space, we shall not be able even to mention many collections, devoted to the work of individual scholars, the development of this or that discipline in a given country, that of the different fundamental concepts, like the number, function, infinitary magnitudes, differential, integral etc. etc., and the activities of the individual institutions, academies, societies, periodicals etc. At times, even a series of articles (for example, those of O.B.Sheinin on the history of the theory of probabilities and its applications) is of no less importance,than this or that book.

Let us conclude the retrospective of the historico-mathematical investigations here, and turn to a retrospective of mathematics itself. The problems selected herein for consideration, make no claim to completeness and, naturally, they express the interests of the author. As far as possible, the following exposition follows the chronological order of development of mathematics and, takes care of its regional specificities.

Neolithic Mathematics. Our judgements about the formation of the earliest mathematical notions, formed in the pre-historic times, are based on archeological data, sometimes, upon the written legends and inscriptions, preserved on the architectural structures and utensils, on the pictures found on the rock surfaces of the cave dwellings ; and, finally, our ideas on the subject are also developed in analogy with the mathematical knowledge of those people and tribes, who are, or were a short while ago, situated at the lowest levels of cultural development. For some time now, the earliest developed culture known to us has been, the one that existed on the Indus river basin, in the middle of the 3rd millennium B.C. — the so-called Mohenjo-daro culture. Mathematical texts from this culture have not survived, and the inscriptions that have remained intact, have not been deciphered. We have extremely meager data about the arithmetical and geometrical knowledge of this culture, and still less — about its history; it appears to be close to the culture of Sumer. However, results of the investigations into the culture of Mohenjo-daro, are not in confirmity with the hypothesis regarding the introduction of this culture in the Indus Valley by some of the Aryan tribes (i.e. these studies do not confirm the hypothesis of B.L.van der Waerden ; on this more later). Recently, an even more ancient culture has been discovered in Upper Egypt, but it remains almost uninvestigated.

The earliest preserved Egyptian representations of numbers date back to the first half of the 4th millennium B.C., but the two, aforementioned , so far preserved, basic mathematical papyruses date back to the first centuries of the 2nd millennium B.C. The Babylonian cuneiform texts are divided into three basic categories : 1) the most ancient economic texts of Sumer, 2) the tables for multiplication, division and other operations, often also meteorological tables — dating back to the end of the 3rd millennium and, 3) some even later collections of problems — approximately belonging to the 9th-7th centuries B.C. All these written documents are close in time to the Indus Valley Civilization and, all of them go back to even earlier periods; whether or not there existed any direct contact between these civilizations, that however, remains to be established. A somewhat authentic information about the ancient Indian mathematics of the subsequent centuries, belong to a much later period ; it is related to that epoch when the religious books — the Vedas — were composed. It is contained in some essays, enunciating the rules for the *construction* of sacrificial altars, in the so-called "Śulva-sūtras", written, probably, in the 6th and subsequent centuries B.C.; these have come down to us in several variants. The Chinese culture is also very ancient, but it is practically impossible to isolate the authentic facts from the legends, contained in the later chinese chronicles. [For example, about the awareness of some particular instances of the theorem of Pythagoras in the 12th century B.C., and with its generalised form — in the 6th century B.C.] There is no doubt about the fact, that already in the school of Mo Zi, the philosopher and logician, i.e. in the 4th century B.C. or even earlier, the Chinese attained a high level of mathematical knowledge. By then, probably, many of those problems about which we came to know from the most ancient mathematical and mathematico-astronomical works — "Mathematics in Nine Books" and the "Treatise on Gnomon" — were already formulated and the methods of their solution were found; these books became famous through their editions published around the beginning of the Christian Era.

Sometime ago, in 1983, B.L. van der Waerden attempted a partial reconstruction of the mathematics of the Neolithic epoch. It predates the Egyptian, Babylonian, Indian and Chinese mathematicses and serves as their most important primary source [24]. In the opinion of this leading algebraist and outstanding historian of science, there existed a very highly developed mathematics in the territories of Central Europe and Great Britain, somewhere between 3000 and 2500 B.C. ; afterwards, this mathematics spread towards the South and the East, into the territories of Egypt, Babylon, India and China. Its traces are to be found in ancient Greece too — and this includes the works of Euclid and Diophantus; however, it was the Greeks who radically reorganised this ancient mathematics and created a deductive science based on definitions, postulates and axioms. It will not be possible for us, here, to enter into a detailed analysis of the ideas of van der Waerden. His book contains many interesting and valuable remarks, but his entire conception has been, on the whole, less convincingly argumented. Three basic arguments have been put forward in the author's introduction. The first among them — the presence of Pythagoras' theorem and of its application for transforming a rectangle into a square, in the "Śulva-sūtras", wherein the sides of the right-angled traingles used in the constructions are proportional to the "Pythagorean triplets" of natural numbers. "Pythagoras' theorem" and the extensive tables of Pythagorean number triplets were well known in ancient Babylon. From this, van der Waerden concludes — following A. Zaidenberg (1978) — that there existed some kind of a common source, of the Babylonian algebra and geometry, the Greek geometrical algebra and, the Indian geometry. The second argument—the existence of a large number of similar problems in the ancient Chinese "Nine Books of Mathematics" and in the ancient Babylonian texts, assuming in particular a knowledge of Pythagoras' theorem. And the third argument — towards the end of the 70s, a number of archeologists studied some megalithic structures, erected on some platforms, for ceremonial rituals, as well as for definitely oriented astronomical observations. These platforms are bordered with menhirs, placed along circular, elliptical or oval lines or along the circumference of forms flattened out into circles. It is possible, some times, to determinately inscribe individual integral-numerical Pythagorean triangles into these figures. Such megalithic structures were erected since the middle of the 4th millennium B.C. and were widespread in Central Europe, Great Britain, Ireland etc., in the first half of the 3rd millennium B.C.; and, according to van der Waerden and Zaidenberg, this testifies to the existence of a highly developed mathematics in the Neolithic epoch and it influenced the entire subsequent development of mathematics.

The decisive argument of van der Waerden is as follows : the discovery of the Pythagorean theorem and of the Phythagorean number triplets, were great discoveries, and the great discoveries of mathematics, physics and astronomy are, save in the rare cases, made only once; independent discovery of the said theorems and number triples in ancient Babylon (around 2000 B. C.), India, Greece (where they were well known not later than the 7th-6th centuries B. C.) and China, is improbable. Some unknown people took all these with them in the course of some migrations to the East.

This argument cements the entire conception of van der Waerden. It has been illustrated with the examples of momentarily invented epicycles and eccentric circles, of the establishment of the sphericity of Earth, the heliocentric system of Copernicus, the three laws of Kepler,

and the laws of motion of Newton as well as his law of universal gravitation, the laws of optics etc. Independent discoveries — for example, of the non-Euclidean geometry by Lobachevsky, Gauss and Bolyai — are very rare. There is inexactitude in van der Waerden's enumeration; for example, R. Hooke discovered the law of universal gravitation independently of Newton; it is true, however, that he could not construct a system of celestial mechanics. The defect however, is not with the particular instances of inexactitude; independent discoveries are by no means a rarity in the history of mathematics and of the sciences in general. Here are some examples : the logarithmic tables of Napier and Briggs, the calculating machines of Shicard and Pascal, the analytical geometry of Descartes and Fermat, the differential and integral calculus of Newton and Leibnitz, the theory of elliptical functions of Abel and Jacobi, Dedekind's and Zolotarev's theory of cut, the special theory of relativity of Einstein and Poincaré, Urison's and Menger's topological theory of measure... This list may be indefinitely extended further and, in general, in the given realm of questions, it is difficult to count and mutually compare the probabilities. One way or the other, according to van der Waerden, when a theorem like that of Pythagoras, is found in different countries, then the best course open is to accept the hypothesis of their dependence upon a primary source and to use it, as a heuristic principle.

It stands to reason, that the question of dependence or independence of identical discoveries in different cultural environs, requires to be investigated. Only this much is certain, that the solution of this question must not be based upon highly indeterminate probablistic estimates and unprovable presuppositions about the course of development of humanity. Having put forward his hypothesis and heuristic principle, van der Waerden himself then and there notes many points of contact between the mathematics of China and Babylon or India and Greece; incidentally, these comparisons, made by him, are highly interesting and deserve serious attention. But if such points of contact, yet to be studied in their full scope, did exist, then it is legitimate to ask oneself : were not the theorem of Pythagoras and the Pythagorean triplets born in the civilizations of Mesopotamia, from where they spread out in different directions ? Why assume the existence of a highly developed Neolithic mathematics in Europe, in the 4th-3rd millennium B.C., about which we practically know nothing, when we know for certain that a Sumero-Babylonian mathematics did exist, which is known to us, at least in part ? And what makes the hypothesis of a single source more preferable to the hypothesis of independent discovery of the theorem of Pythagoras, in course of the progress of architecture, that developed upon the ground reality of the general civic and ritual requirements of the people of a number of regions, which did attain similar levels of culture, at approximately the same time ?

About the integral numerical Pythagorean triangles, which may be inscribed within the contours, along which menhirs were placed in a number of instances, it is not at all understandable, as to why the builders of the structures were in need of them. Traces of such triangles were not retained. And the contours themselves — be they spherical, flattened out and consisting of the arcs of circles of different radii, oval or even near elliptical — were outlined, one should think, with the help of simple string contraptions. Right-angled triangles are not necessary for all such constructions.

Ancient Orient. While going over from the unwritten evidences of mathematics to the most ancient written mathematical documents, one must first of all turn to Egypt and Babylon. Here the information at our disposal, is clearly fragmentary, but nevertheless, it does allow us to judge the systems of numeration, methods of computation and predominant types of problems of both the civilizations. It appears, that in Balylon, that component of mathematics was considerably more developed, which we can call algebraic : here we find cycles of problems expressed in quadratic equations or in systems reducible to them — the Egyptian papyruses do not contain such problems; we have already mentioned the fact that the theorem of Pythagoras and the Pythagorean numerical triplets were known to the Balylonians. One of the Egyptian papyri dating back to the beginning of the 2nd millennium B.C., contains a simple algebraic problem, where the sum of the squares of the unknown quantities is given, there exits a given linear interconnection among the unknown quantities and the solutions — 6, 8 and 10 — happen to be double of the simplest Pythagorean numerical triplets; but the text does not mention any triangle, thus it would be hasty to conclude about any acquaintance with the general theorem of Pythagoras and with the Pythagorean numerical triplets in ancient Egypt.

During the last few decades, there were only a few substantial discoveries in this section of the history of mathematics : the most interesting among them being the "Balylonian numbers" detected by A.A. Waimann (1957) — these are numerical triplets, expressing the ratios of length of some line segments, parallel to the bases of a trapezium, which divide it into paired bands of equal area; these numbers turn out to be Pythagorean numbers, their sums, and differences. But the study of the mathematics of Babylon and Egypt produced a number of interesting reconstructions of those methods with the help of which various problems were solved there. In both the cases, the texts contain only calculations, providing the solution or even, straight off, the answer, without any explanation : the enunciation is prescriptive in character and does not include any such element, which we would have now termed theoretic. It is clear, however, that the solutions of many problems could not have been obtained purely empirically. This is true, for example, about the rule for calculating the volume of a truncated pyramid (Egypt) or, about the solution of quadratic equations (Babylon); it is not likely, that the rule for the summation of the sequence of the squares of natural numbers and, the correlation among the Babylonian and Pythagorean numerical triplets (Babylon) or, the (approximate) equality of the area of a circle with that of a square, a side of which is equal to 8/9 of the diameter of that circle, were detected accidentally. All the texts of the ancient Oriental mathematics, known to us, highlight only one side of it : these are either manuals for solving a definite type of problem, or collections of exercises with answers, and sometimes with verifications. There is no doubt, that there were mathematicians with a command over the tools of arithmetical, algebraic or geometrical deductions. This apart, we do know that when the texts under consideration were being composed, mathematics had already attained such a level of development, that apart from the problems generated by the direct requirements of economic, political and technological practice, it also handled those problems which were bereft of all practical significance, those which arose in course of the development of mathematics itself. The computation of the area of a rectangle with given sides, is an elementary practical problem, solved arithmetically. Reflection on it

gives rise to an abstract algebraic problem : the determination of the sides of a rectangle in terms of a given area and perimeter.

Some historians of science (A. Sabo, I.G. Bashmakova, P.P. Gaidenko and others) put forward the view, that only in Ancient Greece did mathematics arise as a science, when significant portions of this discipline began to be constructed in the form of deductive axiomatic systems. They opine, that there exists no ground for thinking, that mathematical knowledge was formulated in ancient Egypt or Babylon, into such systems; there, many results were obtained empirically or through unsubstantiated generalization from particular modes of calculation or measurement. Notwithstanding this, the following remark of N. Bourbaki [21] is fully convincing : it is not possible to view the entirety of Babylonian algebra as a simple collection of exercises, solved empirically, to the touch, and if it does not contain any "proof" in the formal sense of the word, then some sort of, not yet fully realized, logical arguments were put forward. By using the analogue of a mathematical terminology, used in another context, we may call ancient Oriental mathematics — "piece-wise deductive".

It has already been said in the beginning of this paper, that the words "proof" and "science" do not have any univocal meaning when they are viewed in their history, different contents were put into them at different times. The refusal to call the mathematics of Egypt and Babylon — science, because there are no proofs in their written documents, is as groundless, as the exclusion of impressionism or the abstract school from painting, because they are "not realistic", and of the works of Eluard or Khlebnikov from poetry, as these do not resemble those of Verlen or Blok. The same is true also in respect of the ancient mathematical texts of China and India.

Ancient Greece : Hellenism. After Egypt and Babylon, it is natural to turn to Ancient Greece. There are several aspects of the problem of the sources of Greek mathematics, the first among them being the question of Oriental influences. Taking it up in the case of Egypt, B.L. van der Waerden now, as they say, elevates it, in the ultimate analysis, to a hypothetical European culture of the Neolithic epoch; O.Neugebauer takes into consideration the emergence of the theories of irrationality, proportions and integrations within Greece and thinks that the Greek geometrical algebra shows Babylonian influence, which became stronger in the beginning of Hellenism, and he raises doubts about the roles of the Ionic school and of Pythagoras ; I.G. Bashmakova is of the opinion that Pythagoras is the creator of mathematics as a science; A.Sabo gives precedence to the influence of the Eleatic school and the introduction of the rule of contraries; L.Ya. Jmuid has recently raised doubts about the presence of Oriental influence ... It is apparent that opinions are changing and, clearly, the discussion around this question will continue. Perhaps, the question of formation of Greek mathematics should be considered within the wider frame-work of social and ideological development of the entire Mediterranean culture. Here the reader is recomended to get acquainted with the materials published in the collection : "Metodologicheskie problemy razvitiya i primeneniya matematiki" ["Methodological Problems of Development and Application of Mathematics"] (M., 1985), especially with the section : "Metodologicheskie aspekty stanovleniya matematicheskovo znaniya" ["Methodological Aspects of Formation of Mathematical Knowledge"]

While dealing with the problem of formation of the mathematical deductive method, in that specific form, which it assumed in Ancient Greece, i.e. in the first place, in the

axiomatization of geometry (but not of arithmetic, the reason behind which has been investigated by S. A. Yanovskaya in 1958), from the very first steps one runs against the non-univocality of the possible interpretations of the ancient idea of the infinite, especially in the early stages, and of the corresponding terminology. What gave rise to this idea? How to understand the "apeiron" of Anaximander [apeiron — a concept introduced by Anaximander of Miletus (c. 610-546 B.C.) to denote boundless, indefinite, qualityless matter in a state of contsant motion — Tr.] or the "aporias" of Zeno of Elea (c. 490-430 B.C.) [aporia — a problem which is difficult to solve, owing to some contradiction in the object itself or in the concept of it — Tr.] ? There is no doubt about the need for discussing these questions, but we are still far away from their unanimous solution.

Having mentioned the names of Anaximander and Zeno of Elea, one has to state that the problem of infinite did have a decisive impact upon the entire methodology of Greek mathematics, upon its various aspects. 60 years ago H.Weyl wrote, that mathematics is the science of infinity and if an intuition of the infinite was characteristic of the Oriental world, where it did not give rise to any question, then the Greeks reconstructed the polar opposition of the finites and the infinites into powerful instruments for the cognition of reality. Unfortunately, the opinions about the problem of infinity and about the infinitesimal methods of the ancient Greeks, are so divergent, that one has to refrain from characterizing them in the present paper.

In the recent times, unfortunately, the study, almost only, of the "Arithmetic" of Diophantus, has come to occupy one of the foremost positions in the study of the history of Hellenic mathematics. I.G. Bashmakova was the first to produce a deeper study of this work, utilising the tools of modern algebraic geometry (1972-); soon it was independently extended to the study of R.Rashid — the so called Arab Diophantus, by J.Sesiano (1974-) and, to that of the so-called "Diophantine analysis" upto the epoch of Fermat (I.G. Bashmakova and E.I. Slavutin, 1984); these studies threw a new light upon the formation of this field of mathematics, which played an important role in the development of theory of numbers and algebra. In a recent book on the development of the theory of numbers (1983), A. Weil has related the entire work on indeterminate analysis upto the time of Vieta [1540-1603] and Bachet [1581- 1638], with the pre-history of the theory of numbers, since, therein, the attention was fixed, not only on the search for the integral, but also on that for the rational solutions. Weil has considered the works of Diophantus mainly in connection with those of Fermat, whom he considers to be one of the founders of the modern theory of numbers, together with Euler, Lagrange and Legendre.

The "Arithmetic" of Diophantus exerted a direct or (and) indirect influence — mediated through some unknown links — upon the development of Arab algebra. It has been a great influence. We must mention here the fact, that so far the very emergence of the "Arithmetic", has almost always been viewed as an isolated event in the development of the mathematics of the Alexandrine epoch. This work determined a trend of thought different from the classical one, proposed by Euclid, Archimedes and Apollonius. To all appearence, it was the result of a synthesis of the Classical and the Oriental traditions; the creation of the empire of Alexander of Macedonia and, after its fall — the emergence of several Hellenic states, created the general historical preconditions for this synthesis. There are common elements in the

works of Diophantus and in those of Hero of Alexandria, who lived nearly two centuries before the former. However, there still exists a very big gap in the primary sources known to us, which can not be filled by the data on hand about the problem of piers, related with the name of Archimedes. The ancient method of solving this problem is still not known to us. The possible connection of "Diophantine Analysis" with the mathematics of India and China also remains unknown. In India and China too, the integral and rational numerical solutions of different kinds of indeterminate algebraic equations, did occupy an important place.

The Middle Ages. As we go over to the Middle Ages, we must first of all state that this term is unsatisfactory, even if because of the fact that in different regions its natural boundaries belong to different centuries. In the absence of a better name, however, we shall be using this term. In places it began at the junction of the pre-christian and the christian eras, at others even later and, came to an end in the 15th-16th centuries. The states that existed in this period, over considerable parts of the territories of China, India, the Arab countries and Europe, were in the main similar types of feudal (? — Tr.) economic and political formations [while discussing the ancient and the middle ages, the question of the Asiatic etc. modes of production can not be/should not be avoided — Tr.]. They attained almost the same levels of technical and material culture. It stands to reason, that the exchnges of material and spiritual values that took place among these regions, were by no means regular and, were interrupted by wars and internal disorders. A natural consequence of all this has been the emergnce of similar practical problems before the mathematics of these four regions. In some of the works — written in the period 1958-1961 by the present author, often in collaboration with B.A. Rozenfeld — a conception of medieval mathematics has been proposed, wherein it has been considered as a single whole; but the specificities inherent to the mathemtics of each of these regions have been taken note of there; these specifics were largely the consequences of even earlier scientific, philosophical and religious traditions prevalent in these regions. That is why, it would be better to consider the changes that have taken place in the retrospective of mathematics, during the last few decades, separately, within the frame-work of each region. It must be mentioned initially, that during the last few decades considerable success has been achieved in the study of the medieval mathematics of many Oriental countries ; and, to a great extent this was made possible by the decolonization of the territories under the control of the imperial powers, the emergence of independent states in Asia and Africa, as well as owing to the fast progress in those Republics of Central Asia and of the Caucasus, that were backward areas before the October revolution in Russia and, were in the best of circumstances — second grade areas of the Russian empire.

China. Our knowledge of the development of mathematics in China has greatly increased in the recent years. European scholars obtained their first solid informations about Chinese mathematics through an English language book (1913) by the Japanese scholar I. Mikami. In the first half of the 20th Century, work was conducted in this field in Europe and in China, mainly independently of each other. During the 30s-60s, considerable contributions were made by Li Yan and Tsyan Baotzun, and by Mikami, who continued his investigations; however, they wrote mainly in Chinese and Japanese and, for a long time their books and papers were accessible to only a few European or American historians of science (now the number of sinologists and historians of mathematics have increased). That is why

the publication of a generalised work on the mathematics of China, by the English sinologist (and biologist) J. Needham, was an event of great significance; Needham wrote this book (1959) in collaboration with the Chinese mathematician Wang Ling. Later on the work gathered momentum in China and in Europe. Many of the classical works of Chinese mathematics were translated in the European languages, and books were written about the great mathematicians of China. From among these translations one must first of all, mention the Russian edition of the tracts of the so- called "Ten Books", which have been provided with commentaries. The last of these "Ten Books" was written in the 7th century and, the earlier "Nine Books of Mathematics" were published some where near the beginning of the Christian Era. E.I. Bereozkina's translations of these works were published in the period 1957-1985; and in 1980 she initially summed up the results of her investigations in a special monograph, wherein she has considered the attainments of the mathematicians of China upto the beginning of the 14th century, but somewhat more briefly. K.Vogel prepared a German translation of the "Nine Books of Mathematics" (1968) and, this is accompanied by his own commentaries. A collection of papers on the same book and on its most important commentary, composed in the 3rd century by Liu Huei, has been published in 1982, in the Chinese language, along with an English resume. The authors of these papers — Bai Shanshu, Li Di, Shen Kanshen and others — made a significant contribution to a fuller study of the "Nine Books of Mathematics". [The present author is greatful to A.K. Volkov, for translating parts of the big chapters of this collection, as well as Shen Kanshen's review (1985) of E.I. Bereozkina's book (1980) mentioned above — from Chinese. Kanshen has justly stressed the importance of the commentaries of Liu Huei, in his review of Bereozkina.] For nearly one and a half thousand years the mathematicians of China, on very many instances, took their cue from these "Nine Books of Mathematics", and this explains the special attention that has been paid to it ; see : the bibliography prepared by the German sinologist G. Kogelshats (1981).

The epoch of pre-decline flowering of mathematics — above all of algebra and theory of numbers — in ancient China of 13th century, has been the subject matter of a number of important investigations. A Belgian scholar U. Libbrecht (1973) made a detailed study of a treatise by Tsin Tsziu Shao ; Leim Lai Young, who works in Singapore, published (1977) an English translation of a treatise by Yang Hue; J. Go (1977) published a French translation of the works of Zhu Shijie, and used therein the special symbolism devised in that epoch. K. Sheml (1982), just like Go, used a semi-symbolic language, in his doctoral dissertation on an algebraic treatise of Li Ye — a contemporary of Qin Jiushao. Unfortunately, this dissertation, as well as a much earlier work by another French sinologist K. Shrimpf (1963), on the mathematics in China upto the 7th century, has not been published.

In this connection, special mention must be made of the papers of Ho Pen-lok (Malaya) on Qin Jiushao, Zhu Shijie, Li Zhi (or Li Ye), Liu Huei and Yang Hue, in the 3rd, 8th and 14th volumes of the American Dictionary of Scientific Biography [22].

As a result of all these investigations our knowledge of the development of mathematics in China, has been greatly extended. It appears now that its arithmetico-algebraic component is richer than what it was thought to be, in the beginning of this century. At the same time, a number of questions remain unsolved and far from all the important primary sources have been studied till date. Undoubtedly, there had been mutual interaction among the mathematicses

of China, India and the Arab countries, which was reflected even in Europe ; the migration of similar or identical problems, and not merely the deductions constructed upon the shaky principle of *post hoc ergo propter hoc*, testify to this effect. Modern historians of mathematics do take note of all these moments, but by far not all of them are clear and definitive.

P. Liukei has advanced a hypothesis about the transference of the rule of two false positions and of the methods of extracting the square roots and cube roots of rational numbers, to the Arab countries from China; but it lacks certitude. Perhaps the Belgian sinologist L. van Hee was the first to put forward the idea, prevalent till recent times, that Liu Huei solved the problem of measuring inaccessible objects and the distances upto them, by basing himself upon the similarity of traingles. According to a recently advanced hypothesis, Liu Huei used some methods which are characterestic of the Greek geometrical algebra. We have already mentioned the noteworthy proximity, of the treatment of the basic ideas of geometry, in one of the works of the Moist school 4th century B.C., with a somewhat earlier ancient Greek treatment of the same. Exactly in the same way, the similarity between Liu Huei's methods of approximate calculation and those found in Archimedes' "Measurement of Circle" — notwithstanding their inessential technical differences — has been mentioned more than once. All of this leads to an idea about the existence of scientific contacts between China and the Hellenic countries : trade was conducted between China and the Roman empire. In the more recent years an observation of D.R. Wagner (1978), to the effect that the ancient Chinese mathematicians used the so-called principle of Cavalieri, in their studies of the problem of cubature of a sphere, has become an object of special interest in Europe ; apparently, it was known much earlier, to Mikami (I have not yet been able to verify it). The history of this question is as follows. Liu Huei expressed the volume of a sphere in terms of the volume of a body, contained within the surfaces of two cylinders, inscribed in a cube and having mutually perpendicular axes, but he could not determine the volume of this body. It is very likely that Liu Huer used the so-called principle of Cavalieri, though this principle has not been formulated in any of his texts known to us. We first come across a formulation of this principle in the 5th century, in the writings of Zu Chongzhi. [It is difficult to translate this formulation with exactitude, as the corresponding terms do not have fully determinate mathematical significance in latter Chinese speech; while referring to Anaximander's concept of "apeiron", we have already mentioned the possibility of non-univocal understanding and translation of the ancient terms.] Zu Chongzhi's son Zu Heng applied this principle and found out that the volume of such a body is equal to the 2/3 of the cube, which gives us the cubature of a sphere. It is remarkable that the result of Zu Heng has been formulated in the 2nd proposition of Archimedes' "Epistle to Erastophenes"; in this work the method of indivisibles has been regularly used for heuristic purposes (but not for obtaining any "strict" proof); unfortunately, the conclusion of this proposition, contained at the end of the "Epistle", is not known to us. We do not find a statement of the "Principle of Cavalieri" in the Greek texts known to us, however, Archimedes' quadrature of the ellipse, viewed as the result of compressing a circle, leads to the thought, that in essence, this principle had been used by him intuitively. Personally to me it seems plausible, that there had been a Greek influence upon the infinitesimal methods of the Chinese mathematicians of the 3rd-5th centuries.

On this score Leim and Shen Kanshen display greater restraint (1985), when they say that Liu Huei's text contains no evidence of a Greek influence; such an influence is discernible only later on, in the works of Mei Wendong — a mathematician active in the 17th-18th centuries.

On the whole, however significant may the achievements in the study of Chinese primary sources be, still a very great amount of work remains ahead of us. Perhaps one of our primary tasks is to study all the hitherto known commentaries on the "Nine Books of Mathematics", which shed light upon the methods employed for solving those various problems, which we now classify as algebraic, number-theoretical and geometrical. The Chinese treatises, beginning with the "Nine Books of Mathematics", do not contain any proof, there we find only laconic mention of the methods of solving the problems; the substantiations of these methods are often met with in the commentaries. More than thirty years ago I expressed the opinion, that it would be unjust to judge the mathematics of China on the basis of its collections of exercises, that here and in the case of the mathematics of ancient Orient (and, I add, of India), we must distinguish between the manner of presenting a discipline in the text books, from the creative elaboration of the methods of investigation which preceeds it.Both in China and in Europe, the study of these commentaries are, in essence, in their early stage. Recently, A. K. Volkov examined the commentaries on the rules for calculating some areas (1985) and, therein he noted that in the mathematics of ancient China, the very concept of "proof" and the "systems of proofs", have their own specific characteristics : not an axiomatic theory, but a theory of models happen to be a more adequate analogue of the ancient Chinese logical system, and their criteria for deciding about the correctness of propositions correspond to this; this question deserves a more detailed study. The intensive work that is being carried out in this field in China, USSR (now erstwhile - Tr.), France, FRG (now Germany — Tr.), and in the other countries, will soon yield new results and, one may say, newer postulation of the problems too. The development of mathematics in China beyond the classical period, i.e. in the 14th century and afterwards, has not at all been touched upon here.

India. While dealing with B.L. van der Waerden's hypothesis about Neolithic mathematics, we have already mentioned the latest investigations on the history of mathematics in India. Apart from the more detailed analysis of the works of individual mathematicians like Shridhar or Mahavir (A. I. Volodarsky, 1966, 1969), the reconstruction of the solutions of some problems in the Āpastamba "Śulva-sūtras" (A. I. Raik and V. N. Ilin, 1974), the recent observations of R. Singh (1985) about the so-called Fibonacci numbers of 7th-8th century Indian mathematics or, the works on the history of Indian astronomy by D. Pingri (USA, 1963) and A. K. Bag (India, 1966-), the most interesting attempts in this field were concerned with the exact determination of the connections of Indian science with the science of the other regions and, with its place in the overall progress of mathematics. The aforementioned book by A. I. Volodarsky contains an overall survey of the work done prior to 1977. [Here the reference is to : *Volodarsky A. I.* Ocherki srednevekovoi indiiskoi matematiki (Esays on Medieval Indian Mathematics). M.: Nauka, 1977, 182 pages. — Tr.] It appears that, comparative analysis must be further continued in this direction. It is a fact, that here the investigator has to face a deficiency of exact informations, so much so, that even the emrgence and the earlier stages of development of the now commonly accepted system of decimal positional numeration, remain largely unclear.

Arab Mathematics. During the last quarter century, our ideas about medieval Arab mathematics have changed no less substantially, than those about mathematics in China. The term *Arab Mathematics* has struck roots and comes in handy, though in fact, at issue here is the mathematics of the people of many countries[where at a given period of time Arabic has been the principal language of science — Tr.], stretching from the Pyrenean peninsula, through the northern, Mediterranean regions of Africa, Near East, Central Asia and further ahead, roughly upto the present borders of China and India [in the case of India it should mean upto the India-Burma border of Mughal India and, in case of China — at least upto the eastern borders of Sinkiang or Chinese Turkestan — Tr.]. The sources of Arab mathematics remain largely uninvestigated, they go back to the mathematics of Babylon and Egypt, a large number of Hellenic states, Byzantine, and to the science of ancient Khwarazm, not to speak of the much later influences which came through the contacts with China, India etc. Here one may put forward a number of questions, but in the absence of exact data, it would be more than difficult to answer them. We have left out such questions in the present paper. But one must at once make one observation about the expression "Arab Mathematics".We are unable to find a similarly brief expression, which may be a substitute for it; but one has to stress the fact that even a brief bibliographical survey of the original literature shows the special significance of the contributions of the mathematicians of Central Asia and, this justifies a separate treatment of the history of mathematics of Central Asia of the period under consideration, in a number of books. When one deals with the cultural developments of the Central Asian Republics of the USSR [now CIS — Tr.], which were, quite understandably, closely connected with the culture of those regions, where Arabic or Persian had been the pricipal language of the scholars, then such a separate treatment becomes essential.

A more detailed study of the already known Arab works, and, to a greater extent, an analysis of a large number of mathematical manuscripts preserved in the various libraries and archives, showed that medieval Arab mathematics did attain a scientific level, which is much higher than what it was earlier thought to be. It goes without saying, that the mastery of the Greek scientific heritage, which was one of the consequences of Arab expansion and of the formation of the Arab states in the territories that were earlier under the rule of Rome, was of great significance for the progress of scientific and philosophical ideas in these states, which was often supported (and sometimes opposed) by the rulers of these states as they changed hands.It is enough to mention the scientific school of Bagdad, of the end of the 8th-9th centuries, which blossomed soon after the stabilization of the Caliphate of Bagdad and, the Samarkhand school of the first half of 15th century, during the rule of Ulugbeg. Thanks to the opportunity of quick assimilation of the heritage of Greek ideas and the very turn of thinking, the science of the Caliphate of Bagdad found itself in a situation that was much more favourable than the one in which science found itself in India and, even more compared to the situation in far off China. But an entirely wrong approach to Arab science, including Arab mathematics is prevalent till date; according to this interpretation Arab science is nothing but a transmission point between Greece and Rome on one side and the Europe of the middle ages and of the beginning of the modern times on the other. This conception was clearly formulated by E.Renan, more than hundred years ago, in 1863. It was he who put the expression"the Greek miracle" into circulation and, considered Arab science to be a reflection of Greek science, combined with the influences which came from Persia and India.

In defence of his thesis Renan adduced even philological considerations : he, and not he alone, thought that the Indo-European languages are more suitable for expressing abstract concepts, than the Semitic ones etc. However, Renan was not an historian of science; but his ideas were developed by such important specialists in the field as P.Tanneri and P.Duhem; these opinions are shared by some of the leading scholars even to-day. This Eurocentric conception of history of science, does not correspond to the actual course of scientific progress, and it has been criticised many a times in the Soviet, as well as in the foreign literature. What is special about Arab mathematics is this, that here we find a magnificent development of certain trends which originated in Greek or Indian mathematics and, some important advances in new directions. It would be enough to cite some examples :

The first systematic construction of the decimal positional arithmetic, the principles of which were, however, borrowed from the Hindus. Introduction of decimal fractions and the method of extracting the n-th roots, by using binomial expansions (11th-15th centuries); the different numerical methods for solving algebraic equations, and besides this, an example of approximate solution of a transcendental equation, with the help of successive iterations; extension of Diophantine analysis, solution of indeterminate linear systems, properties of the friendly numbers, Wilson's theorem (9th-10th centuries).

An original theory of ratios and proportions; extension of the concept of number to the positive irrational numbers; arithmetization of the ancient teachings on quadratic and biquadratic irrationalities (9th-13th centuries).

Isolation of numerical algebra together with the algebra of polynomials as an independent science; a developed geometrical theory of cubic equations and, a geometrical theory of the equations of fourth power (15th century, the corresponding treatise is yet to be traced).

Different theories of the parallels, connected with the attempts to prove the 5th postulate of Euclid (9th-13th centuries).

A reconstruction of the 8th book of the "Conic Sections" of Apollonius (9th century).

New quadratures and cubatures (9th-10th centuries).

In this list we have not specially isolated those trends and results, where the Arab mathematicians happened to be pioneers; it is enough to state that where they broke entirely new trails, they went considerably further than their predecessors. We have neither mentioned the names of these mathematicians, nor the names of those historians of science, who have of late elaborated or are continuing to elaborate upon the entirety of this vast complex of disciplines, theories and problems: the lists of either of them would be very large ; one may find these names in the corresponding literature. [See, for example : *Matvievskaya G.P., Rozenfeld B.A.*, Matematiki i astronomy musuilmanskovo srednevekoviya i ikh trudy (VIII-XVII vv.)/ Mathematicians and Astronomers of the Muslim Middle Ages and their works (8th-17th centuries) / ; in 3 volumes (479+650+372 pages), M. : Nauka, 1983. — Tr.]. But in view of the special importance of the question of evaluation of the role of Arab mathematics in the subsequent forward movement of mathematics — a question, which has already been touched upon — it is essential to dwell upon it. As we have noted above, Renan's evaluation of the issue, continues to find its supporters even in our time. B.L.van der Waerden sticks to a clear cut Eurocentric position. He came forward with the following sketch of the emergence of modern science, in a seminar held in Oxford in 1961 [25], while discussing

J. Needham's paper on science in China. He said, Newton's mechanics is the basis of modern science, wherein three threads remain intertwined — each of which has emanated from Greece. The first among them is the planetary astronomy, leading to Copernicus and Kepler, i.e. to the necessary prerequisites of Newton's mechanics. The second thread : Newton's use of Apollonius' conic sections and of the entire structure of the Greek axiomatic geometry, which served as the model for Newton's axiomatic mechanics. The third thread emerges from the Greek mechanics of Archimedes and of some other scholars. There exists no doubt about the enormous significance of the Greek heritage for the sciences of modern Europe, and that includes the work of Newton. But only the Greek heritage was not enough for it. The discoveries of Copernicus, Kepler and Newton are essentially based upon the Arab traditions of a highly developed new algebra and trigonometric system. The intertwining of the Greek and the Arab threads of scientific development and their subsequent creative synthesis in the mathematics, mechanics and astronomy of medieval Europe, had provided the necessary prerequisites here.Not all the attainments of the sciences of the Arab countries, were known in medieval Europe, and a lot of it had to be discovered anew. But often, even a fragmentary introduction of the results of the Arab investigations, served as points of departure for important trends in modern European mathematical thought. Such, for example, has been the case with the Arab theory of parallel lines, an acquaintance with which in the 17th century played an important role in the first stage of evolution of the non-Euclidean geometry.

All the same, if the sources of the sciences of modern Europe go back not only to Ancient Greece, but also to the countries of the East, and the terms "western science" and "eastern science" become admissible only with some stipulations — the mathematics of medieval Europe did have its own specificities, which were only very insignificantly, or not at all, characteristic of the other times or regions. Two of these had a very important or even determining significance for the formation of the mathematics of the modern period.

Here, first of all, we have in view, the creation and systematic perfecting of symbolic algebra, in the 13th-16th centuries. The timid steps taken in this direction, in the Moorish countries, are not going to be taken into account here : thanks to the march of world history, these steps could not be continued, and they failed to exert any influence upon the subsequent progress of mathematics. The formation of symbolic algebra was of immense significance for the entire further development of mathematics, and for developments beyond its boundaries; it was Leibnitz who first evaluated the role of symbolism in human thought. Idea- and time-wise, the progress of symbolic algebra came along with such attainments of the 16th century, as the solution of the equations of 3rd and 4th power into radicals and, the introduction of imaginary numbers. Here the mathematics of Europe broke an unbeaten trail and this led to results of truly universal significance for the entire system of physico-mathematical sciences.

Another characteristic specificity of the mathematics of medieval Europe was connected with the distinctive development of some ancient natural-philosophic and scientific ideas, which, to a significant extent, go back, on the one hand to Aristotle and his school, and on the other — to Pythagoreanism and to Plato. Here, medieval (European) mathematical thought went far beyond the boundaries of that elementary mathematics, which was then known in all the four regions considered in this paper. On the one hand, it was the programme of

mathematization of the entire world of knowledge, put forward by the scholar from Oxford R. Grosseteste and his pupil R. Bacon, and together with it the development of experimental method and of the technical means of scientific investigations. On the other hand, it was the first, but already fully perceptible growth of the infinitesimal mathematics of a new type, elaborated first of all in the universities of Oxford and Paris (Sorbonne). An in principle new development of the infinitesimal ideas took place here, and then also in the other universities of Europe. Together with this renewal and the deepening of the ancient discussions about the nature of the infinite in both of its forms, of the potential and the actual infinity, continuity and discreteness etc., within a short while, as has been pointed out by N.Bourbaki, the foundations of a theory of change of magnitudes, viewed as functions of time, and of their graphic representation, were laid down — true, in a rudimentary form. The English (T. Bradwardine, R. Swineshead etc.) and the French (especially, N. Oresme) scholars of the 14th century made a bold attempt to quantify the basically qualitative natural philosophy of the Peripatetics, with the help of infinitesimal ideas. First of all, a new interpretation was given to those sections of Aristotle's physics, wherein the interrelations of force and motion and, force and resistance has been considered — and this turned out to be especially important for further developments; in other words, a reconstruction of the Peripatetic mechanics was undertaken; after that, all kind of changes of the continuous, and partly also of the piece-wise broken, measurable quantities or, in the terminology of the Peripatetics, the intensification — strengthening and remission — weakening of all kinds of "forms" or qualities, like heat, colour etc., as well as of goodness, sin etc. — the varying intensity of which were dependent upon their extensity — the spread of their intensity over finite or infinite intervals in space or time, were subjected to mathematical treatment. Here the simplest of mechanical movement, i.e. spatial displacement too belongs to the category of form. Generally speaking, not theological or ethical, but rather natural-scientific, and in the first place mechanical intensity, is at the centre of interests here.

A quite vast literature has been devoted to these theories of the Oxford and Paris schools, which were extremely close idea-wise, though coloured, so to say, in different tones (the English worked out their "calculations" at a more abstract-quantitative level; and the French "theory of widths and lengths of forms" made wide use of the graphic representations, which, however, were not alien to the "calculators"). The works of P. Duhem, published in the beginning of the 20th century, laid the foundations of the studies in this field; however, V.P. Zubov (1948) has convincingly demonstrated that Duhem was not at all impartial in his judgements. Here, once again, it is not possible to go into the details, and, by way of evaluating the teachings under consideration, it is enough to state that, therein we already find the formation of an idea of variability — flow (fluxus) of magnitudes, of momentary speed and accelaration, for which suitable, even Latin, terms were introduced and, the basic law and the other properties of uniformly accelarating motion were proved at an entirely abstract level, not connected with physics.

The calculations and the theory of widths and lengths of forms became quite widely known in the 15th and 16th centuries, first through manuscripts, and then through printed publications and, in the university-level teaching of a number of countries. During the last quarter of a century, a very large number of investigations have been devoted to this trend of medieval European mathematics, and this includes the publication of many

manuscripts; together with this, the infuence exerted by this trend upon the formation of the "mathematics of variable quantities" (A.N. Kolmogorov's term) in the 17th century — on Galileo, Napier, Barrow, Newton and his school, and very likely also on Descartes and Leibnitz — has been studied in greater detail. Of course, there has been no mention of the beginnings of analytical geometry and of infinitesimal analysis, of laying their foundations, in the schools of Oxford and Paris; at issue here are the anticipations of and the ideational preparations for the just mentioned sections of mathematics, the foundations of which were laid in the 17th century. In particular, such ideational connections are evident in the terminology of Newton's" method of fluxions" and, in the fact that till date, in different languages, we use the expression"flowing coordinates" (the terms "variable", "function" and "coordinates" were introduced by Leibnitz). Till date, the works of V.P. Zubov (1962 and 1965) remain the best Russian work in this field (the second book was published posthumously) but, naturally, the results of the later investigations could not be included in them.

Modern Age. In what follows, we shall be providing an even more fragmentary survey of the changes, that our notions about the historical past of mathematics have undergone, and we shall be illustrating it with only a few examples. Choice of the examples will be, to a considerable extent, connected with the recent archival investigations and, they are aimed at showing how these investigations are important for making our knowledge, not only of the medieval mathematics, but also of the mathematics of the modern and recent times, more exact.

One of the greatest events in this area has been the publication of the eight volumes of the mathematical manuscripts of Newton, edited by D.T.Whiteside and his colleagues (1967-1981). This edition contains excellent commentaries. It has fundamentally changed our ideas about the scientific career of Newton and about the chronology of his discoveries. We have also come to know of those of his discoveries, which remained unpublished in his lifetime, owing to various, and not always clear, reasons. Thus, Newton discovered the expansions of Taylor and MacLaurin; he was the first to attempt an axiomatization of the method of fluxions; he proposed remarkable examples of asymptotic serial expansions etc. Almost simultaneously, A.R.Hall brought out a 7-volume edition of the complete correspondence of Newton (1959-1977). Here. V. Boss' book on the spread of the ideas and discoveries of Newton in 18th century Russia (1972), deserves special mention.

The work on the scientific legacy of Leibnitz has been less successful; often his manuscripts are found to be chaotic in character and can be read only with great difficulty. [There are nearly 75000 separate works of Leibnitz, preserved in the Leibnitz Archives of Hanover, and many of them remain unpublished till date. On this see : Katolin L., "Mee byli togda derzhkimi parnyami ...". M., "Znanie", 1979, p. 70. — Tr.] Study of Leibnitz's legacy began long ago, but, in spite of many interesting results obtained so far, the principal work remains ahead of us. Some of the important relevant publications in the field are : the 1st volume of Leibnitz's mathematical, natural scientific and technical correspondence, pertaining to the period 1672-1676, published by I.E.Hofmann (1976) ; the same Hoffmann prepared a detailed name index to the entirety of Leibnitz's correspondence (1977) ; E. Knobloch published a dialogue by Leibnitz, which contained, among other things, the first clear expression of the idea of multidimensional space (1976); three volumes of the publications and investigations

of E.Knobloch (1973-1980) are devoted to Leibnitz's works on combinatorics and theory of determinants; the two volumes of the papers read at the Leibnitz- seminars of Hanover, 1966 (1969) and Paris(1978); and the complete chronicle of the life and work of Leibnitz, prepared by K. Müller and G. Krenert (1969). Some of the published works in the field deal with Leibnitz's treatment of the problem of foundations of the differential calculus; here, the emergence and development of non-standard analysis has led to a re-assessment of the earlier evaluations of the relevant contributions of Leibnitz; this holds good for Newton too. Finally, one must mention A.B.Steekan's (1952) investigations on the first analog integrators, inventd by Leibnitz and Newton, as well as by Ch. Huygens and Joh. Bernoulli.

Here one must also mention the deep-going investigations on the history of Jacob Bernoulli's work on the theory of probabilities — based, to a considerable extent upon archival materials — and the publication of his fundamental work in this field of mathematics the "Art of Suppositions"(as well as those related to the early stage of the spread of his ideas), conducted by B.L.van der Waerden, K. Koli and Yu. Henin (1975). Ya. V.Uspensky's Russian translation of the basic theoretical part of this work, which was subjected to deep going analysis by A.A.Markov (1913), has been reissued recently with Yu. V. Prokhorov's and O.V.Sheinin's commentaries (1986); A. Hold (1984) and the present author (1986) too dealt with a number of related questions.

Limiting ourselves only to the archival legacy of the most oustanding mathematicians, we must now go over to L. Euler. Here a great amount of work has been done involving close cooperation among the scholars of USSR, GDR, France and Switzerland. In view of the variety of the work done, we shall have to limit ourselves to listing the basic publications. These are : a complete description of the materials pertaining to Euler preserved in the archives of AS USSR (1962) and a part of his scientific diaries, a complete description of the materials preserved in the archives of AS GDR (1984), 3 volumes of Euler's letters to the Peterburg Academey of Sciences from Berlin (1959-1976), an annotated index of the complete correspondence of Euler — published in Russian (1967) and, a considerably supplemented German edition of it — published as volume I of series IV of the Complete Collected Works of Euler (1975). Volume V of this series contains the correspondence with Clairaut, d'Alembert and Lagrange (1980), and Volume VI — the correspondence with Maupertuis and Freidrich II (1986). So far, volume IV has been issued in another edition, which contains the correspondence with Goldbach (1976). All these volumes are being supplied with a large apparatus of commentaries; the publication of series IV continues. These fuller studies of the archival materials pertaining to Euler and their publication has been connected with some memorial years related to his life — 1957, 1982 and 1983 [L. Euler was born in 1707 and he died in 1783, thus his 250th birth anniversary fell on 1957, 275th birth anniversary — on 1982 and, 200th anniversary of his death — on 1983. — Tr.]and with the holding of various conferences and meetings, as well as with the publication of some collections containing the papers read in these conferences and the papers specially written for these collections. The sum total of all this work has given rise to a much deeper awareness about the works and science-organizational activities of this great mathematician of the 18th century and of a number of his outstanding contemporaries, about the scientific contacts between the Academies of Sciences of Peterburg, Berlin and Paris, as well as those among

the other scientific collectives and educational institutions. We should add here, that the publication of some of the volumes of series II of the Complete Collected Works of Euler, in particular of his works on mathematical physics (1960) and, on the theory of ship (1978), have also considerably enriched our knowledge of the mathematics of 18th century.

The number of new books on the history of mathematics of the 19th and 20th centuries considerably surpass all the above mentioned and related publications. In what follows, we shall be limiting ourselves to brief comments on some of the newer publications on the histories of mathematical analysis, theory of sets and, theory of functions of the real variable.

Thus, the manuscript entitled "Dissertations on the Theory of Mathematical Infinity", presented in 1785 by L. Carnot in the competition organised by the Academy of Sciences of Berlin, anounced at the initiative of Lagrange, has been published (1970,1979). At that time Lagrange was the Director of the Mathematics Class of the Academy and, he did not approve Carnot's manuscript, and that is why it remained unpublished. But it is undoubtedly an interesting work. This was the first variant of Carnot's widely known"Meditations on the Metaphysies of Infinitesimal Calculus" (1797, 2nd ed. 1813). Carnot's basic idea — substantiation of analysis upon the principle of compensation of errors, dating back to G. Berkeley — is one and the same in the manuscript and in the published version, but the "Dissertation" contains some important moments anticipating Cauchy's reform: an understanding of the infinitesimal as a variable, of the connection between this concept and the concept of limit, and even an attempt at synthesizing the theory of limits together with its"strictness" and the infinitesimal calculus along with its algorithmic attainments. It is true that Carnot could not succeed here, and all this was successfully done by Cauchy. New light has also been thrown upon the so-called theory of limits of d'Alembert. This name does not quite exactly reflect the role of d'Alembert in the elaboration of the theory of limits : he was rather a successful propagandist of this theory, which was still in need of some important specifications and development; N. Bourbaki's evaluation of d'Alembert's definition of the limit as "very clear", is an overstatement. Lhuilier's book — written under the influence of d'Alembert, but providing a broader treatment of the concept of limit, as has been noted by E.S.Shatunova (1966) — won the prize in the above mentioned competition.

Archival searches have also thrown a new light upon some aspects of the work of Cauchy. The beginning of the proof sheets of the second part of his famous course of lectures in the Polytechnical School has been found, and this has finally enabled us to date the proof of Cauchy's famous theorem about the existence of solutions for the system of first order differential equations, and at the same time to explain the reason impeding the publication of this second part, and namely the differences of Cauchy with the then more influential professors of the Polytechnic, on the question of level of teaching of analysis in the Polytechnic (K. Jilen, 1981). The materials available in the archives of Paris have also enabled us to specify the exact nature of M. V. Ostrogradsky's and V. Ya. Bunyakovsky's participation in the elaboration of Cauchy's theory of residues, at its early stage (1824-1826) and in its first applications in mechanics and in the theory of heat (investigations and publications of the present author and V.A.Antropova, 1965; V.S.Kirsanov, N.S.Ermolaeva, 1985). Here, one of the manuscripts of Ostrogradsky, presented at the Academy of Paris in the beginning of 1826, deserves special mention : herein Ostrogradsky's famous integral formula has been formulated and proved for the first time and thereby his priority has been

finally established — it has at times ben subjected to doubt; here we find the first generalisation of the method used by Fourier, during 1807-1822, for solving the problem of propagation of heat, wherein the ideas of D. Bernoulli and Euler were developed. Subsequently, the author of this work presentd it in a somewhat revised form, to the Peterburg Academy of Sciences in 1828 and, it was published in the Transactions of the Academy in 1831. In 1827 Ostrogradsky submitted another memorandum to the Academy of Paris — on the propagation of heat in the right prism, having an isosceles right angled triangle as its base. He conveyed his solution to G. Lame, who published his own enunciation of it in 1861, but the Russian translation of Ostrogradsky's memorandum was published only in 1965. Yet another valuable archival material, to some extent close to the one mentioned above, namely, the notes of Ostrogradsky's lectures on the theory of definite integrals, read in the years 1858-1859, in the hall of the Engineering Academy, has been published by V.I.Antropova (1961). Here many special integrals have been computed with the help of Cauchy's theory of residues and an originally enunciated theory of multiple integrals.

In this connection, here it must be mentioned, that in the 19th and early 20th centuries courses of differing volumes on the theory of definite integrals — computed this way or that, when the corresponding prototypes were not elementary functions or their superpositions — were read in many universities. P.L. Chebyshev read it for a number of years in the University of Peterburg, as an introductory course, together with the calculus of finite differences. Recently, N.S.Ermolaeva found the complete notes of the course on theory of probalilities, read by Chebyshev in the years 1876-1878, including both the introductory parts, and she read a paper on it in the International Congress of the Bernoulli Society. This paper is being published in the proceedings of the Congress, and the manuscripts of the said lectures are being readied for publication.

The place of honour in the elaboration of the foundations of the mathematical analysis of 19th century belongs to B. Bolzano, who largely anticipated Weierstrass, Dedekind and G. Cantor — both in his general conception and in a number of concrete results. His remarkable "Studies on the Functions" remained in manuscript form for nearly a hundred years, and was published by K. Rykhlik only in 1930, and from among the works published in his lifetime, special mention must be made of the brochure, containing a "purely analytical" proof of the theorem about the intermediate values of a continuous function — which too failed to draw the attention of the leading mathematicians immediately. There was a gap in Bolzano's proof : the theory of real numbers was not enough for its completion. It has been found out comparatively recently, that evidently Bolzano himself noticed this gap. In any case the text of his elaboration of the theory of real numbers has been preserved; it predates the constructions proposed later on and independently of each other by Weierstrass(1860), Mere (1869), G. Cantor and Dedekind (1872). K. Rykhlik published this text in 1961; he is of the opinion, that Bolzano's theory, which is not quite clear and complete, may be brought up to the level of modern requirements of strictness, without substantial changes; on this all the specialists are not in agreement with him.

Bolzano was not only a predecessor of Weierstrass and Dedekind in the realm of ideas, but it appears — as has been shown by P. Dugak (1973) — that he influenced both of them.Weierstrass set forth his classical system of mathematical analysis, as well as the theory

of analytical functions in his Berlin lectures. Weierstrass' published some isolated new results contained in the lectures, but did not publish the course of lectures, as he was not satisfied with what he attained. Weierstrass' lectures were collected by a diverse international audience, but their contents were spread even wider, thanks to the printed versions prepared by some of his listeners. In 1973 P. Dugak published highly interesting notes of 4 courses of lectures read by Weierstrass during the years 1861-1887, and the corresponding correspondence of Weierstrass with a number of outstanding mathematicians. These materials substantialy helped specifying the stages of development of the foundations of classical analysis of Weierstrass, as well as of his system of theory of analytical functions. Of no less importance is P. Dugak's book on R. Dedekind, published in 1974. Archival materials constitute half of this book. These materials are related to Dedekind, one way or the other; these are : his manuscripts, correspondence (in particular, with G. Cantor) — in part, not contained in the collection of their correspondence published by J. Kavaies and E. Neter, letters of many other leading mathematicians etc. Dugak considerably specified the interrelationship of Canotr's and Dedekind's set theoretic ideas and the significance of their respective contributions to set theory, in the light of the subsequent logico-mathematical investigations right upto K. Gödel and P. Cohen. J. Dieudonné briefly formulated the general conclusion in his forward to this book by Dugak : Cantor's early work on countability, real numbers and topology remain his living and fundamental legacy, and in these fields Dedekind shares with him, in equal measure, the credit for laying the "set theoretic" foundations of the mathematics of our times. Unfortunately, P. Dugak failed to mention F.A.Medvedev's contributions in the elaboration of the history of set theory, who made a detailed and objective study of the connections between G. Cantor's and Dedekind's set theoretic investigations, in a book published in 1965, it is true, however, that he did not have at his disposal all the archival materials, brought into circulation by Dugak, who mentioned Medvedev's book only in passing. It is well known that set theory and theory of functions were developed vigorously in our country. The first shoots of the Moscow school of theory of functions, often called the school of D.F.Egorov and N. N. Luzin, date back to the beginning of this century. Hence, the interest in the life and work of these scholars and of their colleagues and followers, is quite understandable. F.A.Medvedev was the first (1959) to seriously study the given stage of the Moscow school. Subsequently, many archival materials, pertaining to the life and work of the pioneers of this school and of their followers, came to light : Egorov's letters to Klein, those of de la Vallée-Poussin to Luzin, Egorov-Luzin and Luzin-Danjua etc. correspondences, Luzin's preface to Euler-Goldbach correspondence, Luzin's opinions about the work of the famous geometer S.P.Finikov etc. Recently, new and important materials pertaining to the first formative years of the Moscow school of theory of functions have come to light; these are related to : the course of lectures on the theory of functions delivered by V.K.Mlodzievsky in the beginning of the 20th century, the papers read at the students' circle — where P.A.Florensky was one of the most active members and where Luzin took part, the discontinuous functions studied there etc. What is more, now new light has been thrown upon the role of N.V.Bugaev, whose philosophico-mathematical ideas clearly stimulated the interests of the younger scholars and students in the theory of functions. Herein also comes to light the connections with the Moscow school of philosophy. S.S.Demidov's, F.A.Medvedev's, A.N.Parshin's and other publications pertaining to these connections came to light during 1985-1986. In this connection one must mention the analysis

made by the Colombian scholar R.K.Arboleda(1980) of the letters to M. Fréchet from P.S.Alexandrov and P.S.Urison — who founded the Moscow school of topology in the begining of the 20s of this century. Reminiscences of P.S. Alexandrov were published during the years 1971-1980. These are not archival materials, but we should not pass them over in silence, since here we find a brilliant description of mathematics in Moscow with the global developments of mathematics in the background, from the middle of the first decade of this century and for a stretch of more than sixty years. D.E.Meinshov's reminiscences of his student days and of the early years of his scientific career (1983), constitute a valuable supplement to the memoirs of Alexandrov.

Scientific correspondences were of first order significance during the 17th-18th centuries, when scientific periodicals were extremely weakly developed, scientists rarely met one another, lived in different towns and, scientific congresses and conferences did not take place at all. These correspondences did retain their place in scientific life even later on, right upto our times, though to a lesser extent and, to the examples just cited, here I shall mention only three : the correspondence of S.V.Kovalevskaya with G. Mittag-Leffler — edited by P. Ya. Kochina and E.P.Ozhegova — important, not only for the biography of this outstanding woman and mathematician, but also for investigations into the life of the international mathematical community during the 80s of the last century; the correspondence between V.A.Steklov and A.Knezer, in the beginning of the 20th century, on questions of mathematical physics and related themes — published by I.I.Markush and others (1980); and the correspondence of A.A.Markov with A.A.Chuprov, 1910-1917, on questions of probability theory and mathematical statistics, prepared for publication by Kh.O.Ondar (1977).

Conclusion. This survey of the historico-mathematical investigations of the last few decades, is far from complete; but even this survey shows, that these investigations have not only considerably supplemented our knowledge of the past of mathematics, but that they also entail considerable changes in our general notions about the characteristic traits of the developments of mathematics at different times and in different directions. This survey was divided into several points; in each of them a corresponding summing up has been provided with and, some of the open problems have been indicated. Now a few words remain to be said about some of the over-all changes in that retrospective, wherein the developments of the last four or four and a half thousand years of developments of mathematics had been presented until recently; it is about these years that we may speak sufficiently confidently. At issue here is the further specification of the periodization proposed by A.N.Kolmogorov [26]. It is true, that the periodization proposed here is global in character and, does not pretend to provide the universal characteristic traits of the objects and methods of mathematics at each of the indicated periods, thanks to the unevenness and inexact synchrony, and sometimes owing simply to the non-synchronous progress (at times regress) of mathematics in the different regions or sub-regions considered [26]; the same is true of the present survey. Having this stipulation in view, such global periodization of mathematics, understood as a single science, without its division into sub-disciplines, appeared to be fully satisfactory for a long time. Briefly speaking, A.N.Kolmogorov, made a distinction among four large periods:

1. Birth of mathematics, as it took place, for example, in Egypt, where, evidently, mathematical theory — in the sense of proofs of general theorems — did not exist at

all, and everything was reduced to the collection of arithmetical and geometrical examples of practical importance, solvable at the level of simplest concepts, according to some prescriptions and rules for computation and measurement; however, perhaps, in Ancient Babylon the prescriptions were not confined to what was of direct practical necessity and, there arose more abstract scientific interests and they evolved some general algebraic methods for solving a number of problems (here, we have, almost word for word, reproduced A.N.Kolmogorov's expressions, from his article in the second edition of GSE, v. 26, 1954).

2. The elementary mathematics of the period 7th century B.C.-17th century A.D., when it became theoretic, having the constant magnitudes of arithmetical or geometrical nature as its main object ; approaches to the ideas of an infinitesimal analysis are observed in Greece and then in medieval Europe, but they were not developed; this second period has been divided into two sub-periods in Kolmogorov's article entitled "Mathematics" — encompassing, correspondingly, the ancient (Greece, Hellenism, Rome) and the medieval (countries of the Orient and of Europe) periods.
3. The period of emergence and development of the mathematics of variables, from the 17th century; it enters into [4.] the period of modern mathematics in the 19th century, in connection with extreme extension of the object of mathematics and the generalisation of its concepts and methods; A.N. Kolomogorov refrained from providing any direct global characteristion of this modern period. At the same time he has provided a fully intelligible account of the transition to modern mathematics, which began in the first decades of the 19th century, straight off in the two fields of geometry and algebra. His article began with Engels' definition of mathematics as the science of the quantitative relations and spatial forms of the real world. In the section devoted to modern mathematics Kolomogorov indicated that, the range of "quantitative relations" and "spatial forms" studied, becomes extremely widened in this stage (this is explained with the help of some examples) and that, when these two expressions are so widely understood, even then, that is even at the present stage of development of mathematics, its initial definition holds good. Here Kolmogorov adds that, when the expression "quantitative relations" is interpreted sufficiently broadly, then"spatial forms" may be considered to be special kinds of "quantitative relations". Here it is not possible to enter into a discussion of the wide range of related methodological questions that arise, for example, of A.D.Alexandrov's treatment of geometry (1952) as the science of spatial relations and forms, as well as about the other relations and forms of reality, which are structurally similar to the spatial ones("spatial-like"). B.A.Rozenfeld's proposal to generally call modern mathematics non-Euclidean, is hardly felicitous : this word is very closely connected with the non-Euclidean geometry. This entire periodization is linked with the periodization of the prevailing social formations.

Many Soviet historians of mathematics accepted A.N.Kolomogorov's periodization with some modifications. Now, let me summarize the specific comments regarding the periodization

already made; let me stipulate again, that the terminology used here is conventional and, the chronological and territorial boundaries are diffused.

1. While retaining the term "the period of birth of mathematics", we should stress the fact that not only in Ancient Babylon, but also in Egypt — the prescriptive form of enunciation was followd by works reflecting deductive arithmetical, geometrical and algebraic thought; true, not at the level of construction of systems like the Euclidean geometry. Perhaps, the expression "piecemeal deductive mathematics", more adequately describes the mathematical thought of the aforementioned and similar civilizations.
2. In Greece mathematics was transformed into a deductive science, in the Euclidean — if one may say so — sense, i.e., into a system of disciplines, axiomatically constructed at the level of Aristotelian logic; it was not accidental that the formation of this logic was almost simultaneous with the codification of Euclid's "Elements". However, this axiomatization did not spread to all the branches of mathematical knowledge and, in the 19th-20th centuries it has been substantially completed and developed. But Greek mathematics is different from what preceeded it, also in another, in principle more important, respect — important from the point of view of subsequent development of mathematics : in natural philosophy and in mathematics there arose the idea of infinity and it found application — it became the starting point of infinitesimal mathematics; it is in Greece that the fruitful interaction of philosophy and mathematics was established. Thus, the term "elementary mathematics" is hardly adequate for the content of Greek mathematics. Perhaps, here the more general, though less concrete, name of "the period of emergence of mathematics as a lógico-deductive science", has some advantages.
3. Probably, the term "the period of elementary mathematics" is most appropriate , when one wants to provide a global characterization of the mathematics of the middle ages. But here it is important to have in view the essential traits of mathematics in Europe of 14th-16th centuries and the progress of the non-elementary ideas and methods spoken above.
4. In reality, the 17th-18th centuries are characterised, first of all, by the primacy gained therein by the mathematics of the variable magnitudes. The problem of naming the period of modern mathematics in terms of its contents, happens to be more complex. Perhaps, it would have been proper to borrow a term from modern mathematics, and speak of this period as the period of "non-standard mathematics". However, the question arises : would it be correct to speak of the last two centuries, as one single period of development of mathematics? Does not the mathematics of the recent times of scientific and technological revolution, with its characteristically fast progress of informatics and of the other noticable (discrete,finitary etc.) tendencies, constitute the first stage of a new period of development of this most ancient science ? Personally for me, it is very difficult to provide an answer to-day.

Literature

1. *Montucla J.F.* Historie des mathématiques ... P., 1758; 2 éd. P., 1792-1802.
2. *Loria G.* Guida allo studio della storia delle matematiche . Sec. ediz. Milano, 1946.
3. *May K.O.* Bibliography and Research Manual of the History of Mathematics. Toronto : Univ. Toronto Press, 1973.
4. *Cantor M.* Vorlesungen über Geschichte der Mathematik. Leipzig, 1880-1907. Bd. I-IV.
5. *Tseiten G.G.* Istoria matematiki v drevnosti i v sredine veka. 2- e izd. M.;L., 1938; *On zhe.* Istoria matematiki v XVI i XVII vekakh. 2-e izd M.;L., 1938.
6. *Klein F.* Elementary Mathematics from an Advnaced Standpoint. Tr. E.R.Hedrick and C.A.Noble, 2 vols. New York : Macmillan, 1932 and 1939 [a Russian tr. of the same was published in 3 vols, in the years 1933-1935].
7. *Tropfke J.* Geschichte der Elementarmathematik . B., 1921-1924. Bd. I-VII.
8. *Tropfke J.* Geschichte der Elementarmathematik. 4 Aufl. B. ; N.Y. 1980. Bd. I.
9. *Cajorie F.* History of Elementary Mathematics with reference to the Methods of Teaching. 2nd Russ. ed. Odessa, 1917.
10. Enzyklopädie der mathematischen Wissenschaften. Leipzig, Bd. 1-6. 1898-1934; 2 Aufl. 1952-1968.
11. Enciclopedia delle matematiche elementari e complementi con estensione alle principali teorie analitiche e geometriche... Milano, 1930-1950. vol. 1-3.
12. Problemy Hilberta (Gilberta). M. 1969.
13. *Klein F.* Lectures on the Development of Mathematics in the 19th Century. Russ. ed. M.; L., 1937, Part I.
14. *May K.O., Roebuck L.* World Directory of Historians of Mathematics. 2nd ed. Toronto,1978.
15. Istoria matematiki s drevneischikh vremeon do nachala XIX stoletia. M., 1970-1972. T.1-3; Matematika XIX veka : Mat. logika. Algebra. Teoria chisel. Teoria veroyatnostei. M., 1978; Matematika XIX veka: Geometria. Teoria analit. funktsii. M., 1981.
16. Abrégé d'historie des mathématiques. 1700-1900/ Ed. J. Dieudonné. P., 1978. T 1-2.
17. Istoria otechestvennoi matematiki. Kiev, 1966-1970. T. 1-4.
18. Ocherki razvitia matematiki v SSSR. Kiev, 1983.
19. *Kline M.* Mathematical Thought from Ancient to Modern Times. N.Y., 1972.
20. *Bourbaki N.* Éléments d'historie des mathématiques . Russ. Ed. M. ,1963.
21. *Struik D.J.* A Concise History of Mathematics. N.Y., 1967. [2nd Russ. ed. M., 1969.]
22. Dictionary of Scientific Biography. N.Y., 1970-1980. vol. I- XVI.
23. Scienziati e tecnologi dalle Arigine al 1875. Scienziati technologi contemporanei. Milano, 1975, vol. I-III.
24. *Waerden B.L. van der.* Geometry and Algebra in Ancient Civilizations. B., 1983.
25. Scientific Change/ Ed. A.C.Crombie. L., 1963
26. *Kolmogorov A.N.* Matematika // BSE. 2-e izd. 1954. T. 26.

Source : Zakonomermosti razvitia sovremennoi matematiki. "Nauka". M.,1987. s. 28-74.

About the Author : Yushkevich, Adolf (Andrei) Pavlovich (1906-). Senior historian of mathematics, Russia.

Other Publications :

1. Istoria matematiki v sredine veka. M., 1961.
2. Omar Khayyam. M., 1965 (in collaboration with others).
3. Istoria matematiki v Rosii do 1917. M., 1968.
4. Christian Goldbach. M., 1983 (in collaboration with others).

NON-STANDARD ANALYSIS
AND THE HISTORY OF CLASSICAL ANALYSIS

FEODOR ANDREIVICH MEDVEDEV

The history of classical analysis is perhaps the most investigated part of the history of mathematics, and this is quite natural. Mathematical analysis has been viewed as the "simplest and most universal language ... most suitable for expressing the invariable relations of natural phenomena" [1, p. XXIII]. That is why the greatest representatives of mathematical thought, and the most modest workers in the field of mathematics, applied their strength to its construction, from the 17th to the 19th century, as well as, during the greater part of the present century. This new discipline, in the process of its construction, not only served as a felicitous means for describing the phenomena of the external world, but also provided a possibility for advancing profound philosophical reflections about the differential picture of the world, about the causal connections in it, about the laws of nature and thought. That is why, historians of science have paid the greatest attention, namely, to the history of analysis.

A major specificity of the approach of the historians to the study of the formation of this branch of mathematics has been, and to a considerable extent still is, to view it as a single integral theoretical discipline. To a certain extent this view reflects an aspect of a more general notion, according to which "mathematics grew as a single whole" [2, p. 13].

However, of late such a view about mathematics has been shaken or has at least been questioned. "The basic conclusion that may be drawn from the presence of several conflicting approaches to mathematics is as follows : there exists not one mathematics, but many mathematicses" [3, p. 358] ; not only some individual mathematicians and historians of mathematics, but even some philosophers share this conclusion [4, pp. 186-187]. An analogous hypothesis suggests itself also in respect of mathematical analysis : it began to take shape after the construction of the intuitionist and constructivist analyses, and became especially clear after the creation of non-standard analysis, towards the middle of this century. Each of the systems mentioned, is sufficiently independent of, and definitely different from, any other of them — in terms of the composition of basic concepts, modes of advancing arguments about them and, computational procedures ; and it is hardly probable, what is more, impossible (and even if possible, then not necessary) that they be united in a single theoretical construction. This especially comes to the fore, when we view analysis, not so much as a theoretical doctrine about its basic concepts, but rather as calculus, as formal system. [Such an outlook on mathematical analysis has been considerably developed, at the level of historical studies, in a book [5] by C.H. Edwards.] In so far as, "it is possible to propose a large number of formal systems, for describing one and the same fragment of reality" [6, p. 87], there is no ground for preferring one of them in advance : it would be expedient to use different calculi in different situations.

A distinctive and even strange specificity of theoretical constructions — and not merely of the mathematical ones — is this, that at a definite stage of their construction, and at times from almost the very beginning, the adherents of the corresponding theoretical schemas become tempted to view them as the universal and the only possible schema and to declare all the others as false. Such, in particular, was the situation in the history of mathematical

analysis, in the second half of the 19th century, when this mathematical discipline attained the highest level of its development and when the gloss of Cauchy-Weierstrassian rigour was put on it. The analysis of Cauchy-Weierstrass was considered to be the only legitimate theoretico-analytical construction, and all that preceded it came to be regarded as mere approximations to it, and largely mistaken at that : "... in the text-books of Cauchy at long last we are on firm ground" [7, p. 207].

The works on the history of analysis come out mainly as descriptions of the development of some integral, one and only possible thing, which served as a kind of ideal for a single science, to which the constructions of the previous epochs approximated. From the point of view of this ideal science (which is in practice reducible to a construction, put forward in some standard text book, followed by the author of the corresponding "history of analysis" or part thereof), the problems that emerged in the course of history had singular correct solutions; and it is these that were mainly of interest to the historians. Indeed the approaches to them could be different: to a large extent these are unsatisfactory and even patently wrong, and then, in the course of further searches they are usually rejected, or at best they are used as auxiliary devices, serving as the raw material, from which the ideal science was built, or they provide heuristic indications, which lead to the discovery of the absolute truths of the ideal science.

In particular, the "evolution of "rigour" in analysis can be summed up as a continuous ascent from unclear and vague concepts, their gradual elucidation, and then the arrival of a stage of stability — after which it was impossible to have *any* dispute regarding what constitutes a correct proof in analysis" [8, p. 50]. In the opinion of J. Dieudonné, the analysis of Cauchy-Weierstrass happened to be this stage of stability.

The analysis of Cauchy-Weierstrass took shape in the 19th century. However, before that mathematical analysis went through more than two thousand years of development — from the first quadratures and cubatures of the ancient Greeks to the analysis of Newton-Leibnitz, crowned with the works of J. L. Lagrange and L. Euler. Here we shall mention only some of the landmarks along this road.

It is well known, that in ancient Greece the first quadratures and cubatures were carried out with the help of some infinitesimal procedures [9]. These procedures were so fruitful, that even after the elaboration of the method of exhaustion by Eudoxus, which tended to exclude the infinitesimals from mathematical arguments and soon became the official doctrine on the corresponding questions, infinitesimal considerations continued to be in wide use, and among the users there were adherents of the method of exhaustion; the assertion of S. Ya. Lureo, that if the followers of the method of exhaustion "did not get hold of ready-made solutions, already discovered by the atomists, then they themselves preliminarily found them out — by stealthily applying [the method of] atomistic decomposition" [9, p.159], is best illustrated by the example of Archimedes.

With the renewal of interest in the problems of analysis in the 17th century, infinitesimal considerations again came to the fore (in the works of J. Kepler, B. Cavalieri, J.-P. Roberval and of many others), though the spell of the method of exhaustion also continued. It was considered to be an irreproachable, totally rigorous mode of mathematical reasoning in

respect of the problems of quadrature, cubature, centres of gravity, maximums and minimums, and tangents; and mathematicians did not see in it those serious logical flaws, which may be observed today; for example, this, that therein, neither were the magnitudes under consideration defined, nor were their existence proved. Though this spell was retained for a long time, right upto A. Cauchy and even later, nevertheless the method of exhaustion more and more retreated to a secondary position. It was substituted by the *infinitesimal calculus*. Namely, thus was created the grandiose house of mathematical analysis — the most extensive and the most fruitful mathematical discipline.

Almost from the very beginning of its construction, strange reproaches began to be sounded regarding the illogicality of the arguments, involving the infinitesimals, provided in this calculus — these reproaches continue even to-day [3, pp. 151-177]. However, much more illogical was the very demand, that the modes of reasoning used in analysis are to be subordinated to the norms of existing logic. Mathematical analysis is based on the concept of function — which is a special instance of relation (and for an extended understanding of function, they are one and the same). Logic of relations did not exist earlier and it was constructed only in the second half of the 19th century. [It is true that individual advances were made in that direction. Such attempts date back to Aristotle, * G.W. Leibnitz, I.H. Lambert and to some others, but the construction of the theory of relations as a section of logic began only with A. De Morgan, or even later — with C.S. Peirce.] That is why, the arguments employed by the mathematicians in analysis, naturally, did not fit into the frame-work of the arguments conducted in accordance with the canons of older thought. Mathematicians did not like to wait till the emergence of the corresponding logic, in fact they paved the path for it with their new calculus, having constructed a formal system, which is a special instance of a large fragment of the logic of relations (this was done by Euler, Lagrange and their followers).

In the final analysis this formal system was infinitesimal, and Lagrange's heroic attempt at liberating mathematical analysis from the infinitesimals turned out ot be totally unsuccessful, in spite of his indisputable achievements. [In J. Grabiner [10] one finds a good description of many achievements of Lagrange.] What is more, even Cauchy's reconstruction of analysis carried the mark of that very infinitesimal. On this, one must dwell in greater detail.

In correspondence with the established tradition, earlier we have united the constructions of Cauchy and Weierstrrass into an aggregate. Indeed, there is much in common in their approaches, which provides a definite basis for such unification. But in essence the constructions of Cauchy and Weierstrass, differ by so many fundamental parameters, that it would have been more correct to speak of two different systems of analysis, created by them. This difference has been quite clearly outlined by N.N. Luzin [11, pp. 305-312], and in his words, generally speaking, this difference lies in the fact that, Cauchy "in principle, introduces variables, and, thus, since then analysis stands enriched by these new magnitudes, used equally rightfully with the constants, just as the imaginary numbers are used on a par with the real ones" [11, p. 305]; Weierstrass, however, "first of all removes all the variables from analysis, any change, motion and everything is reduced to the stationary conditions and to that alone, i.e., to the constant magnitudes" [11, p. 307]. These systems are also different in

* (to Gangesopadhyaya in India — Ed.)

terms of the explications of the concepts of infinity, which are foundational to them. Cauchy based himself upon the concept of potential infinity and, the concept of potentially infinitely small magnitudes are central to his system of analysis, whereas Weierstrass leaned on the concept of actual infinity, and he in fact drove away the infinitesimals from analysis. This is not the place for citing the other differences, many of them are yet to be brought to light; here we shall only mention the fact that Cauchy's conception completed the stage of infinitesimal analysis, whereas in Weierstrass' system, analysis was restructured as a set theoretic discipline; its contours have been quite clearly outlined by Luzin [11, pp. 307-312].

Infinitesimal considerations were put aside in the classical set theory, and the actual infinitesimals were denied every right to exist ; G. Cantor fought against them energetically — on this, see, for example, [12, pp. 294-296]. At the end of the 19th and during the first half of the 20th century, the Weierstrassian system of analysis, based on the theory of sets, became the ideal of analytical construction — it was considered to be singularly legitimate and absolutely true. Armed with it, even the historians of analysis started viewing the previous developments through the prism of such construction. Naturally, there arose all possible distortions. They are quite numerous, and there is no scope here, even for providing a simple list of them. We shall cite only a few examples.

After a very brief and not entirely objective description of B. Cavalieri's methods of quadrature and cubature, O. Teoplitz termed his conclusions — "clever" [13, pp. 55-58], but there itself he characterized them as "untrustworthy to the highest degree". After that he admitted, that in the 17th century the doctrine of indivisibles was "lauded to the skies" and was employed in various forms by P.Guldin, B. Pascal and J. Kepler and, that in the works of G.W. Leibnitz the various trends in the use of the infinitesimal methods came together, "as rays converge in the focus of a lens" and illumined the entire 18th century. Teopliz concludes that L.Euler, D. Bernoulli, B. Taylor and others "constructed the new edifice of mathematics basing themselves upon such non-strict and heuristic methods " [13, p. 59]. Thus, it appears that a large mathematical theory was built upon the precarious foundations of doubtful heuristic methods, and only a "clever instinct" saved Cavalieri and, of course, the other mathematicians from making false moves.

While describing the mathematical analysis of the 18th century, J. Dieudonné, on the one hand, calls Euler one of the two "giants of that century" [14, p. 20] (the other one being Lagrange), and on the other — presents him as nearly the greatest fumbler : he did not define the concept of a "continuous" or "regular" function "more precisely"; did not give an "exact definition" of his "mechanical functions"; his conception of numerical serieses was "shaky and imprecise"; he "could not formulate" the definition of the concept of the sum of a series, basing it on the concept of limit, and though "he knew well, that when the general term of the series

$$a_0+a_1+\cdots+a_n+\cdots,$$

does not tend to zero, then one should not speak of the "convergence" of this series in the usual sense, nevertheless he thought that it was often possible to calculate the "sum" of this series" [14, pp. 21-22]; Euler "did not make a clear statement" about one of the divergent serieses; "Euler's interpretations of the different meanings of the word "sum" of a series

appear to be very confused, leading to contradictions", "Euler did not at all understand the difficulty inherent in his definition of the "sum" of a series" [14, p. 23]; Euler "did not always remember what he wrote a few years back", "he did not elucidate the phenomenon of non-uniform convergence, when he ran into it" [14, p. 30], etc.

In short, it appears, that only with Cauchy did the mathematicians — to use an expression of Bourbaki, indicated above — find themselves "on a firm ground", and till then they wallowed "in the quagmire of mathematical analysis" of the Newtonian- Leibnitzian-Eulerian kind [3, p.151], and Cauchy too, quite often fell into it.

Demolition of such mistaken ideas about the development of mathematical analysis began after the construction of the so-called non-standard analysis, in the middle of the present century. [The construction of this analysis is to a great extent connected with the name of the American mathematician A.Robinson [1918-1975], and his work published in 1961 [15] is considered to be the first work in the field. This is not entirely correct. Even if we refrain from mentioning its pre-history, which dates back to the 19th and the first half of the 20th century, we may mention, for instance, the work of C. Schmieden and D.Laugwitz, published in 1958 [16] and the papers of the latter : [17] , [18]. A.F. Monna began working in this direction [19, p. 4], since 1950.] The sole fact, that the infinitesimally small magnitudes and numbers, that were assiduously driven away to build the Weierstrassian analysis, again got back their right to citizenship in mathematics, itself compels us to re-examine the ideas so far formed, about the course of development of analysis upto Cauchy and Weierstrass, about the character of its infinitesimal apparatus and, about the modes of reasoning employed therein. Now the infinitesimal procedures are no more considered to be "non-strict, heuristic methods", but are rather viewed as quite honest means of advancing mathematical reasoning, which not only enable us to re-state the well-known results with great simplicity and elegance, but also help us discover new results; as an example of the latter, one may mention the so-called solution-configurations for a determinate class of ordinary differential equations [20].

This re-examination still continues. It began with the efforts of the founders of non-standard analysis — Laugwitz (1965) and Robinson (1966) [21 & 22]. Now the front in favour of such re-examination is quite broad, and the mere listing of the works in the already indicated direction, would take a lot of space. In particular, many results of Euler, so far considered to be "non-strict" and to have been obtained by doubtful means, turned out to be rationally interpretable within the frame-work of these new ideas [23 & 24]. Now they are no more the products of some special and secret intuition, but are provided with a rational reconstruction in a different system of analytical thought. Now there has even arisen a danger, of too direct an interpretation of some isolated notions of Leibnitz, Euler and of some other mathematicians, of their unnecessary modernization in the spirit of non-standard analysis. W. Felsher has characterized this danger quite clearly [25, pp. 179-180]. However, one may state — by generalizing, to some extent, Laugwitz's understanding of some of the results of Euler [23, p. 4] — that, we must examine the attainments of our predecessors from the modern points of view, while keeping in mind the fact that they could not have thought, namely in the way we do; all the same, we can rationally interpret what they did, namely, with the help of modern ideas.

Further, one of the basic principles of non-standard analysis is the principle of transfer / translation, according to which every true statement of the classical analysis (the classical theory of sets) is true in the non-standard analysis (non-standard theory of sets) too, and conversely [see : for example, [20, p.81]; and see : [26, pp. 202-203] for a more detailed formulation]. Thus, non-standard analysis appears to be yet another model for that very class of analytical phenomena and relations among them, for which the model of classical analysis was built. It is understandable, that in an atmosphere of supremacy of the classical model, the latter played the role of a paradigm in the sense of T. Kuhn or, that of a research programme in the sense of I. Lakatos, and the historians of analysis examined the results obtained by their predecessors through the prism of the basic propositions of this paradigm or programme. In particular, since in it [the classical model] there was no room for the actually infinitesimally small, so all manipulations with them, met with by the historians, were viewed with suspicion, and the data obtained with their help, were interpreted as to have been found non-rationally — intuitively or through the mediation of unreliable heuristic methods. And this led to the, not quite correct, prevalent interpretation of historico-scientific facts, herein indicated (and largely, not even indicated).

After the construction of non-standard analysis, classical analysis of the 19th century ceased to be the only correct analytical construction, its universalization and absolutization appeared to be illegitimate forms of activities. At least two formal systems possessing equal rights turned out to be equally suitable "for describing one and the same fragment of reality". In its turn, the notions about some mythical "absolute strictness" — turned out to be a myth. That is why there arose the necessity of substantial corrections in the historiography of mathematical analysis — corrections, that have been made complicated owing to the presence of intuitionist and constructivist analyses.

Everything said and done, non-standard analysis permits us to give an answer to a ticklish question, which arose in connection with the earlier approaches to the history of classical analysis : if we admit that the infinitely small and infinitely large magnitudes are contradictory concepts, then how could the grandiose edifice of one of the most importanmt mathematical disciplines be built upon them? And H.J.M. Bos — one of the keenest of modern historians of analysis — has been compelled, inspite of certain prejudices against the non-standard analysis, to declare that "the recently constructed non-standard analysis provides an explanation, as to why analysis could develop upon the precarious foundations of the infinitely small and infinitely large magnitudes" [27, p. 13].

Literature

1. *Fourier J. B. J.*, Théorie analytique de la chaleur (1822) // Oeuvres. T.1. P., 1888.
2. Istoriya matematiki s drevneischikh vremeon do nachala XIX stoletiya. M., 1972. T. 3.
3. *Kline M.*, Mathematics : The Loss of Certainty. New York, 1980. [Russ. Tr. : Moscow, 1984].
4. *Panov M.I.*, Metodologicheskie problemy intuitsionistskoi matematiki. M., 1984.
5. *Edwards C.H.*, The Historical Development of the Calculus. N.Y., 1979.
6. *Ershov Yu.L.*, Nekotorye voprosy primeneniya formalizovannykh yazykov dlya issledovoniya filosofskikh problem // Metodol. problemy matematiki. Novosibirsk, 1979.

7. *Bourbaki N.*, Élements d'histoire des mathématiques. P., [Russ. Tr. : M., 1963].

8. *Dieudonné J.*, La notion de rigueur en mathématiques // Conférence prononcé le 8 décembre 1982 au Séminar loi de philosophie et mathématiques. P., 1983. Reprint.

9. *Lureo S. Ya.*, Teoriya beskonechno malykh u drevnykh atomistov, M.; L., 1935.

10. *Grabiner J.V.*, The Origins of Cauchy's Rigorous Calculus. Cambridge; L., 1981.

11. *Luzin N.N.*, Differentsialnoe ischislenie //BSE. 1-e izd. 1934. T.22; *On zhe.* Sobr. soch. M., 1959. T. 3.

12. *Kantor G.* (*Cantor G.*), Trudy po teorii mnozhestov. M., 1985.

13. *Teoplitz O.*, Entwicklung der Infinitesimalrechnung. B., 1949.

14. *Dieudonné J.*, L'analyse mathématique au dix-huitième siècle // Abérgé d'historie des mathématiques. 1700-1900. Algèbre, analyse classique, théorie des nombres. P., 1978.

15. *Robinson A.*, Non-standard Analysis // *Konikl. Akad. Wetensch. Amsterdam. Ser. A.* 1961, Bd. 64, N 4.

16. *Schmieden C., Laugwitz D.*, Eine Erweiterung der Infinitesimalrechnung // *Math. Ztschr.* 1958. Bd. 69.

17. *Laugwitz D.*, Anwendungen unendlich kleiner Zahlen, I : Zur Theorie der Distributionen // *J. reine und angew. Math.* 1961. Bd. 207.

18. *Laugwitz D.*, Anwendungen unendlich kleiner Zahlen, II : Ein Zugang zur Operationsrechnung von Mikusinski // Ibid. Bd. 208.

19. *Bertin E.O., Blij F., van der, Hornix E.*, Curriculum vitae de A. F. Monna // Symposium dédié à A.F. Monna. Utrecht, 18 déc. 1979// *Communications of the Math. Inst.* Utrecht, 1980. Vol. 12.

20. *Zvonkin A.K., Shubin M.A.*, Nestandartnyi analiz i singulyarnye vozmuscheniya obyknovennykh differentsialnykh uravnenii// *Uspekhi mat. nauk.* 1984. T. 39, vyp.2.

21. *Laugwitz D.*, Bemerkungen zu Bolzano Grössenlehre // *Arch. Hist. Exact. Sci.* 1965, Bd. 2, N 5.

22. *Robinson A.*, Non-standard Analysis. Amsterdam, 1966.

23. *Laugwitz D.*, Eulers Begründung der Analysis aus der Algebra. 1983. Prepar. N 728.

24. *Laugwitz D.*, Fall and Resurrection of Infinitesimals // Textes UMFUFA, 1984. N 88.

25. *Felscher W.*, Naive Mengen . und abstrakte Zahlen, II : Algebraische und relle Zahlen. Mannheim etc. 1978.

26. *Stroyan D.K.*, Infinitezimalnyi analiz krivykh i poverkhnostei // Spravochnaya kniga po matematicheskoi logike : Teoriya modelei. M., 1982, Ch. 1.

27. *Bos H.J.M.*, Differentials, Higher Order Differentials and the Derivative in the Leibnitzian Calculus // *Arch. Hist. Exact. Sci.* 1974. Bd. 14, N1.

Source : Zakonomernosti razvitiya sovremennoi matematiki. "Nauka". M., 1987; s. 75-84.

About the author : Medvedev, Feodor Andreivich (1923-), Cand. (sc). Areas of specialization : history of set theory and theory of functions. Published a number of books and papers in these areas since 1965.

THE NEW STRUCTURAL APPROACH IN MATHEMATICS AND SOME OF ITS METHODOLOGICAL PROBLEMS

GEORGI IVANOVICH RUZAVIN

The structural approach in mathematics has found its most complete development in the works of a group of French mathematicians writing under the pseudonym of Bourbaki, and of late in the works of MacLane and, of the other scholars engaged in the elaboration of the algebraic theory of categories. This approach offers an opportunity to take a new look at many major problems of philosophy and methodology of mathematics. The most important among these problems are : the specificity of the object and method of mathematics, the place of mathematics in the system of scientific knowledge and, the relation of mathematical structures with objective reality.

The general problems of the application of mathematics in the other sciences and in practical activities become more clear, when they are veiwed from the point of view of abstract structures. The question of the interrelationship of "pure" (theoretical) mathematics with applied mathematics simultaneously finds a more thoroughgoing solution.

1. FORMATION OF THE CONCEPT OF ABSTRACT STRUCTURE AND THE EMERGENCE OF A NEW APPROACH TO MATHEMATICS

Progress in mathematics has always been connected with the growth in the abstractness of its concepts and theories. Modern mathematics uses ever deeper abstractions to study, not only the quantitative, but also the more complex structural relations; the traditional quantitative relations among magnitudes happen to be a constituent part of these more complex relations.

The beginning of this new approach to the subject-matter of mathematical investigations is, to a considerable extent, connected with the discovery of non-Euclidean geometry by N. I. Lobachevsky and J. Bolyai. It is difficult to overestimate the general scientific and theoretico-cognitive significance of this discovery. It not only undermined the centuries old belief in the possibility of one and only. one geometry of Euclid, but also fundamentally changed our earlier notions about geometry, about mathematics as a whole. Above all, the theoretico-cognitive lesson offered by the discovery of the non-Euclidean geometries consisted in the following : it convincingly demonstrated that the axioms of geometry are neither empirical descriptions and inductive generalizations of the properties and relations of the real physical space, nor are they *a priori* synthetic judgements — as I. Kant thought them to be.

Mathematicians, following B. Riemann, began to view these axioms as hypotheses of a kind, whose applicability to the study of the surrounding space must be established after providing suitable interpretations for the basic concepts of a geometry. Within the framework of mathematical investigations these concepts (of a point, straight line and plane) themselves remain as abstract as, say, the algebraic formulae. No one doubted the fact that the symbols of these formulae may indicate any number, and subsequently any vector, matrix, function or other object. However, for a long time the statements of geometry remained

associated only with the properties and relations of physical space. It is that is why, namely, that the discovery of the non-Euclidean geometries had such a decisive significance for the emergence of a new approach to the subject-matter of geometrical investigations.

In so far as axioms can describe the properties and relations of objects having the most diverse concrete contents, we cannot pass judgements about their truth or falsity, basing ourselves upon any one system of objects, serving as their interpretation. Besides, intuitive self-evidence also cannot serve as a criterion for truth, since that which is self-evident to one, may not appear to be self-evident to another. That is why, the demand that the axioms should be intuitively self-evident is not a mathematical, but rather a psychological demand. From the logico-mathematical point of view, the most important criterion, which must be satisfied by any system of axioms, is the simultaneity or formal non-contradictority of the system of axioms.

If a system of axioms is contradictory, it will not have any interpertation and, consequently, will have no scientific worth whatsoever. The demands for completeness and independence of a system of axioms are not that obligatory, if only owing to the fact that a dependent axiom can always be translated into a class of theorems, and the criterion of completeness is applicable only to the comparatively simple axiomatic systems.

The transition from the concrete, contentful axiomatics like, for example, the axiomatics of the Euclidean elementary geometry, to the abstract axiomatics like the axiomatics of Hilbert, and then on to those fully formalized axiomatics, wherein symbols substitute terms, and propositions are transformed into folmulae — is quite a new step in the development of the axiomatic method; it is sometimes called a revolution in this method. As we have noted above, it is, namely, this approach which provides an opportunity for viewing axioms as abstract forms ; these forms may be used for investigating the properties and relations of various things that differ in their concrete contents.

In the formation of the ideas about abstract structures, a significant role has been played by the set theory, which emerged towards the end of the 70s of the last century. This theory was crafted, in the main, in the works of the great German mathematician G. Cantor, directed at providing a satisfactory foundation to classical mathematics. This theory views the objects of all mathematical theories like the number, function, vector, matrix etc., in isolation from their mathematical contents. For the set theory these are but elements of such infinite sets, which may be handled with definite rules. Such an extremely general and abstract approach provided an opportunity for viewing the subject-matter of the most diverse mathematical disciplines from a single point of view. Namely, that is why, with the passage of time the set theory came to be looked upon as the foundation of the entire house of classical mathematics.

In the end, a synthesis of the set-theoretic ideas and the axiomatic method led to a new conception of the abstract mathematical structure. This conception was floated towards the beginning of this century. "One is tempted to acknowledge that the modern concept of "structure" was in the main formed around the year 1900; actually, it took another thirty years of study to make it fully clear" — wrote N. Bourbaki [5, s. 33]. This work was done by that talented team of French mathematicians, which uses the collective pseudonym

of N. Bourbaki. They characterize a structure as follows: " In order to define a structure, one or more relations, containing their elements are at first specified... then it is postulated that the given relation or relations satisfy certain conditions (these are listed and they constitute the axioms about the structure under consideration). To construct an axiomatic theory of a given structure, is to deduce logical conclusions from the axioms about that structure, without admitting any other presupposition in respect of the elements under consideration (in particular, staying clear of any kind of hypothesis regarding their "nature" [5, s. 25].

From this, it is evident that concrete interpretations of the objects of mathematical investigations are quite unimportant for the said investigations. One can view these objects as elements of abstract mathematical structures, all the essential properties of which are specified with the help of axioms, N. Bourbaki have classified the mathematical structures into three basic types, in accordance with the character of these properties; these are: the algebraic, order, and topological structures. More complex or multiple structures are formed by combining these initial structures. The structure of various mathematical theories may be studied and thereby they may be classified, with the help of these more complex structures. If earlier mathematical theories were simply elaborated one beside the other, in the course of their historical emergence, then now it is possible to reveal their deeper resemblance. Thus, for example, the set of natural numbers, which serves as the basis for the construction of mathematical analysis, contains all the three generative structures and hence, the earlier isolation of algebra, geometry and analysis turns out to be groundless.

Of late the algebraic theory of categories is drawing ever greater attention of the specialists. Before the emergence of this theory, the set was considered to be the most general concept of mathematics. That is why, while substantiating mathematics, all other mathematical concepts were sought to be defined by using the terminology of set theory. Even the conception of N. Bourbaki was no exception to this, based as it is, in the final count, on an axiomatic theory of sets. The concept of category is not only an alternative to the concept of set, it is also the further concretization and development of the idea of mathematical structure.

A category is made up of a certain class of objects and a definite class of morphisms, wherein each ordered pair of objects is contrasted with a corresponding set of morphisms. An operation with the morphisms, with the help of which from two given morphisms of a category, a third unique element is found out from among the set of morphisms, is called the composition or production of morphisms. It must satisfy the conditions of associativity and identity [4, s.9-11]. Simply speaking, in the theory of categories the sum total of objects are considered together with their structure and with some representations among them, retaining the given structure. If sets are viewed as the objects of a category, and representations among them serve as morphisms, then we get the category of sets. Thus, the concept of set turns out to be a special case of the more general concept of category. In contrast to the static concept of a set, here the principal attention is turned towards the character of representations, which retain the definite structural specificity of the objects, and thereby the active, constructive aspect of mathematical knowledge is underlined.

2. THE ROLE OF ABSTRACT SRUCTURES AND CATEGORIES IN THE MODERN UNDERSTANDING OF THE SUBJECT MATTER AND METHOD OF MATHEMATICS

The ideas and methods of the theories of structure and category provide a better opportunity to understand the qualitative change, which is taking place in the very subject-matter of mathematics, as well as in the application of its methods in the other sciences.

Upto almost the middle of the last century mathematics was viewed as a science of magnitudes and spatial figures. Of course, therein not the concrete physical, chemical etc. properties of the magnitudes, but the properties and relations common to all of them were taken into consideration. Since every magnitude can be expressed numerically with the help of a suitably chosen unit of measurement, in the past, often the essence of mathematics was considered to be located in the investigations regarding the properties of and dependencies among numbers [7, s. 15]. The study of spatial figures in geometry was also, in the main, limited to their metric properties. And though by the middle of the last century, there did exist in mathematics such theories and separate disciplines, wherein the questions of measurement did not play any important role (for example, in the projective geometry, group theory etc.), nevertheless, the view that mathematics is a science about the metric properties of and relations among magnitudes, was dominant among mathematicians.

With the emergence of the new abstract divisions of mathematics, a structural approach towards the objects of mathematical investigations took shape; it became ever more clear that the subject-matter of mathematics is not limited to the study of the properties and relations obtainable among magnitudes and spatial figures. One of the important methodological conclusions, emanating from these latest results of mathematics, is this that notwithstanding the practical significance of the metric relations among magnitudes and their representations in numbers and functions, in the theorerical sense, they constitute only a part of the more extensive and deep-going teachings about mathematical structures and categories.

In their attempts at underlining the difference between the modern and the classical mathematics, some scholars often view the modern as the "qualitative" and the classical as the "quantitative" mathematics. However, such a contraposition is, in essence, based on an identification of the concepts of magnitude and number with quantity, and of the abstract structures and categories — with quality. One cannot agree with this position. It is understandable that nobody will object to the position that the concepts of structure and category are qualitatively different from the magnitudes or figures of the three dimensional space. There is, also, no denying the fact that modern mathematics has raised the investigations about the real world to a qualitatively new height. But this does not mean that now mathematics has gone over to the study of the qualitative specificities of objects and processes. It is evident that by contrapositing modern mathematics to the classical,the deeper and more abstract character of the concepts and theorics of modern mathematics are sought to be highlighted, the broadening of its scope and sphere of application is underlined — but by no means a transition to the qualitative methods of investigation is indicated. In contrast to the methods of the concrete natural and social sciences, the methods of the theories of structures

and categories are mathematical methods, and not the methods of special sciences, which include obsrvation and experiment. Had the converse been true, mathematics would have been then turned into a branch of the natural sciences and thereby it would have lost its character as a "generally significant and abstract science", to which F.Engels drew our attention.

The concepts and methods of the theories of structures and categories have made the process of application of mathematics in the other sciences, technology and practical activities considerably easy. Actually, having proposed the suitable abstact structures, the scholar or the practical worker can limit oneself only to verifying whether the objects under investigation satisfy the axioms of the structures under consideration or not.The entire further tedious and difficult work of deducing the conclusions from them becomes unnecessary, since one can straight off use all those theorems which were obtained while studying the mathematical structure considered. Thus, the abstract structures and categories of mathematics may be compared with ready-made forms, that may be used while investigating the phenomena and processes having the most diverse contents.

Abstract structures can be successfully used for constructing mathematical models; we may especially use those among them, which aim at revealing not only th nu eri adsstrk c) dependencies among magnitudes , but also the re atitj of a n n-mete c char cter.The study of uch non-metr c relations is f considerable sig ificance f r thos sciences, where owing to the complexity of the object under investigation, and sometimes also owing to the unelaborated stage of a theory, it is imnt ible to presen the results umerically.That is why, there one is often required to turn to the abstract s ructures of order. In their investigations about the dif erent types of rel tio s obta nable mong individuals an groups in social coll ctives, psychologists an sociologists have egu to app y t ori ndsstrk ory of g aphs, whic constitutes the simplest formdstrnslebrdssclategory.

The experience of applying the latest structural methods in the exact natural sciences convincingly shows the future possibilities open for th s line of mathematdsat on f the sciences. n fact, the use of the abstract structures of mathematics in such branches of the exact natural sciences as the theory of relativity and the quantum mechanics, theory of elementary particles and cosmology, quantum chemistry and molecular biology etc., is dictated by the very level of development of these disciplines. The concepts and theories of these disciplines, not only very often do not permit visual representations, but also do not admit of their description in the language of classical mathematics. That is why, there one is required to turn to the ideas and methods of the abstract structures and categories of modern mathematics.Thus, these abstract structures go to highlight the remarkable idea of V.I. Lenin regarding the fact that scientific abstractions,laws and theories do not push us away from objective truth, but rather take us closer to it. "Thought proceeding from the concrete to the abstract — provided it is *correct*... does not get away *f r o m* the truth but comes closer to it. The abstraction of *matter*, of a *law* of nature, the abstraction of *value* etc., in short *all* scientific (correct, serious, not absurd) abstractions reflect nature more deeply, truly and *c o m p l e t e l y*" [3, s. 152 ; Eng. ed., v. 38, P. 171].

Yet another important role of the mathematical structures lies in the fact that they serve as an exact language for the abstract description of the most diverse phenomena and processes.

It is enough to note the fruitfulness of the method of mathematical hypothesis in the process of formation of the quantum theory in physics. In this connection Dyson wrote : "For physics, mathematics is not only an instrument, with the help of which it can quantitatively describe any phenomenon, but is also an important source of such ideas and principles, on the basis of which new theories are generated" [6, s. 112]. This specificity of modern mathematics, as an exact language, for abstractly describing the interconnections among phenomena, chracterizes its role as a synthesizer in the general process of development of scientific knowledge.

While discussing the question of inter-object interconnections, it is necessary, in the first place, to turn our attention, to the role of mathematics, namely, as an exact language for expressing the dependencies, that we come across in physics, astronomy, chemistry and in the other branches of natural science. The mathematical language of formulae, equations and fuctions allows us to express the interrelations and laws of the phenomena investigated in every special science, in the most exact and general form. But for finding out the adequate mathematical language one must take the specific, qualitative character of these phenomena into consideration. All these go to show, that in the real practice of scientific cognition, there exists a dialectical interconnection and reciprocity between the quantitative mathematical methods and the qualitative methods characteristic of every spcecial science. The better we know of the qualitative specificities of the phenomena, the more successful we become in using the quantitative and mathematical methods, for analysing them. The establishment of the quantitative regularity of phenomena, is always based upon the ability to reveal that which is similar and common in what is inherent to the qualitatively different phenomena. And this is possible only by studying the phenomena within the frame-work of the special disciplines.The entire powerful apparatus of mathematics turns out to be effective only in that case, when that which is similar and common in the phenomena under investigation, is preliminarily discovered and formulated in the form of sufficiently deep going general concepts and qualitative dependencies. In fact, if this or that science proposes only the simplest of inductive generalizations of facts and empirical laws, wherein the connections among those magnitudes are established, that are immediately observable in the experiments, then it is impossible to count on the application of the latest methods of mathematics, for their quantitative analysis. The history of physics, chemistry, astronomy and of the other sciences clearly testify to the fact that the progress of theoretical investigations in them was accompanied by an extensive application of the mathematical methods. Often the demand for elaboration of these theories promoted the emergence of new mathematical methods, a clear example of this is the emergence of the infinitesimal analysis.

3. THE MATHEMATICAL STRUCTURES AND THE REAL WORLD

Just as the question of the relation of consciousness and being, is basic for philosophy as a whole, likewise the question of the relation of the mathematical structures and the real world happen to be central for the philosophy of mathematics. In contrast to the positivist approach, the school of N. Bourbaki, not only does not ignore this problem, but, on the contrary underlines the fact that, "the interrelationship of the universe of experiment with that of mathematics" is

the basic philosophical problem [5, s. 258].This school also does not deny the existence of close ties among the structures of mathematics and empirical reality, though it considers the reasons of their existence to be entirely unexplainable.

The difficulties connected with the understanding of the objective nature of the abstract structures of mathematics, are rooted in the very specificities of mathematical knowledge, which bases all its propositions upon the laws and principles of logic and not on experiment. It was not accidental, that is why, for G.Frege, B. Russell and their followers to have attempted to seek the foundations of the entire pure mathematics in logic. But therein they ignored the doubtless fact that mathematics, as an independent science, needs its own initial concepts and postulates. Otherwise, as has been correctly pointed out by A. Poincaré, it would turn into a grandiose tautology. It is also important to pay attention to the fact that many western scholars consider logic itself to be an a-priori-science about the forms of thought. And in so far as logic plays the most important role in the formation of abstract structures, often these structures are themselves also viewed as a-priori-forms.

The strictly logical and deductive character of the constructions and substantiations of mathematics are, ultimately, determined by the specificities inherent to the processes of abstraction and idealization in mathematics. Firstly, in mathematics abstraction proceeds significantly further than, say, in the natural sciences. In the concepts of the geometrical point, line, the variable, the function and in the case of all the other mathematical concepts in general, we abstract from the concrete contents and qualitative specificities of objects and processes. Secondly, many abstractions of modern mathematics emerge through a series of successive stages of abstraction and subsequent generalization. It is, namely, thus that all the mathematical structures have been formed. Thirdly, the relative independence of the purely theoretical development is, perhaps, more characteristic of mathematics, than of any other science. In contrast to the experimental sciences, mathematics does not contain any empirical terms or experimental methods for verifying its propositions. These propositions must be proved, i.e., logically deduced from a small number of axioms, accepted without proof. All the hereinmentioned specificities of mathematical knowledge especially clearly came to the fore with the transition in mathematics from the study of the quantitative relations among magnitudes and figures to the investigations into structures of the most diverse kind, which often have only a distant similarity with the traditional objects of classical mathematics. In connection with this, the most widespread notions about the nature of mathematical knowledge and the relation of mathematical objects and structures with the real world, were subjected to criticism.

The empirical notions about mathematics — according to which mathematical knowledge is essentially identical with the natural-scientific knowledge — were the first to be criticised. Moreover, the empiricists misinterpreted natural-scientific knowledge itself. For instance, the followers of classical empiricism viewed its theory to be inductive generalisation of experience. The defenders of similar inductive-empirical notions wanted, according to N. Bourbaki, "to compel mathematics to arise from the experimental truths". Evidently, the new stage of development of mathematics refutes these notions. However, N. Bourbaki so strongly stress the dominant role of rational-theoretic thought in this process, that they entirely forget about the objective source of emergence of mathematical ideas and

theories. They do agree that, "of course, on cannot deny that most of these forms had a completely determinate inductive content when they emerged; but once they were consciously deprived of this content, it was possible to give them all their efficacy, which constitutes their strength, and it was made possible for them to acquire new interpretations and to fulfil their own role in data processing" [5, s. 259]. Clearly, all this is true, but from this it at all does not follow that experimental reality goes into the mathematical structures as a result of some predetermination or pre-established harmony.

Exactly in the same way, these structures should neither be considered to be *a priori* constructions of the human mind, nor conventions or argeements devised for ordering empirical data or for "economy of thought" *a la* the subjective-idealist philosophy of E. Mach. One cannot deny the existence of elements of convention, agreement and even of quite understandable economy of thought in these structures, in so far as any abstraction, as pointed out by Engels, is a shortening, and thereby it rids us of a mass of details. But these elements do not play a self-sufficient role and can be correctly understood only in that case, when a structure is considered in the process of its historical emergence and development, wherein the empirical and the theoretical, the contentful and the formal and, the concrete and the abstract factors dialectically interact with each other.

When we deal with the ready-made mathematical structures, then at the first glance they do indeed appear to be *a priori* forms of knowledge, which turn out to be applicable to the study of very different contents. But why does it so happen ? Even the school of N. Bourbaki does not deny the fact that the initial concepts and structures of mathematics do have a fully determinate intuitive content; and this is evident from the aforementioned quotation. Many western philosophers of science stress the priority of form over content in all kinds of ways and that is why they view the mathematical structures as pure forms. One has only to attentively follow the genesis of these forms historically and logically, for the said illusion to vanish.

In fact, isn't there any connection and continuity between the primary structures of mathematics, which have an entirely determinate intuitive content, say the three dimensional space of Euclid, and the many dimensional or even infinite-dimensional abstract spaces ? Don't these structures emerge thanks to the singling out of the deeper and more important properties and relations of the mathematical objects and structures under investigation ? Considering the point of a many-dimensional space to be a vector, for which the ordered sequence of real numbers serve as co-ordinates, we postulate that for them the basic laws of operations over vectors, hold good. Owing to this it becomes possible to establish the connection and to differentiate between the three-dimensional and the many-dimensional spaces: all the laws of operations involving the vectors of the ordinary three-dimensional space do not hold good in the many-dimensional space. No less important is the fact that owing to this, the continuity in the development of mathematical knowledge, and the possibility of testing the conclusions of the many-dimensional space with the help of those of the three-dimensional, are established, since in the limiting case such conclusions must correspond to the results of ordinary geometry.

The situation is quite analogous with any abstract structure in general. The deepest properties and relations of the abstract mathematical objects are formulated in the axioms

of these structures. It is, namely, that is why that these structures turn out to be applicable to the investigations of the most diverse phenomena and processes of the real world, and not only of those which served the emergence of the primary structures.

Having begun the work with such ready-made structures, a mathematician usually forgets those intuitive prototypes, which provided the impetus for the formation of the entire chain of successive abstractions and generalizations. And of course he is not bound to remember them, since that would complicate his work. Having proposed an abstrct structure, he deduces the logical conclusions from the axioms, seeks various interpretations for them, establishes its connections with the other structures etc. In other words, the modes of application and elaboration of the mathematical structures are directly opposite to the historical path of their formation. Historically speaking, mathematical knowledge went from the separate, concrete system of objects, or interpretations, towards revealing their general structures,i.e. from the particular and the separate to the general, whereas in the process of further investigations it moves from the ready-made abstract structures towards the observation of various other interpretations.

Marx termed this process of development of mathematical knowledge, wherein its final point is taken as the beginning of the further movement of thought, the "inversion of method".He has illustrated it in detail in the "Mathematical Manuscripts", in the light of the emergence and use of the basic concepts of differential calculus. In the article entitled "On the Differential", he has shown, how, in course of the elaboration of the calculus, the symbolic differential co-efficient becomes an independent point of departure, but "with this, the differential calculus too appears as a specific kind of calculus, already operating independently upon its own ground...." [2, s. 55-57].

These ideas of Karl Marx about the "inversion of method" in the course of mathematical cognition, provides us with an opportunity to proceed correctly towards the solution of the problem of objective contents of the abstract structures of modern mathematics. At the first glance, their emergence appears to be *a priori,* but in reality it is the product of a protracted development, in course of which, the deepest and the most important properties of the structures are gradually brought to light. But as soon as this cycle of development comes to an end and leads to the formation of the corresponding mathematical structure, this final point becomes the beginning of a new stage of mathematical cognition, connected with the elaboration of the theory of the given structure and with its application in the other sciences.

In so far as the entire historical process of emergence of the structures, usually goes on outside the field of vision of the modern mathematician, it is easy for him to have an illusion about the *a priori* character of the abstract structures or about some pre-established harmony among them and the empirical reality.

Roots of the idealist notions about the nature of mathematical cognition consist in this that, therein those connections between the abstractions and reality are ignored, which are realized in the process of application of mathematics in the natural sciences, technology and in the social-human sciences. Here, the contra-positing of "pure" mathematics and its application, is also to be largely blamed. This has promoted the cultivation of the idealist notions to the effect

that, it is not the case that the abstractions and structures of mathematics are in agreement with the real world, but, conversely, it is the world and its regularities that come to conform to these abstractions and structures of mathematics. Criticising similar idealistic views about mathematics F. Engels pointed out that here,"as in every department of thought, at a certain stage of development the laws which were abstracted from the real world, become divorced from the real world, and are set up against it as something independent, as laws coming from outside, to which the world has to conform" [1, s. 38].

Another widespread point of view on the mathematical structures is connected with their conventionalist treatment. The famous French mathematician A. Poincaré is the founder of this approach; under the influence of the discovery of the new, non-Euclidean geometries he began to think that the axioms of geometry are conventional agreements and, while choosing them the mathematician is guided exclusively by the demands of convenience. We have seen that the abstract structures are defined by their axioms and, to that extent, here the mathematician enjoys considerable freedom, of choice, and of mutual combinations of the axioms and, that is why, in these structures the conventional moment is clearer than in the ordinary geometrical systems.

Understandably, one can not deny the fact, that in the formation of the abstract structures, as in that of any other mathematical concept, the elements of choice and agreement do have a place. Without such choice, mathematical creativity would become meaningless, but freedom of choice does not signify a rule of arbitrariness. It is confined within the framework of necessity and, in mathematics in particular — constrained by the demands of logical non-contradictority of the axioms of the structure. But how can we be sure of their non-contradictority? In the light of the example of the geometry of Lobachevsky we have seen that the non-contradictoriness of its aximos can be proved by constructing its model with the help of the geometry of Euclid. In its turn, the non-contradictoriness of the geometry of Euclid can be proved with the help of an arithmetical model.

This process of proving the relative non-contradictoriness of the more abstract and new theories with the help of the old theories, with which we are more accustomed, is very characteristic of mathematical cognition. It is testified, firstly, by the fact that there exists a continuity and close inter-connection among the new and the old mathematical theories. Secondly, the freedom in the process of creation of the new mathematical structures, constrained by the demands of logical non-contradictoriness of the system of axioms, in essence signifies this, that the conventional elements play a subordinate role in mathematical cognition. The mathematician may substitute some axioms by others or seek out more general premises for his conclusions, but ultimately the correctness of his results are controlled by logic and by such well substantiated and corroborated theories, as the elementary geometry of Euclid and arithmetic, the truth of which have been tested in the centuries old practice of mankind.

Literature

1. *Marks K., Engels F.*, Soch. 2-e izd., T.20 (Collected Works, Eng. ed. v. 25).
2. *Marks K.*, Matematicheskie Rukopisi. M., 1968.
3. *Lenin V.I.*, Poln. sobr. soch. T. 29 (Collected Complete Works, Eng. ed., vol. 38).

4. *Bukur I., Delyanu A.*, Vvedenie v teoriu kategorii i funktorov (Introduction to the Theory of Categories and Functors). M., 1972.

5. *Burbaki N. (Bourbaki N.)*, Ocherki po istorii matematiki (Essays on the History of Mathematics). M., 1963.

6. *Dyson F.J.*, Matematika v fizicheskikh naukakh (Mathematics in the Physical Sciences) // Matematika v sovremennom mire (Mathematics in the Modern World). M., 1967.

7. *Foss A.*, Sushnost matematiki (Essence of Mathematics). M., L., 1923.

Source : Zakonomernosti razvitiya sovremennoi matematiki. "Nauka", M., 1987, s. 155-169.

About the Author : Georgi Ivanovich Ruzavin (1922-), philosopher and mathematician. Areas of specialization : Philosophy of Mathematics and Logic of Probability.

Other works :

1. O kharaktere matematicheskoi abstraktsii, 1961;
2. Osnovnye etapi razvitiya formailnoi logiki (soot. P.V.Tavantsova), 1962;
3. Induktsia i veroyatnost, 1962 ;
4. Veroyatnostnaya logika i eo roil v nauchnom issledovanii, 1964;
5. Induktsia i veroyatnost, 1965;
6. Induktivnye vyvody i veroyatnostnaya logika, 1967;
7. Semanticheskaya kontseptsia induktivnoi logiki, 1967;
8. O prirode matematicheskovo znaniya, 1967;
9. "Matematicheskie Rukopisi" K. Marksa i nekotorye problemy metodologii matematiki // *Voprosy Filosofii*, 1968, No. 12, s 59-70.

REFLECTIONS
ON SEVEN THEMES OF PHILOSOPHY OF MATHEMATICS

VLADIMIR ANDREIVICH USPENSKY

PREFACE

An All-Union symposium was organised in the town of Obninsk, from the 26th to the 29th of September 1985, on the theme "The Regularities and Modern Tendencies of the Development of Mathematics". I took part in it. I was invited by Vladimir Invanovich Kuptsov. To him, to a large extent, goes the credit for the relaxed, creative and business-like atmosphere of this symposium. The papers were followed by intensive discussions, which continued in the so-called "round table" meetings. I did not read any paper, but took part in the discussions several times. Mikhail Ivanovich Panov thought that what I said was good enough for publication and, it is he who gave them the shape of papers, to be included in a collection [of some of the papers read at the symposium] edited by him. It is thus that these "Seven Reflections" came into being. Here are the themes :

1. Is it true that in mathematics everything is defined and proved?
2. Is it possible to define the concept of natural number?
3. Is it possible to define the Series of Natural Numbers (written with capital letters)?
4. Is it possible to axiomatically define the concept of a series of natural numbers (written with small letters)?
5. Is it possible to prove, that Fermat's Great Theorem can neither be proved nor disproved?
6. What is a proof?
7. Can mathematics be made understandable?

1. Is it true that in mathematics everything is defined and proved?

Mathematicians are, as a rule, proud of the fact that they are mathematicians. For them, the source of their pride lies in their discipline — not so much in the usefulness of mathematics, as in the fact , that it is an unique field of knowledge, not resembling any other. And even the non-mathematicians are in agreement about its exclusiveness (thus not only the mathematicians themselves, but to their satisfaction, even those around them, recognise the greatness of the mathematicians). Indeed, it is considered to be generally acknowledged, that at least the three following traits belong to mathematics, and to it alone. Firstly, in mathematics, unlike in the other disciplines, all the concepts are strictly defined. Secondly, in mathematics — and again unlike in the other disciplines — everything is strictly proved from axioms. Thirdly, no other discipline has attained that level of respectful trepidation, at which mathematics remains not understood. Tutors of mathematics are hardly more in number, than those of all the other school-level subjects taken together, and yet there is nothing that one (of them)can really say about modern "higher" mathematics: it would be enough to open any monograph, or better still a paper in any journal.(Please note, that it often goes unnoticed, that the third trait indicated above clearly contradicts the first two).

When something becomes very well-known, then a suspicion creeps in : isn't this "something" a myth (well-known ideas do indeed possess an autonomous self-support mechanism). Let us attempt an, as far as possible, unbiased critical examination of the three , just indicated, well-known traits of mathematics.

We notice, firstly, that it is not possible to define all the concepts of mathematics. One is defined through the other, this other through a third etc.; we must stop at some place. (Mrs Prostakova rightly observed — "A tailor learned his trade from another, that one from a third, but from whom did the first tailor learn it?") A story goes, that once the famous mathematician from Odessa S.I. Shatunovsky was introducing ever newer concepts in a lecture and, while so doing he was repeatedly being asked : "And what is this and what is that ? "; he lost his patience at long last, and asked in retort: "And what is 'what is'? ".

Let us consider the structure of a defining dictionary in any language — Russian, English or any other. In it, one word is defined through another, that one through a third etc. But since the words of a language are finite in number, emergence of a circularity is unavoidable (i.e., there emerges a situation, when, in the final count, a word is defined through and by itself). [Here it would be useful to think of a graph, in which the words are placed at the apexes and, when in the dictionary entry defining the word X one meets the word Y, then in that case an arrow goes from the apex X to the apex Y.] There is only one way of getting rid of such a circle: some words are to be left undefined. And that is what is done in some of the dictionaries.[For instance, the words " thing" (in its principal meaning) and "all" have been left undefined in the defining dictionary of English language compiled by Hornby and Parnwell [8]. Unfortunately, such a dictionary has not yet been compiled for the Russian language .] Clearly, such is also the case with the concepts of mathematics. And if, namely, one does not wish to permit a vicious circle, then one has to leave some of the concepts undefined. The question arises — how are these concepts going to be assimilated? Answer : through immediate observation, from experience, from intuition. There is no need to remind the reader, that the formation of general, abstract concepts in the human brain, is a complex process, which belongs more to the realm of psychology, than to that of logic. These concepts, which are assimilated not from verbal definitions, but rather from immediate personal experiences, are naturally called the *primary concepts* or categories of mathematics. The concepts of point, straight line, set, natural number etc. are examples of such categories.

One is certainly required to be careful while preparing a list of the categories (primary concepts) of mathematics (such a list can hardly be made fully precise). Otherwise the number of primary concepts will become unjustifiably large and, the principle of "Occam's razor" will be violated. Here let us consider, for example, the concept of a sphere. It is well-known that a sphere is the locus of points in space having a given fixed distance from a given point — which is the centre of the sphere. However, we can hardly find anyone, who came to know what is a sphere, first of all, from this definition. We must concede that a person assimilates the concept of a sphere in childhood — from the examples of a ball, a globe, a ball-bearing and a billiard ball. One learns the aforementioned definition of a sphere only in the class-room. And

there, the teacher does not always find the time to explain it to the learners that the sphere known to them since the early childhood and, the sphere about which they were taught at school —happens to be one and the same sphere. In consequence there grows the notion that : "everything is upside down in their physics and mathematics; perhaps even their sphere too goes upwards". The words just quoted were uttered by a "quite intelligent student ", in justification of a statement made during a lesson, to the effect that a sphere put on an inclined plane starts rolling upward. This remarkable episode has been described in [10]. But, from what has been said above does it follow that since one comes to know about the concept of a sphere from experience, and not from a verbal formulation, the concept of a sphere should be considered a primary concept, one of the categories of mathematics?

It would appear, that the situation becomes more clear in the case of the more complex concepts of mathematics, which are further removed from experience,like for example, the concept of a group — surely, one would not regard the concept of a group to be a primary concept. However, the process of formation of the concept of a group in the brains of professional mathematicians is, perhaps, not very different from the process of formation of the concept of a sphere in the brains of people in general (which includes the mathematicians and the non-mathematicians) : just as the concept of a sphere emerges as a result of numerous observations of various spheres, so did the concept of a group emerge as a result of examination of concrete groups — and only then was this concept fixated in a verbal formulation (obviously, at issue here is the emergence of the concept of a group in the collective experience of mathematicians, and not in the experience of an individual mathematician). That is why not the mode of emergence of a concept,but rather the mode of transmission of informations about it within a system of knowledge, which should be considered to be the characteristic indicator of its primacy (categoricity). For elucidating what has been stated above, we shall imagine a situation where the carrier of a system of knowledge — in our case knowledge of mathematics — has to transmit his knowledge to others. Then he can tell others, what is a sphere or what is a group, by using the verbal definition of the corresponding concept. And that is why these concepts are not categories. If, however, one is required to communicate, what is a set, or what is a straight line or what is a natural number, then that is done differently. For example, it is said that : all the chairs in this room constitute a set, and all the Ostriches beyond the Polar Circle constitute a set (academician P.S. Alexandrov's example), and all the irrational numbers within the interval [0,1]constitute a set; and later on, after having provided sufficient number of examples, it is said that : "these are all sets" — and thus there emerges the general concept of a set. Analogously: zero, one, two, three, four, five etc. are all natural numbers, and thus emerges the general concept of natural number.[It is high time for putting an end to the anachronism of beginning the series of natural numbers with one. In a pencil-box there are always some natural number of pencils — perhaps zero. A natural number is the cardinality (of the number of elements) of a finite set, in particular — that of an empty set.] (We see that while explaining the concept of natural number, there appears the word "etc." — implicitly or explicitly, and it

could not have been otherwise in the case of primary concepts: first, a sufficient quantity of examples are indicated and, then we have the word — "etc.")

Thus, the first among the myths about mathematics — that, "in mathematics everything is defined" — collapses. Let us proceed to the second : "in mathematics everything is proved from axioms". In order to get convinced that such is not the case and, to thus blast this second myth too, it would be enough to open the classic text-book of school geometry by A.P. Kisvpfx.psr any text-book of mathematical analysis for the technical colleges, or any university level text-book of the theory of numbers. In all these text-books we come across theorems being proved, but hardly any axioms (save the axiom about the parallel lines — the fifth postulate of Euclid). The situation is somewhat enigmatic. Indeed, if there are no axioms, then on the basis of what re the proofs — say, of the theorems of the theory of numbers — put forward ? Evidently, on the basis of common sense and some notions about the basic properties of the natural numbers; though these notions are identical for all persons, they have not been explicitly formulated in the form of a list of axioms. (How far they may be so formulated! Well, that is the theme of our next reflection).

It must be stated in all honesty that, in reality, in mathematics one pretty often comes across theorems, which are proved without basing the proofs upon any kind of axioms. The situation with the third trait of mathematics indicated by us — namely, with its non-understandability — happens to be more complex. It would be very easy to say that it is a myth; but if in respect of the first two traits it was enough to ask mathematics itself — one asks and gets a negative answer — then here, of course, it is pointless to turn to mathematics with the question: whether or not it is understood. A survey of social opinion indisputably situates mathematics at a prized spot in terms of the level of non-understandability. The reasons behind such an opinion happen to be the theme of a separate large-scale investigation. Any explanation of this phenomenon, it must be admitted, can only be that much objective, as much it is possible, in general, to be objective about the issues of social psychology. We shall not plunge into such a discourse here. In our last reflection we shall be making some remarks on this theme.

2. Is it possible to define the concept of natural number?

It can certainly be said that a *natural number* is the quantity of items in a finite totality. Evidently this formulation is compatible with the meaning (to be more precise, with one of the meanings) of the verb"to define" according to the "Defining Dictionary of the Russian Language " edited by D.N. Ushakov [5] (" to give a scientific, logical characterization, a formulation of any concept, to lay bare its (scientific)content"), as well as with the formulation found in the Philosophical Encyclopaedia [11] (the "definition" of an object — the results of the investigations about which are reflected in the corresponding concepts — "may be viewed as an (explicit and concise) formulation of the contents of these concepts"). Let us proceed now to the concepts behind the verb "to define" and the word "definition", from the position of a mathematician. And then , we demand that a definition should contain exhaustive information about the concept defined — it must be so exhaustive as to permit a person having no earlier knowledge about a given concept, to form a correct

understanding of it, solely on the basis of the definition provided. In that case can we assume, that a person who does not know anything at all as to what a natural number is all about (we are not talking here about the term, but, namely, about the concept), will be able to assimilate this concept from the first sentence of the present paragraph ? That is very very doubtful : when one really does not know what is a number, then it is quite likely that he may take the words "quantity of items" to signify, say, their total weight and, the very concept of finite totality of items gets diffused when one considers very large totalities. Probably everyone would agree that trillion to the power trillion is a natural number, however, it remains a fact that this number is greater than the number of atoms in the whole Universe. It is not clear, as to how far appropriate it would be to talk about a finite totality of trillion to the power trillion quantity of items[16].

Thus, we shall be captiously demanding an exhaustive completeness from a definition, i.e. we shall be demanding that the concept being defined be expressed with the help of generally accepted syntactical constructions through other concepts, which serve as starting points for the definition under consideration. Taking into consideration what has been said above, let us attempt the following formulation : a natural number is the cardinality of a finite set. Three basic concepts are operative in this definition : 1) set, 2) cardinality and, 3) finite. The just mentioned formulation indeed appears to be the definition of a natural number, within the frame-work of those theories wherein these concepts are already somehow interpreted (in particular, declared to be uninterpretable or primary). Namely such a definition — such, in the ideational sense, right upto the inessential details — has been accepted in the "Éléments de mathématique" of Nikolai Bourbaki. (In this connection I would like to remind the reader that in Bourbaki's theory the full name of one [unity] requires tens of thousands of symbols, for the purpose of being written down) [6,p.188]. Common sense, however, refuses to accept the concepts of set, cardinality and finite to be simpler than the concept of a natural number. Here we have the typical example of a definition of the simple through the complex.

One should not take the above-mentioned statements to be a criticism of N. Bourbaki and of the other authors, who put forward analogous formulations. Evidently, they, like other people, have some *a priori* notion of a natural number (apparently, *a priori* in respect of the definitions they propose, and not in respect of experience). They do not intend to give an explaining definition of the concept of natural number (i.e., a definition that could be used to teach a novice). Their aim is more modest and technical: to give a definition of this concept within the frame-work of an expounded axiomatic set theory. The concept of a function can be defined through the concept of a pair, and the concept of a pair can be defined through the concept of a function. It is clear that these intellectual constructions have hardly anything in common with the problem of explaining to the uninitiated, what is a pair and what is a function. The aim of the entire foregoing discussion is to lead the reader to the following almost self-evident idea. Let us set aside the mathematical and logical problematique, connected with the search for a definition of (it would be more correct to say — the attempts at representing, modelling) the natural number, within the framework of this or that axiomatic theory. Let us take up instead, the attempts at providing a "naive" explanation of the concept of

a natural number — an explanation, that would enable one who does not know, to know what it is all about. Very soon we shall get convinced that such attempts are fruitless. We must admit that the natural number is a primary, undefinable concept — it is one of the categories of mathematics.

3. Is it possible to define the Series of Natural Numbers (written with capital letters)?

Having failed in our attempts at defining the natural number (or, on the contrary, having succeded in finding out that this concept is an undefinable category), let us now turn to the concept of the Series of Natural Numbers. When written with the big or capital letters, the Series of Natural Numbers is the totality of all the natural numbers. If we know what is a natural number and understand the words "totality of all", then we know what is the Series of Natural Numbers. Conversely, if we know the Series of Natural Numbers, then we can easily define a natural number as one of its elements. That is why the concept of the Series of Natural Numbers is as undefinable as the concept of natural number. (However, the sentence "The Series of Natural Numbers is the set of all natural numbers" may be viewed as a legitimate definition of the concept of the Series of Natural Numbers through the primary, undefinable concepts of a "natural number" and the "set of all".)

The reader would exclaim — "How come? And what about the axioms of Peano? Don't they define the Series of Natural Numbers ?" Of course they don't, and what is more, if one understands the Series of Natural Numbers as we do — i.e., as the unique totality of some univocally understood essences called the natural numbers — then the axioms of Peano do not even pretend to do that. Indeed, let us see, how the axioms of Peano look: "Zero is a natural number, and zero is not the successor of any number, etc." Thus, these axioms are based on the concepts "zero" and "successor of" (here, immediate succession is at issue). But they do not explain, and they can not explain, what these concepts signify (i.e., what is "zero" and what is "successor of"), they only indicate the connections between these concepts. However, these axioms have been so formulated, that if the zero of these axioms is the ordinary Zero of the Series of Natural Numbers, and if the words "successor of " signify the immediate succession of one number after another in the Series of Natural Numbers (such that Zero is succeded by Unity, and after Unity comes the figure Two etc.), then all these connections will be satisfied in the Series of Natural Numbers.[In order to stress the uniqueness, i.e., the absolute singularity of the terms of the Series of Natural Numbers — Zero,One (Unity), Two (the figure Two) etc. — we write them with the capital letters. The words " Zero", "One" (or "Unity"), "Two" (or the figure "Two") etc. are proper nouns in the absolute sense (just like the words "Sun", "Moon" and "Earth"), each of them has unique meaning — the quantity of elements of the empty, unit, two-element etc. sets. And the "zero" axiom of Peano indicates a proper noun only relatively, within the limits of the given context, to be more precise — in the context of that structure, which has been described by these axioms. There are many such structures, and each one of them has its own zero.] In other words, Peano's axioms turn out to be true, correct statements upon their natural interpretation in the Series of Natural Numbers. But evidently, they will be true, not only in the Series of Natural Numbers, but also in all the other structures isomorphic [on the concepts " isomorphism" and "isomorphic" we

request the reader to go through the second of the two articles entitled "Isomorphism" in the 3rd edition of the Great Soviet Encyclopaedia [14] — V.A. Uspensky; readers of the present translation will find this article on "Isomorphism" at the end of the present theme **3.** — Tr.] to the Series of Natural Numbers. For example, if the term "zero" in the axioms of Peano is interpreted as the smallest prime number, and the words "successor of"— as the [result of] a transition from one prime number to another immediately next to it, then under such an interpretation all the axioms of Peano turn out to be true. It appears that these axioms do not even permit us to distinguish the Series of Natural Numbers from the totality of all the prime numbers. I repeat, they do not pretend to do so. They claim to, it is said," define the Series of Natural Numbers right upto isomorphism". To be more precise, the axioms of Peano define not one, but at once many mathematical structures, moreover they are all isomorphic to the Series of Natural Numbers and, consequently, isomorphic to each other. To be more precise, the axioms of Peano define the entire class of such structures. We shall call any such structure a series of natural numbers (written with small, or lowercase letters!). Thus, the Series of Natural Numbers is one of the serieses of natural numbers.

Briefly speaking, isomorphism of two mathematical structures is the mutually-univocal correspondence among the totalities of elements of the first and the second structure, retaining the operations and relations defined on these structures. In our example the isomorphism between the structure N (the Series of Natural Numbers with the operation "to follow") and the structure P (prime numbers with the operation "to follow") provides the following endless table:

0	1	2	3	4	5	6	...
2	3	5	7	11	13	17	...

In this correspondence the operation "to follow" is indeed retained: 6 follows 5, and simultaneously 17 follows 13, and in general in the upper row y follows x if and only if the corresponding terms of the lower row p_y and p_x (namely, in this order!) follow one after the other (follow in the sense defined for P).

It is sometimes said that the Series of Natural Numbers is the series

0, 1, 2, 3, 4, 5, 6 ..., 126, ...

but likewise it can be said that the Series of Natural Numbers is the series
zero, one, two, three, four, five, six... , one hundred and twenty six,....
or the series

$\overline{0}$, I, II, III, IV, V, VI..., CXXVI,....

[Isn't the persistent exclusion of zero from the series of natural numbers explained by the absence of the symbol $\overline{0}$ in the traditional collection of symbols? Briefly speaking, aren't we situated at the level of the Latins on this question?]

Evidently, none of these serieses happens to be the Series of Natural Numbers (which consists of abstract quantitative categories and can not be depicted), these are but the serieses of names designated for its terms, i.e. for the natural numbers. At the same time each of these serieses of names may be viewed as one of the serieses of natural numbers, written with small letters.

The situation with the Series of Natural Numbers is universal in character. For example, we have an analogous situation with the three dimensional Euclidian space, in which we live. Let us digress from the fact that most probably we live in non-Euclidian space, and generally speaking, we live not in the mathematical, but in the physical space — and these are different objects. [In this connection we must mention the fact that, most probably, the "physical" Series of Natural Numbers is something different from its mathematical model — the "mathematical" Series of Natural Numbers. On this issue see the deep-going but insufficiently appreciated essay [16] by P.K. Rashevsky.] Let us abstract from reality and imagine that we live in an entirely concrete three-dimensional Euclidian Space (we are again using capital letters, as we wish to stress the uniqueness of this space). However, it can not be defined with the help of any number of axioms, it may only be "indicated with a finger". On the other hand, there are numerous systems of axioms (the most famous among them belongs to Hilbert) [3], defining this space "right upto isomorphism". The phrase within quotation marks indicates the fact that the given system of axioms defines an entire class of mutually isomorphic spaces, and that our "real" Euclidian Space happens to be one of them.

In general, no system of mathematical axioms can ever define any structure univocally, in the best of the cases they define it right upto isomorphism. (We speak here of " the best of the cases" as there are very important systems of axioms, which define the class of non-isomorphic structures. For instance, the axioms of group theory define the mathematical structures called the groups, but all of them are not mutually isomorphic.)

Let us sum up. It is not possible to axiomatically define the Series of Natural Numbers. We may try to axiomatically define the concept of a series of natural numbers — i.e., the concept of any arbitrary structure, isomorphic to the Series of Natural Numbers. We shall be discussing these attempts in our next reflection.

ISOMORPHISM

It is one of the fundamental concepts of modern mathematics. It initially arose in algebra in connection with the algebraic systems, such as groups, rings and fields, but proved to be extremely significant for the understanding of the structure and domain of possible applications of every branch of mathematics.

The concept of isomorphism applies to systems of objects on which operations or relations are defined. As a simple example of two isomorphic systems consider the system R of all real numbers under the operation of addition $x = x_1 + x_2$ and the system P of positive real numbers under the operation of multiplication $y = y_1 y_2$. It turns out that the internal "lay out" of these two systems of numbers is identical. To show this we map the system R onto the system P by associating to the number $y = a^x$ $(a > 1)$ in P, the number x in R. Then the product $y = y_1 y_2$ of the numbers $y_1 = a^{x_1}$ and $y_2 = a^{x_2}$ which correspond to x_1 and x_2, will correspond to the sum $x = x_1 + x_2$. The inverse mapping of P onto R is given by $x = \log_a y$. From any proposition concerning the addition of numbers in the system R we can obtain a corresponding proposition concerning the multiplication of numbers in the system P. For example, since in R the sum

$$S_n = x_1 + x_2 + \cdots + x_n$$

of the terms of an arithmetic progression is given by the formula

$$S_n = \frac{n(x_1 + x_n)}{2},$$

it follows that in P the product

$$P_n = y_1 y_2 \cdots y_n$$

of the terms of the corresponding geometric progression is given by the formula

$$P_n = \sqrt{(y_1 \; y_n)^n}$$

(raising to the n-th power in P corresponds to multiplication by n in R and extraction of the square root in P corresponds to division by two in R).

As regards their properties, isomorphic systems are essentially the same. From an abstract mathematical standpoint, such systems are indistinguishable. Any system of objects S' that is isomorphic to the system S may be regarded as a "model" of S (modeling a system S by means of a system S'), and the study of the properties of S may be reduced to the study of the properties of the "model" S' of S.

The following is a general definition of the isomorphic system of objects such that each system has a number of relations and each relation involves a fixed number of objects. Let S and S be two given systems of objects. Let

$$F_k(x_1, x_2, \cdots), \quad k = 1, 2, \cdots, n$$

be the relations on S and let

$$F'_k(x'_1, x'_2, \cdots), \quad k = 1, 2, \cdots, n$$

be the relations on S'. The systems S and S', with their respective relations, are said to be isomorphic if there exists a one-to-one correspondence

$$x' = \Phi(x) \qquad x = \psi(x')$$

between the elements of S and S' such that

$$F_k(x_1, x_2, \cdots)$$

implies

$$F'_k(x'_1, x'_2, \cdots)$$

and vice versa. The correspondence is said to be an isomorphic map, or an isomorphism. [In the example cited above, the relation $F(x, x_1, x_2)$, where $x = x_1 + x_2$, is defined on the system R, and the relation $F'(y, y_1, y_2)$ where $y = y_1 y_2$, is defined on the system P, a one-to-one correspondence is given by the formulas $y = a^x$ and $x = \log_a y$.]

The concept of isomorphism arose in group theory where the fact that the study of the internal structure of two isomorphic systems of objects represents one rather than two problems was first understood.

The axioms of any mathematical theory determine the system of objects studied by the theory only upto isomorphism: a mathematical theory based on axioms, that is applicable to one system of objects is always fully applicable to another. Therefore, every axiomatic mathematical theory allows not one but many "interpretations" or "models".

The concept of isomorphism includes, as a particular case, the concept of homeomorphism, which plays a fundamental role in topology.

A particular case of an isomorphism is an automorphism, which is a one-to-one mapping

$$x' = \Phi(x) \qquad x = \psi(x')$$

of a system of objects with given relations $F_k(x_1, x_2, \cdots)$ onto itself such that $F_k(x_1, x_2)$, implies $F_k(x'_1, x'_2, \cdots)$ and vice versa. This concept also arose in group theory but later proved significant in most disparate branches of mathematics.

References

Kurosh, A.G. *Kurs Vysschei Algebry*, 3rd. ed., M-L.,1952.

Entsiklopedia elementarnoi matematiki, bk. 2. Ed. P.S. Alexandrov et al. M-L., 1951.

[Source: *The Great Soviet Encyclopedia*. A Translation of the Third Edition. Vol. 10, p. 465. Macmillan, 1976.]

4. Is it possible to axiomatically define the concept of a series of natural numbers (written with small letters) ?

So then,we get down to the attempts at axiomatically defining the concept of a series of natural numbers, which is a structure isomorphic to the Series of Natural Numbers. As soon as we utter the word "isomorphism", already thereby it is proposed, that the relations and operations to be retained under this isomorphism have been indicated. Consequently, first of all we must indicate precisely, the relations and operations we wish to examine in the Series of Natural Numbers and in the serieses of natural numbers isomorphic to it. Among these operations we may include zero-place operations(i.e., individual constants ; for example, the individual constant "zero" may be viewed as a zero-place operation) and one-place relations (i.e., properties). These earmarked operations and relations are to a significant extent arbitrarily indicated. For example, the Series of Natural Numbers (and thereby any series of natural numbers isomorphic to it) may be viewed :1) as a structure only with the order relation "< ", or 2) as a structure with an earmarked element "zero" and the operation "transition to the next ", or 3) as a structure, wherein, apart from the relations and operations already mentioned, the operations of addition and multiplication have also been earmarked.

For our purposes it would be most graphic not to indicate any operation, but only to stipulate the order relation "< ".Thus we shall be viewing every series of natural numbers as a set, on which the binary order relation "< "has been defined. We shall be investigating, namely, the properties of such a mathematical structure.

Let us enumerate these properties. When the relation "< " is understood as an ordinary relation of order among natural numbers, then every property of the relation "< " in an arbitrary series of natural numbers must (on the strength of the presence of isomorphism) hold good also in the usual Series of Natural Numbers. After this remark, let us now formulate some of these properties.

1. The relation "< " is transitive . Symbolically :

$$\forall x \forall y \forall z (x<y \wedge y<z \Rightarrow x<z)$$

2. The relation "< " is anti-reflexive. Symbolically:

$$\forall x \neg (x<x) \ .$$

3. The relation "< " is symmetric. Symbolically:

$$\forall x \forall y (x<y \vee y<x) .$$

The totalilty of these three properties simply affirms that "< " is a relation of strict linear order.

Before going ahead further, let us stop and think: strictly speaking, why are we listing these properties ? Here's why. We hope that having listed a number of properties, we shall be able to axiomatically define a series of natural numbers . In greater detail, our plan is as follows. At first we write out some of the properties characteristic of the Series of Natural Numbers. Then we declare these properties to be axioms and define a series of natural numbers as an arbitrary mathematical structure, satisfying the listed axioms. We do not exactly claim that a set defined with the given binary relation "< " satisfies our axioms (such a claim would be quite unrealistic), but we do claim that all such sets (with the given relation) turn out to be mutually isomorphic. In so far as the Series of Natural Numbers will satisfy our axioms (we shall be so chosing the axioms), the Series of Natural Numbers will be one of the pairwise isomorphic structures, satisfying these axioms, and this means that all these mutually isomorphic structures will also be isomorphic to the Series of Natural Numbers .If we succeed in attaining the goal just enunciated, then we should think that we have been able to axiomatically define a series of natural numbers.

Keeping in view the aim that we have put forward , can we remain satisfied with the three properties-axioms listed out ? Evidently, no. All linearly ordered sets satisfy these axioms, among them many are non-isomorphic and, consequently, wittingly non-isomorphic to the Series of Natural Numbers N . For example, the set of all real numbers R with the usual order relation will satisfy the three listed axioms. By comparing N and R we note that N has at least two such properties, which are absent in R .

These are :

4. N contains the smallest element. In symbols:

$$\exists x \forall y (x=y \vee x<y).$$

5. In , N after every element x immediately follows some y. ("Immediately" means there is no third element between x and y.) In symbols :

$$\forall x \exists y (x<y \wedge \neg \exists z (x<z \wedge z<y)).$$

Fig.1

These five axioms significantly narrow down the range of linearly ordered sets satisfying them. The Series of Natural Numbers, as well as the set of real numbers

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{6}{7}, \cdots \qquad (*)$$

(considered in the usual order), comply with these axioms. The existence of this, different from N, structure $(*)$, satisfying the axioms 1-5, still does not constitute an hindrance to the view that these axioms provide an axiomatic definition of a series of natural numbers : since this structure is isomorphic to N (and, thus it can be identified as a series of natural numbers).

Figure 1 provides a graphic depiction of the order in $(*)$ (and in N). However, it is easily noted that the structure (i.e., the set plus the order relation):

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{6}{7}, \cdots 10, 10\frac{1}{2}, 10\frac{2}{3}, 10\frac{3}{4} \cdots \qquad (**)$$

too satisfies the axioms 1-5 . Figure 2 gives a graphic depiction of this ordered structure. In this structure two elements (0 and 10) do not have any immediate predecessors. Let us fixate this situation in the following axiom 6 .

6. If two elements x_1 and x_2 do not have any immediate predecessor, then they are equal. In symbols:

$$\forall x_1\ \forall x_2 \quad \{[\neg \exists y_1 (y_1<x_1 \wedge \neg \exists z_1 (y_1<z_1 \wedge z_1<x_1))] \wedge$$
$$\wedge [\neg \exists y_2 (y_2<x_2 \wedge \neg \exists z_2 (y_2<z_2 \wedge z_2<x_2))] \Rightarrow x_1=x_2 \}.$$

Axiom 6 eliminates the structure $(**)$,but does not eliminate the structure

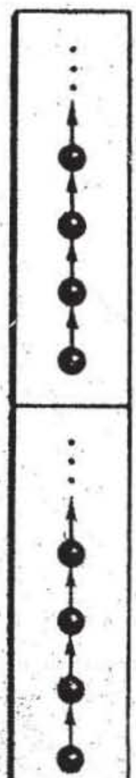


Fig.2

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \cdots$$
$$\cdots 9+\frac{1}{m}, 9+\frac{1}{m-1}, \cdots 9\frac{1}{4},$$
$$9\frac{1}{3}, 9\frac{1}{2}, 10, 10\frac{1}{2}, 10\frac{2}{3}, \cdots, 10+\frac{n-1}{n}. \qquad (***)$$

It is evident that the structure (***) is not isomorphic to a series of natural numbers .

Like the horizon, our goal is moving further and further ... It appears to be unattainable. The following noteworthy fact appears to be the case: however many axioms may we write out, using the logical symbols, the symbol "$<$ " and the variables, covering the elements of the structure being defined — there will always be a model of the totality of listed axioms, non-isomorphic to a series of the natural numbers . In view of the fundamental importance of this fact (signifying the impossibility of axiomatically defining a series of natural numbers by using the means indicated), let us describe it in greater detail.

The following symbols belong to the alphabet of the formalized symbolic language,which we are using for listing the axioms:

1) the symbols of punctuation: the left hand side bracket "(" and the right hand side bracket ")",
2) the logical symbols " $\rceil$ ", "$\wedge$", "$\vee$", "$\Rightarrow$", "$\forall$", "$\exists$", "$=$",
3) the individual variables x, y, z, u, v, w, x_1, y_1, z_1, u_1, v_1, w_1, $\cdots$,
4) the symbol "$<$ ".

Formulas are composed with the help of these letters, according to the natural , but easily formulated syntactical rules. Simplest examples of such formulas are :

$$x<y \vee y<x;$$
$$\forall x\,(x<x);$$
$$\exists x \exists y\,(y<x \Rightarrow y<<x);$$
$$\exists y\,(x<y);$$
$$\forall x \exists y\,(x<y).$$

Now, let us take any set, with any binary relation denoted by "$<$" defined on it (it is not obligatory for the relation to be one of strict order). We shall be calling all such sets with the relation "$<$ ",a *structure with the label* $<$. Thus, a structure with the label $<$ consists of a set (called the *carrier* of the structure) and the relation "$<$" . Let us fix a carrier of the structure as the domain of change for each individual variable. Then every formula becomes either a sentence , as the second, third and the fifth formulas of the just mentioned list are, or sentential forms, like the first and the fourth formulas of the same list. Those formulas which turn into sentences are called *closed* ; we shall be considering only them in future. It is not difficult to notice, that the property of "being closed" in respect of a formula, does not depend on the structure , wherein we are examining the said formula; this property can be defined purely syntactically, according to the external form of the formula.(Closure consists of this that all the variables must be bound by quantifiers.) It is said in respect of a (closed) formula — when considered in a given structure — which becomes a true sentence, that it is *true in the given structure* or that it is *satisfied in the given structure* ; and about the structure it is said, that it is *satisfied by the given formula.*

The structure N — our usual Series of Natural Numbers with the usual order relation — may be singled out from among the structures with the label $<$. we shall call any closed formula,

turning into a true sentence when interpreted in the structure N — an axiom. Thus, however many — finite or infinite number of — axioms may we write out, there will always be such a structure with the label $<$, which, firstly, satisfies all the listed axioms, and is secondly, isomorphic to N.

Thus , it so turns up, that a series of natural numbers can not be defined axiomatically: since to define N axiomatically is to list such a system of axioms, as would define N upto isomorphism (this, in its turn, means, that any two structures satisfying all the listed axioms, are isomorphic).

"But excuse me "— the reader will again object — "the axioms of Peano do define the Series of Natural Numbers upto isomorphism. Peano's system of axioms is categorical, and this signifies, that all the models of it are isomorphic". [Any structure satisfying each of the axioms of a system is called the *model* of that system or list of axioms.] A little patience ! We shall look into the axioms of Peano too.

But now we shall discuss another question. Not merely the order relation "$<$ " , but a numerous set of other relations and operations are defined on the Series of Natural Numbers. Among them there are the two-place (or binary) relation of divisibility of two numbers; the three-place (or ternary) relation "$x + y = z$ " ; the one- place (or singular) relation of "being a prime number" (let us recall that we treat properties as one-place relations), [here we are using the etymologically more correct term "singular", following W.V.O. Quine, instead of the now widely used term "unary"; see: 7, note 29]; the two-place operation of addition; the two-place operation of multiplication; the two-place operation of involution ($0^0 = 1$); the one-place operation of following immediately (as is customary, we shall indicate it with the prime symbol, such that, $0' = 1$; $13' = 14$); the constants $0, 1, 2, 3, 4, \cdots$ (let us recall that we treat the constants as zero-place operations) ; the four-place operation $[\log_{u+1} (z\,! + y^{x \cdot z + u})]$ (here, as usual, through $[a]$ we indicate the integral part of the number a); and many others. We have adduced only a few examples, and altogether a countless number of operations and relations are defined on N. In order to define the concepts of a structure isomorphic to N, we must at first separate out some operations and relations from among them (theoretically it is possible to take into consideration all of them) and examine the isomorphism, namely, in respect of these isolated operations and relations. Indeed , that is why there does not exist the concept of a series of natural numbers as such, but only the concept of a series of natural numbers in respect of a given list of operations and relations.Earlier we have examined the concept of a series of natural numbers in respect of a list, wherein there were no operations at all, and there was but one relation — the relation of "lesser than".

In the context of our investigations, the operations and relations singled out in the set are called — *labelled*, and the list of such operations and relations — a *label*. To be more precise, not the list of these operations and relations themselves, but the list of their names, is called a *label*, (this distinction is very important in itself) but for our purposes it is not quite essential, and it would be easier for us not to notice it. A set with some singled out operations and relations, forming the list σ, is called the *(mathematical)structure with the label* σ. Now we can say that any series of natural numbers is a structure with this or that label σ. That is why, we should be speaking not about a series of natural numbers in general, but about a series of natural numbers with the label σ. So far we have examined the case, when

$$\sigma = \{<\}.$$

Perhaps the poverty of this label is the reason behind the failure of our attempt at axiomatically defining a series of natural numbers? Let us broaden the label and see what happens. First of all, let us add to "<", the constant "0" (for denoting the smallest element in respect of the order "<") and the prime symbol " ′ " to indicate the operation of immediate succession. In the Series of Natural Numbers N, these objects come under the following axioms (properties) 7 and 8 (compare the properties 4 and 5, which follow from the properties 7 and 8) :

7. $\forall y\,(0 = y \vee 0 < y)$;

8. $\forall x\,(x < x' \wedge \neg\exists z\,(x < z \wedge z < x'))$.

Any series of natural numbers with the label $\{0, {}', <\}$ is by definition isomorphic to N, since isomorphism is considered in respect of $\{0, {}', <\}$. That is why any such series of natural numbers consists of the elements 0, $0'$, $\cdots$, ordered as follows:

$$0 < 0' < 0'' < 0''' < \cdots$$

R e m a r k s. We must be aware of the fact that every series of natural numbers has its own 0, own ′ and own <, i.e., own element indicated by 0, own operation signified through " ′ " and, own relation denoted by "<". Strictly speaking, for every series of natural numbers we shall devise its own symbols for these objects — for example, if we are considering a series of natural numbers M, then it is necessary to add this letter " M " as an index to the symbols "0", " ′ ", and "<". This strictness provides some convenience . However, the absence of strictness also gives rise to some convenience .In the given case, the convenience from lack of strictness is considered to be greater, and that is why one and the same "0" is used to signify various elements (but in every series of natural numbers it denotes one and only one element; in particular, the cardinality of the empty set in the Series of Natural Numbers). Analogously for " ′ " and "<". These remarks are valid not only for a series of natural numbers, but also for any structure with the label $\{0, {}', <\}$, not bindingly isomorphic to N.

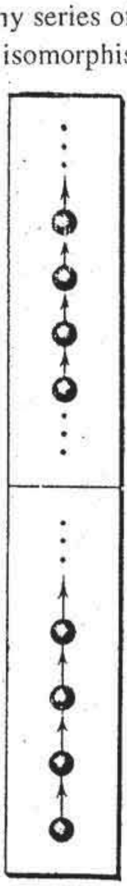


Fig.3

Now we shall see, how an arbitrary structure with the label $\{0, {}', <\}$, subordinated to the axioms 1-8, looks (axioms 4 and 5 follow from the axioms 7 and 8, but that is not a great calamity). Evidently, it is a linearly ordered set, wherein 0 is the least element, $0'$ is the element immediately following 0 (such that there is nothing between 0 and $0'$). $0''$ is the element that immediately follows $0'$ etc. All these elements — 0, $0'$, $0''$, $0'''$ — form the initial cut of our structure. This initial cut is called the *standard* part of the structure, and the remaining part (it may even be empty) is called *non-standard*. The standard part is isomorphic to N. Had there been nothing but this standard part in any structure with the label $\{0, {}', <\}$, subordinate to the axioms 1-8 , then we would have attained our goal:

axioms 1-8 in their totalilty would have given us the axiomatic definition of a series of natural numbers we are looking for, to be more precise — they would have given us the axiomatic definition of a series of natural numbers having the label $\{0, ', <\}$.

This, however, is not the case. The structure graphically depicted in figure 3 — say, the one like (***), where

$$0' = \frac{1}{2},\ \left(\frac{1}{2}\right)' = \frac{2}{3},\ \left(9\frac{1}{4}\right)' = 9\frac{1}{3}\ \text{etc.},$$

satisfies the axioms 1-8, but it is not isomorphic to N : it contains a non-empty non-standard part (in figure 3 this non-standard part has been depicted in the upper rectangle, in (***) this non-standard part consists of the elements of the form $9 + \frac{1}{m}$ and $10 + \frac{n-1}{n}$). What is more, it turns out that no system of axioms can give us a series of natural numbers with the label $\{0, ', <\}$, since the structure depicted in figure 3 will always be a model for such axioms.

Fig. 4

Perhaps it is still a case of poverty of the label ? What will happen if we add addition and multiplication and consider a series of natural numbers not having the label $\{0, ', <\}$, but one with the label $\{0, ', <, +, \cdot\}$?

Would it be possible to make a list of axioms for such a richer label, which would define the concept of a series of natural numbers having this label — i.e., from *a l l* the structures with this label, would it be possible to single out those structures which are isomorphic to N in respect of $0, ', <, +, \cdot$? It appears that no, it can not be done. Whatever be the totality of axioms — finite or infinite — made out by us, for this totality there will always exist a structure [with the label $\{0, ', <, +, \cdot\}$], non-isomorphic to N. [When we speak of axioms, we have in view a symbolic language, like the one described above for the label $\{<\}$; only now, together with "<", the alphabet contains "0", "′", "+" and "·".]

What is more, w h a t e v e r b e t h e l a b e l chosen and w h a t e v e r b e t h e s y s t e m o f a x i o m s chosen for this label, there will always exist a model of this system of axioms, not isomorphic to N. Such, non-isomorphic to N, models are called non-standard, and the axioms listing the properties of a series of natural numbers (especially, when + and · enter into the label), are called the *axioms of arithmetic*. That is why, we may re-state what has been stated above as follows : t h e r e e x i s t s a n o n - s t a n d a r d m o d e l f o r a n y s y s t e m o f a x i o m s o f a r i t h m e t i c .

if the axioms 1-8 or something equivalent to them enter into our axioms, then it is possible to single out the standard part $0, 0', 0'', \cdots$ in any model; in this case, non-standardness of a model signifies the non-emptiness of the non-standard part. This non- standard part may turn out be more complexly constructed , than the case depicted in figure 3. From the standpoint of order, the non-standard part depicted in figure 3 is similar to the set Z of all whole numbers. In the case of natural axioms for a label that includes the operation of addition, the non-standard part of any denumerable structure (i.e., of one containing denumerable number of elements), satisfying these axioms, assumes a look, which we have (not very successfully) attempted to depict in figure 4. In this diagram we have attempted to somehow depict the following idea : a very large (denumerably infinite) number of examples of the sets of whole numbers Z are taken up, and these examples are arranged like the set of all rational numbers Q.

Thus, it is not possible to produce a system of axioms, defining the concept of a series of natural numbers (with any label whatsoever). To our knowledge, a more detailed interpretation of this statement is as follows : whatever operations and relations defined on N, may be chosen, there can not be such a system of axioms, all models of which are isomorphic to N, in respect of these operations and relations.

And now we shall answer the question: "But what about the axioms of Peano ?".

With inessential changes, the classical axioms of Peano are us under. Here, the label $\{0, '\}$ is being considered .Three axioms have been formulated:

I. $\neg \; \exists x \; (x' = 0)$;

II. $\forall x \; \forall y \, (x' = y' \Rightarrow x = y)$;

III. The Axiom of Induction.

We have, till now, only named the third axiom of induction, but have not described it. Now we describe it:

$\forall P \{[P(0) \wedge \forall x \, (P(x) \Rightarrow P(x'))] \Rightarrow \forall x \; P(x)\}$.

When we look at the axiom of induction we notice that together with the usual individual variable it contains yet another variable P. we shall explain the meaning of this variable . First of all, let us recall that the semantics of a formula (i.e., the meaning attached to it) emerges only after the mathematical structure corresponding to the label is produced. In particular, in order to find out the meaning of the axioms of Peano (of the formulas I-III), we must produce some structure with the label $\{0, \; \}$, i.e., a set with a singled out element, indicated by "0 " and a singled out one-place operation, indicated by " ′ ". Then the domain of change of the variabale x is at once defined (like that of any individual variable) — it is the set of all elements of the structure under consideration. What is the domain of change of the variable P like ?

The variable P is of a special type, we have not met with the like of it hitherto in our enunciation. Its domain of change consists of all the possible properties (= one-place relations), defined on the structure under consideration, i.e., the properties of the elements of this structure.

The concept of property is a primary concept; it is grasped from examples. The property of being even is defined on the natural numbers — every number may be either even, or odd. It is

inessential that there are even as well as odd numbers; we may construct a situation, where all the numbers are even; what is important is that it should make sense to ask in respect of every number, whether it is even or odd. The property of being green is not defined on a series of natural numbers ; for a number the talk of "being green" is pointless. The Series of Natural Numbers possesses the properties formulated above, as a whole. Relations, too, may possess properties: example — the relation of transitivity. But at the given moment we are interested only in the properties of the elements of the structure under consideration (for which the axioms of Peano are satisfied). It is these properties, namely, that can provide the values of the variable P.

The fact that an element a possesses the property Q is described as $Q(a)$. If a property Q is defined on the elements of some set M, then it is possible to introduce for consideration a sub-set of this set, K — consisting of those and only those elements of M, which possess the property Q

$$x \in K \Leftrightarrow Q(x) . \qquad (+)$$

Conversely, for every sub-set K it is possible to introduce the property Q : of "being an element of K", and again the correspondence (+) will be satisfied.

Thus, a property and a sub-set are almost the same : "the language of properties " and " the language of sub-sets" are trivially inter-translatable. For example, the axiom of induction would look as follows in the language of sub-sets:

$$\forall P \{ [0 \in P \wedge \forall x (x \in P \Rightarrow x' \in P)] \Rightarrow \forall x (x \in P) \}.$$

Thus, in the axiom of induction, the domain of change of P is the totality of all the properties defined on the structure under consideration. We shall now see, how this axiom is utilised to ascertain the fact that a structure satisfying the axioms of Peano, is isomorphic to N. Thus, let a structure with the label $\{ 0 , ' \}$ satisfy the axioms I-III. Axioms I-II ensure the presence of a standard part $\{ 0 , 0' , 0'' , 0''' \cdots \}$ in the structure. Now let us apply the axiom of induction, having taken as a value of the variable P, the property P_0 of the elements of the structure : "to belong to the standard part". The axiom says that something is true for all P_0 , in particular for this P_0 . Thus, it occurs that

$$[P_0 (0) \wedge \forall x (P_0 (x) \Rightarrow P_0 (x')] \Rightarrow \forall x \, P_0(x).$$

The premise enclosed within square brackets is evidently true (0 belongs to the standard part, and if x belongs to the standard part, then x' too belongs to it); that is why $\forall x \, P_0(x)$, i.e., all x (all elements of the structure !) belong to the standard part. We have already noted that the standard part is isomorphic to N. This concludes the proof of this that the structure under consideration is isomorphic to N.

Thus, any structure satisfying the axioms of Peano, is isomorphic to N, and consequently, these axioms define the concept of a series of natural numbers of the label $\{ 0 , ' \}$. Apparently, this situation contradicts our repeated announcement to the effect that it is not possible to formulate a system of axioms with such properties.

There are no contradictions here, and here's the reason why. Earlier we were speaking only about the properties of the Series of Natural Numbers, which could be expressed through a definite linguistic means — in other words, we were talking about some axioms written in a definite language. This language contained only one type of variables — the individual variables $x, y, z, \cdots$. The essence of these individual variables lies in the fact that upon being interpreted on any structure, each one of these variables gets one and the same set as its domain of change — it is the set of all elements of the structure under consideration. Another kind of variable — the variable P — takes part in the axiom of induction. Its values are not the elements of the structure under consideration, but a property of these elements (in other words — the one-place predicates defined on these elements, whence the variable P itself is called a *predicate*, to be more preceise — a *predicate variable of valency* 1). Thus, the axiom of induction is a formula of a n o t h e r, e x t e n d e d l a n g u a g e; this language is more extended than the narrow language so far considered. (Narrow because it contains only individual variables). And when we said that there is no system of axioms, which fully characterizes a series of natural numbers, then we had this earlier, narrow language in view.

Of course, an explanation has been provided, but it hardly satisfies any one. What if it is not possible to write out a system of axioms for a series of natural numbers, in some language? It is, as they say, "not a fact from the biography of a series of natural numbers, but rather one from the biography of that language". Simply put, a narrow language is bad, and look, now we have found a good, extended language, in which it is possible to write the adequate axioms for a series of natural numbers.

Everything, however, is not that simple. Crudely speaking, the situation is just the opposite: a narrow language is "good", an extended one — "bad".

Let us attempt an explanation of the situation. We shall begin with terminology.

The formulas, wherein all the variables are individual are called *elementary* formulas, and the language that permits of only the elementary formulas, is called an *elementary* language. In the given context, the synonym for the term "elementary" is the term "1st order" or "first order". All the axioms considered above, save the axiom of induction (i.e., the axioms 1-8 and I-II) were elementary axioms, i.e., elementary formulas. There exists no (neither finite, nor infinite, and besides of any label) system of elementary axioms, which would satisfy the Series of Natural Numbers N and, all the models of which would be isomorphic to N.

There are non-elementary formulas, but they belong to a non-elementary language. Variables of a more complex nature are permitted in this language — predicate variables of valency 1, properties (= one-place relations) serve as their values; predicate variables of valency 2, binary (= two-place) relations serve as their values etc., and also, functional variables (any one-place operation, like, say, "to follow", may serve as the value of a functional variable of valency 1, and any two-place operation, like, say, addition, may serve as the value of a functional variable of valency 2). The axiom of induction is an example of a non-elementary formula. A more precise non-elementary language, having the possibilities just described, is called a *2nd order language*: this means, that it admits of variables covering relations and operations (what sort of relations and operations, that must be defined on the elements of the structure), but does not consider more complex variables, as the values of which may serve, say, the properties

of operations or operations on relations (or the properties of relations such as "transitivity"). The axiom of induction serves as an example of non-elementary formula of a 2nd order language (or simply, as an example of a 2nd order formula).

A second order language is the simplest of all non-elementary languages.

One would think — and the presence of the axioms of Peano somehow confirms this — that it is possible to have a system of non-elementary axioms of 2nd order (i.e., axioms, written in the form of formulas of this non-elementary language), defining the concept of a series of natural numbers, in the following precise sense:

1) N is a model of this system; and

2) any model of this system is isomorphic to N.

However , here arises an unexpected , but quite fundamental semantic (one may even say, epistemological) difficulty. The fact is this, that already in the case of a 2nd order language (not to speak of the more complex non-elementary languages), the very concept of a model loses its essential clarity. The following example, connected with the so-called problem of the continuum, illustrates this situation.

It is well known that the quantity of elements of any set is called the *cardinal number* or the *cardinality* of that set. The concept of a cardinal number or cardinality is a generalisation of the concept of natural numbers, in so far as the natural numbers are the cardinalities of finite sets. From among the infinite cardinalities the following two are singled out: the cardinality of the set of all natural numbers and the cardinality of the set of all real numbers. The first is indicated by $\aleph_0$ (read "aleph- null") and is called the denumerably-infinite (or infinitely denumerable) cardinality; the second is indicated by c (small gothic "c") and is called the cardianality of the continuum. Evidently, $\aleph_0 < c$. The famous *continuum problem* consists of explaining whether or not there exists any intermediate cardinality, i.e., a cardinality satisfying the inequality

$$\aleph_0 < m < c.$$

The famous continuum hypothesis consists of this, that such a cardinality does not exist. (On the strength of the results of K.Gödel and P. Cohen) it is well-known that it is neither possible to prove, nor disprove the continuum hypothesis. While speaking of "proving" and "disproving", we have in view all the conceivable means permitted in modern mathematics. Thereby, the question of the very meaning of the continuum hypothesis remains unsettled. Indeed, the meaning of such a statement is taken to be vague — its truth or falsity can not be determined in any way. This extraordinary situation is radically different from such situations, often to be met with, when we simply do not know something (though we understand the question very well). [And the security of clarity in understanding a question lies in the clarity of understanding the possible answers.]

It appears that we may write out a formula of the 2nd order, which then and only then has a model (i.e., such a structure , for which it becomes true), when the continuum hypothesis is true.

It is also possible to write such a formula of the 2nd order, the existence of a model of which is equivalent , on the contrary, to the existence of an intermediate cardinality, i.e., to the truth

of negation of the continuum hypothesis. [*A note for the specialists.* Examples of such formulas, known to the present author , contain predicate symbols of valency 2. However, if in the axiom of induction we change the prime symbol " ′ " into a predicate symbol, then this axiom too would contain a predicate symbol of valency 2.] Thus, for the formulas of 2nd order, the question of existence of their models may turn out to be as hazy as the continuum hypothesis itself.

That semantically so vague a language would be able to serve as a satisfactory means for axiomatically defining something — in particular , a series of natural numbers — appears to be doubtful.

And indeed, if we analyse the use of the axiom of induction in the process of proving that any model of the axioms I-III is isomorphic to N, then we shall see, that here , in essence, we are using that very concept of a natural number, which we still only intend to define axiomatically. Our property P_0 signifies "to have the form 0 ″· · · ′". The dots in the expression "0 ″· · · ′" are only a substitute for the general notion of a natural number. And it is not possible to depict the property P_0 without an *a priori* notion of the natural numbers or without substituting that notion by dots or by the expression "etc.".

5. Is it possible to prove that Fermat's Great Theorem can neither be proved nor disproved ?

We have mentioned the continuum problem at the end of our last reflection. It is one of the major problems agitating the intellect of mathematicians. In the famous report entitled "Mathematical Problems", read by the great Hilbert in the year 1902, at the International Congress of Mathematicians in Paris, it was mentioned as the first problem. We have already noted that the continuum problem turned out to be unsolvable : it is neither possible to prove, nor disprove the continuum hypothesis.

While enumerating 23 basic problems of mathematics, Hilbert did not mention the problem of proving (or disproving) the Great Theorem of Fermat. Evidently, Hilbert did not consider it to be important enough. At the same time there is no doubt that it is the most famous among the unsolved problems of mathematics. And besides, unfortunately, it is unique among the unsolved problems, known to the wide mass of non-mathematicians. We wrote "unfortunately", since professional mathematicians spend an appreciable percentage of their time, studying and refuting the essays of the Fermatists — the name given to people, who do not have the necessary mathematical preparation, but who think that it is, namely, they who have proved Fermat's theorem.

Strictly speaking, Fermat's theorem should not be called a theorem. The "Matematicheskaya Entsiklopedia" [22] defines a theorem as a "mathematical statement, the truth of which has been established through proof".

A proof has not yet been found for Fermat's "theorem". [However, not every one supports this point of view. Thus, Viktoly Invanovich Budkin states in p.45 of his book "Methodology of Cognition of "Truth". A proof of Fermat's Great Theorem" (Yaroslavl: Upper Volga Publications, 1975. pp. 48): "13 generations have passed, and yet Fermat's Great Theorem still remains unproved. Only in the present work, a complete proof of the theorem is being given in its general form".]

What is more, the same "Matematicheskaya Entsiklopedia" contains an article entitled "Fermat's Theorem", in its 5th volume (the same volume that contains the aforementioned definition of a theorem). We too shall be using this generally accepted but imprecise term — though we admit that it would have been more correct to speak of the hypothesis of Fermat.

Many factors contributed to the popularity of Fermat's theorem among the non-professionals. These include : 1) the authority of its author : it has been stated by one of the originators of the theory of numbers — the famous French mathematician Pierre de Fermat; 2) the respect due to age: it was stated at around 1630; 3) the romantic circumstances of its formulation: Fermat wrote it down on the margins of the 1621 edition of the "Arithmetic" of Diophantus [the eighth problem of the second book of the "Arithmetic " of Diophantus reads — " To decompose a given square into two squares "; Fermat made the following comment on this problem — "On the contrary, it is not possible to decompose any cube into two cubes, any biquadratic into two biquadratics, and in general no power greater than the square, into two powers having the same index. I have discovered a truly wounderful proof of this, but these margins are too narrow for it"; this proof could not be found among the papers of Fermat]; 4) the setting up of a prize of 100,000 German marks for providing a proof of Fermat's theorem, in 1908 by Wolfskel (naturally, the "pleasant" fact of the institution of a big prize became much more well-known, than the "unpleasant" fact of its complete depreciation as a result of the post-first-world-war inflation); and 5) the simplicity of its formulation.

Of course, the first four factors could not have been effective in tandem, had not the formulation of Fermat's theorem been so popular. It is as follows : W h a t e v e r b e t h e i n t e g e r n, g r e a t e r t h a n 2, t h e e q u a t i o n

$$x^n + y^n = z^n$$

h a s n o p o s i t i v e i n t e g r a l s o l u t i o n.

We see, that the equation present in the formulation of Fermat's theorem may be viewed as an equation with three unknowns — x, y and z. In sofar as n may assume any of the values 3 , 4 , 5 , 6 etc., here, in fact, we have an infinite series of equations, and it has been stated that none of them has solutions in such integral x, y and z that $x>0$, $y>0$ and $z>0$. From the point of view of logic it would be more natural to consider the equation

$$x^n + y^n = z^n$$

as an equation with four unknowns x, y, z, and n. Then, Fermat's theorem would state that this equation has no integral solutions for $n>2$, $x>0$, $y>0$ and $z>0$.

The search continues for the proof(s) of Fermat's theorem*. Theoretically speaking , the search for its refutation could also have continued, but that is not happening. The situation with the hypothesis called "Fermat's theorem", is significantly different from the situation in respect of the continuum hypothesis : we know, that for the continuum hypothesis it has been proved that it can neither be proved, nor disproved (to be more precise , in 1939 Gödel showed that it can not be disproved , and in 1963 Cohen showed that it cannot be proved). For the hypothesis (theorem) of Fermat, such a proof — the proof that it is neither possible to prove, nor disprove it — does not exist. The question arises : whether this proof does not exist s o f a r (with the

* A more recent example : Andrew Wiles' (June, 1993) claim to have provéd the Taniyama-Weil conjecture, entailing the solution of Fermat's problem. Experts are now examining this proof.—Ed.

hope that it will be obtained in future) or is it in principle impossible ? Had this proof been obtained, it would, undoubtedly, have been of great use for mathematics , as it would have closed, once and for all, the floodgates in the face of the flow of ignorant attempts to prove the theorem of Fermat. Unfortunately, such a proof is not possible. It is true, that there remains a theoretical possibility of proving that Fermat's theorem cannot be proved. The appearance of such a proof too would have closed the aforementioned floodgates — but then, perhaps, there would have emerged a flow of attempts to disprove Fermat's theorem (for example, by way of producing, in an oblique manner, four astronomically large numbers n , x , y , z for which the required equation would be practically unverifiable).

Thus, we are assuming, that (a) there exists a proof to the effect , that Fermat's theorem can not be proved; (b) there exists a proof to the effect, that Fermat's theorem cannot be disproved.

Now, our aim is to show, that (a) and (b) are incompatible, i.e., it is not possible for these two statements to be true at the same time. In fact we find that (b) is incompatible even with the weaker-than-(a)-statement (a_1) : "Fermat's theorem cannot be proved ". We shall show, namely, that from (b) the existence of a proof of Fermat's theorem follows and thereby (a_1) is negated.

First, some preliminary remarks. Let us agree to call, any four natural numbers n , x , y , z such that $n>2$, $x>0$, $y>0$, $z>0$ and $x^n+y^n=z^n$, the *Fermat four*. Fermat's theorem states that the *Fermat four* do not exist. Disproving any theorem is to prove its negation. [Thus] disproving Fermat's theorm would mean proving that the *Fermat four* exist. [As before we are using inexact terms and identifying the word "theorem" with the word "statement" and not with the expression "proved statement".]

Lemma 1. If it cannot be proved that the Fermat four exist, then they do not exist.

Remarks. Let A be a statement. There is no reason for thinking, that if it cannot be proved that A , then A is not true. However — and herein lies the essence of the lemma — that is the case, as soon as A is the statement that "the Fermat four exist".

Proof of lemma 1. We shall proceed from the opposite. Indeed, we shall assume that the Fermat four exist. Let us write out any of them — let it be the four natural numbers a , b , c , d. Let us verify that they really are the Fermat four, i.e. let us verify that the inequalities $a>2$, $b>0$, $c>0$, $d>0$, and the equality $b^a + c^a = d^a$, are satisfied. Presence of the four numbers a , b , c , d together with the indicated verification constitutes the **existence proof for the Fermat four.**

Lemma 2. If Fermat's theorem cannot be disproved, then fermat's theorem is true.

Remark. There is no reason why this must be true of any theorem.

Proof of Lemma 2. Lemma 2 is simply a reformulation of Lemma 1. "To disprove Fermat's theorem" is "to prove that the Fermat four exist", and to assert that "Fermat's theorem is true " is to say that "the Fermat four do not exist".

The lemma 2, which we proved, has the structure " if P then Q". That is why, if P has a proof, then Q too has a proof (the proof of Q consists of joining the proof of the lemma with the proof of P). That is why, we have the following

Corollary of lemma 2. If there exists a proof to the effect, that Fermat's theorem cannot be disproved, then there also exists a proof to the effect, that Fermat's theorem is true, i.e.,simply put, a proof of Fermat's theorem.

In view of the importance of this corollary, let us Formulate it once more : *if there exists a proof to the effect that Fermat's theorem cannot be disproved , then Fermat's theorem can be proved.* Thus, if (b), then Fermat's theorem can be proved, and this is the promised negation of the statement (a_1).

The contradiction thus obtained concludes our arguments to the effect , that (a_1) and (b), and even more so (a). and (b), are incompatible.

Here arises the following natural question: but why these arguments cannot be repeated for the continuum hypothesis ? Indeed, Fermat's hypothesis (theorem) states that the Fermat four do not exist, and the continuum hypothesis states that there exists no set having a cardinality intermediate to $\aleph_0$ and c. Now let us replace the Fermat four by a set of intermediate cardinality, and Fermat's theorem — by the continuum hypothesis and, let us once more adduce the arguments just adduced. We are bound to stumble somewhere , as the statements (a′) and (b′), obtained from (a) and (b), by substituting the words "continuum hypothesis " for the words "Fermat's theorem", are both true. And where shall we stumble ? Here's where — in the proof of lemma 1(evidently, not in the initial formulation, but with the replacement of the words "Fermat four" by the words "set of intermediate cardinality"). The aforementioned proof of lemma 1 was based upon the following idea: that it is in fact possible to produce the four numbers a , b , c , d and to assure oneself that they are the Fermat four. But what does it mean to produce a set ? Objections may be raised, that strictly speaking, we do not produce the numbers as quantitative categories, it is not possible to produce them, we can only write their names (for example, in the form of zero with the prime symbols or in the form of decimal notation). However, the fact remains, that each natural number has a name, but such is not the case with the sets : t h e r e a r e m o r e s e t s, t h a n t h e r e a r e n a m e s (if we understand the latter as finite combinations of the symbols of some alphabet). But even if we limit ourselves to the sets having names, and produce in place of the sets — these names, there remains, all the same, a major difficulty : how to verify that the set produced has an intermediate cardinality ? The verification to the effect that the four numbers are the Fermat four, is not complicated in principle (if we digress from the number of steps and the necessary space) : we just have to put the numbers in the equation and compare the left hand side with the right hand side. But there exists no way of determining the cardinality of the produced set or of determining whether or not this cardinality satisfies the inequality $\aleph_0 < x < c$.

The theme under consideration is most intimately connected with the famous incompleteness theorem of Gödel. This theorem states that *whatever be the proposed concept of a formal proof, there would be such a statement about the natural numbers, that neither it itself , nor its negation may be formally proved within the frame-work of the proposed concept.* We begin with the self-evidence of the fact that it is possible to define formal proof variously. These definitions differ from one another in respect of the collection of permissible axioms and the rules of deduction. It is possible to have such notions about formal proof, wherein there is no use at all, either of the axioms or of the rules of deduction. Briefly speaking, the approaches to the concept of formal proof may be very very different. But all these approaches have a fundamental generality expressed in the following principles:

1) every formal proof is a text — i.e., a finite chain of symbols, chosen from some alphabet;

2) in respect of every text, composed of the letters of an alphabet under consideration, it is possible to algorithmically identify, whether or not it is a formal proof , and if yes, then what, namely, does it state;

3) only true statements can have formal proofs.

On the strength of the third principle, the production of a formal proof of some statement guarantees its truth and, consequently, may be considered to be its proof. The converse, of course, is not being proposed: it is not being proposed that every true or even essentially provable statement has a formal proof, in terms of a pre-given concept of formal proof.

An analysis of Gödel's incompleteness theorem shows, that the statement therein discussed always has the form $\exists x\ u(x)$, where u is some property of the natural number x. This property depends upon the concept of formal proof under consideration, but it is always algorithmically verifiable (just as it is possible to algorithmically verify the property of "being the Fermat four", in respect of four given numbers); [being algorithmically, verifiable means — there exists an algorithm, which verifies for any c, whether or not $u(c)$ is true]. Thus Gödel's theorem states that neither $\exists x\ u(x)$, nor $\neg\ \exists(x)\ u(x)$ has a formal proof.

Let us make our demands about the notion of formal proof even more strict. Let us demand, namely, that as soon as the statement $\exists x\ u$ turns out to be true for some algorithmically verifiable property u, then and there this statement $\exists x\ u$ possesses a formal proof. This demand is quite natural : it is realized upon formalization of the following steps indicated above: 1) the production of some c ; 2) the verification that this c satisfies the property u ; here it is essential, that c may in fact be produced and, that $u(c)$ may in fact be verified.

Our demand follows from two even more natural demands:

1) if the (algorithmically) verifiable property u is valid for a number c, then $u(c)$ has a formal proof;

2) for any property u whatsoever, if for some c the statement $u(c)$ has a formal proof, then the statement $\exists x\ u(x)$ also has a formal proof.

Now, with the help of arguments analogous to those used in connection with Fermat's theorem, we arrive at the following conclusion: if neither the statement $\exists(x)\ u(x)$, nor its negation $\neg\exists x\ u(x)$ has a formal proof, then from this information alone about the given situation it is possible to find out which of these two statements is true: namely, it is true that $\neg\exists x\ u(x)$.

Indeed, had it been true that $\exists x\ u(x)$, then this statement would have had a formal proof; perhaps it is not true that $\exists x\ u(x)$, and it is true that $\neg\exists x\ u(x)$ [the words "true" and "correct" are synonyms, but the word "provable" has another meaning (even other meanings)].

Let us appreciate the paradoxicality of the situation once more: *from the sole fact that neither A, nor not-A has a formal prof, it is possible to conclude ,which of these two sentences is in fact true.*

6. What is a proof ?

When we read a book written some fifty years ago, then the arguments found there, appear to us to be largely bereft of logical rigour.

Jules Henri Poincare, 1908.

(Nauka i metod , kn. II, gl. 2, § 4; [2,s.356]).

In the previous reflection we came across the terms "proof" and "formal proof". It is sometimes thought that a formal proof is a proof that is formal. We would prefer to take a different look at these concepts.

A formal proof is a mathematical object, like, say , a matrix or a triangle. It is a finite chain of the symbols of some pre-fixated alphabet, i.e., as they say in mathematics, a *word* according to this alphabet. In the given instance, when we speak of a "symbol", we do not have in view the meaningful, contentful side, but only the external, graphic aspect of it is taken into consideration. To stress this circumstance, in mathematics, when the external, graphic aspect is had in view, then they speak not about a "symbol" [or "sign"], but about a "letter"'. Usually, the letters of the alphabets of various (Russian, Latin etc.) languages, numerals and the punctuation marks are considered to be letters. It would be reasonable, to consider the gaps among the words to be letters too (words in the ordinary, and not in the mathematical sense); we may devise some special symbol for it, for example # . This creates a possibility for viewing a text, i.e., a sequence of words, also as a word (in the exact mathematical sense indicated above). Thus, a formal proof is first of all a word in some alphabet — in the alphabet of formal proofs. It is clear, that this does not exhaust the concept of formal proof in the least: we simply wanted to stress that the concept of formal proof belongs to the class of words — just as the concept of triangle belongs to the class of geometrical figures.

What sort of words may be considered to be formal proofs ? That is the theme of a special discourse; it is beyond the cycle of topics we wish to discuss here. We stress here that it is possible to give various definitions of the concept of formal proof, each of which would lead us to its own set of formal proofs. In the previous reflection we have enunciated some general postulates, to which any reasonable definition should be subordinated. It must be mentioned, however, that sometimes yet another step is taken in the side of generality and it is not demanded beforehand , that only true statements should have formal proofs, thereby the concept of formal proof is fully separated from the concept of truth. And afterwards this discarded requirement is introduced in the form of a supplementary property (which a formal proof, generally speaking, may not have): namely, if all statements having a formal proof are true, then the set of formal proofs is called *semantically non–contradictory.* A more precise general notion of formal proof is enunciated with the help of the concept of deduction; see, for example, [21].

We would like to stress once more, that not the contentfully understood statements themselves, but only their representations (i.e., again words) may (or may not) have formal proofs , in some precisely given logico-mathematical language.

The definition of the concept of formal proof — perhaps, it would be better to say : the definition of the sets of formal proofs — within the broad limits (conditioned by the general limiting properties of the sets of formal proofs, indicated above), happens to be arbitrary. Here, we have in view that " juridical" arbitrariness, which distinguishes mathematical definitions in

general. For example, we have the "juridical" right to arbitrarily define a class of functions and to call it, " as we wish ", say — continuous.

It is another matter, that any reasonable mathematical definition usually claims to correspond to some intuitive notions, to reflect them. Legitimacy of a definition still does not signify its reasonableness. Thus, the mathematical concept of a continuous curve reflects (with some sort of precision) our intuitive, contentful notions of the trajectory of a moving point. Analogously, the concept of formal proof reflects the intuitive notions of a contentful proof.

It may be said that the concept of formal proof is a mathematical model of the concept of proof — in the same sense , in which the concept of continuous curve is a mathematical model of the concept of trajectory.

It still remains to be explained : what a proof is . We have indicated at the very beginning of the present cycle of reflections, that it would be incorrect to assume that in mathematics everything is proved ; however, there is no doubt about the fact that the concept of proof plays a central role in mathematics . " From the time of the Greeks, to say 'mathematics' is to say 'proof' " — thus begins Nikolai Bourbaki his " Éléments de mathématique " [6, p. 23]. At the same time we have noted that the concept of proof does not belong to mathematics (only its mathematical model — the formal proof, belongs to mathematics). It belongs to logic , to linguistics and, above all — it belongs to psychology .

Thus, one of the most important terms in mathematics, the term "proof", has no precise definition. An approximate definition of it is as follows : *a proof is a persuasive argument, which so persuades us that with the help of it we become capable of persuading others* [12].

Having grasped a proof, we become aggressive to a certain extent, ready to convince others with the help of the arguments which we have grasped . If we are not so ready , then it signifies that we are yet to grasp the presented argument as a proof, and even if we have given it the recognition of a proof, then we have done so simply to brush aside something.

We find that the concepts present in our definition of a proof are either logico-linguistic ("argument"), or psychological ("persuasive strength ", " readiness") in nature. This fully meets the essence of the matter : the very notion of proof is inseparably connected with the linguistic means and with the social psychology of human society. And both of them change in the course of history . Linguistic formulations of proofs change. Our notions of persuation change.

The notion of persuation depends not only on the epoch, but also on the social surroundings. Unfortunately, I am unable to recollect now, where I read a passage on the following theme. The Cardinals of the time of Galileo, were quite intelligent, some of them saw with their own eyes the mountains on the moon through Galileo's telescope, and could follow the logic of Galileo's arguments. However, for them, their own views, based on an a *priori* dogma, were more convincing than any experiment and any logic . [In an article by S.P. Bozhich [13], we find an interesting analysis of how an *a priori*, predetermined notion about the ways of proving things prevents the recognition of certain facts.]

The notion about the persuasiveness of this or that argument depends on many factors. Revealing these factors happens to be an important task of logic and psychology. For example, the division of concepts (to be more precise, of terms) into sensible and senseless ones, happens to be one of these factors. The concepts of phlogiston and thermogen were considered to be

meaningful in the 18th century, but they are now considered to be meaningless. Einstein discovered that the concept of simultaneity of two events is meaningless , as an objective concept independent of the observer (to be more precise, he discovered that simultaneity is not a two-place relation between two events, but a three-place relation involving the 1st event, the 2nd event, and the observer, as its terms). On the other hand, such an "evidently meaning-less concept" as the infinitesimal number, now fits into an exact meaning within the framework of a new branch of mathematics — the so-called non-standard analysis. With the changes in the notions about meaningfulness or meaninglessness of concepts, the notion about the very essence of scientific truth also changes. The notion of evidence too changes. Once everyone knew that " higher forces " unleash storms, now everybody knows that storms are caused by atmospheric electricity. In the case of inert gases, their property of not taking part in any chemical union was so evident, that this property was fixated in the very name "inert"; when , in 1962, these gases were for the first time found to take part in chemical unions, then , apparently, the chemists were not ashamed, rather — they happily stated that "for explaining the structure of these unions, we did not need any, in principle, new notion about the nature of chemical bonds " (The Great Soviet Encyclopedia, 3rd ed. the article on the "Inert Gases ").

It is common ground that human knowledge changes with the march of history. Here one would like to stress that not only the facts themselves go into the composition of knowledge, but also the initial positions and presumptions, on the basis of which this or that fact becomes a component of a system of knowledge : these are the notions about meaningfulness and meaninglessness , about obviousness and non-obviousness, of the possible and the impossible, of the part and the whole, persuasiveness and the lack of it, the proved and the unproved and, the authentic and the inauthentic. It is possible , that all these notions change more slowly than the simple notions about facts ,but in essence , they are historically as relative as our notions about facts.

Mathematics is sometimes perceived as a stationary rock towering above the waves of changing notions about the other disciplines. Of course, there are grounds for such a view of mathematics. At the same time, the notion of some absoluteness of mathematics is evidently exaggerated. If mathematics is absolute, then it is so only at the level of everyday experience — just as Newtonian physics is absolute in its application to the phenomena of "medium size" (and yet another — Einsteinian — physics operates at the level of the small and the very big) [see the already mentioned article of P.K. Rashevsky [16] : on "diffusion in the large " notions of the natural number].

In particular, the socio-historical conditionalities of the notions of proofs are on the whole extended upon mathematical proofs.

To illustrate what has been said above, I shall now briefly narrate my understanding of the concept of proof in ancient Egypt, ancient Greece and in India.

We do not have much authentic information as to how mathematical proofs were enunciated and understood in the ancient period. The texts that came down to us are in many cases fragmentary : moreover the terms contained therein often have debatable interpretations ; for example , on the interpretation of ancient Egyptian mathematical texts, see the remarks of the translator in : [4, p. 139]. There is a lot that is conjectural. Every one makes conjectures in the

direction s/he wishes, and the present author is no exception. Taking these stipulations into account, the following outline may be proposed.

The proposed outline is based on the conviction that the notion of proof is a product of the history of societies. We are aware of the simplification involved in our historical excursus,as we describe ancient Egypt as a centralized state — since there have been periods of splintering there, or ancient Greece — as a democracy, since there, too, there have been cases of tyranical or oligarchic rule. But then, any outline involves some simplification.

Ancient Egypt. A centralized theocratic state , with an extraordinarily strong discipline. Continuous construction of pyramids — requiring colossal human and material resources and uniting the strength of the entire land — served as an effective instrument for maintaining centralization, discipline and order. Authority of the Pharaoh and of the priests was incontestable. Authority of the written word was also unquestionable. If a priest , scribe or teacher said or wrote something, then that means — such is the case. If something is written on papyrus, then that is the case. Persuasiveness was based on the authority of the source.

Ancient Egyptian mathematical texts contain ready-made recipes without any substantiation. When we speak of absence of substantiation, here we have in view the modern understanding of the word "substantiation". From the point of view of a person of that time a recipe on a papyrus was fully substantiated , as it came from an authoritative source and was drawn up in the authoritative form of a record on papyrus. The fact of being recorded on a papyrus, was in itself the proof. In reality, this fact was enough for convincing others with the help of it. A number of recipes for computing the areas of triangles and quadrangles have been non-univocally interpreted in our time; the disputes about how to understand the terms contained in these recipes, still continue [4, ch.IV, §2, a]. Depending upon these interpretations, these formulae may be taken to be either exact, or approximate, or totally incorrect. When we speak of incorrect formulae, here, we have in view the representation of the area of a tringle through half of the product of the base and a side of it. This is what academician L.S. Pontryagin has to say on this score : " The first mathematical manuscript known to us — is the manuscript of Ahmes, composed some 2000 years before our era. It contains some algebraic and geometrical rules — for example, for computing the area of a triangle ... However, the Ahmes Papyrus contains a mistake. According to him the area of an isosceles triangle is equal to the product of its base and half of a side — but to-day every school-student knows that it is not true" [25]. However, many a researcher thinks that the corresponding ancient Egyptian term should not be translated as a side , it should be taken to mean height (and then the formula contained in the papyrus turns out to be true. However, even if this term did in reality signify, not the height, but a side, the corresponding (according to our modern point of view incorrect) formula should be considered as proved according to the ancient Egyptian understanding of the word "proved ": as this formula is convincingly substantiated by the fact that it (of course, not as a formula, but as a recipe expressed in words) is contained in an authoritative document.

The situation was somewhat different in *ancient Greece.* (In comparison to Egypt) here we have comparatively small state formations together with popular assemblies. The orators, who spoke in these gatherings did not carry any *a priori* authority. They had to convince the listeners by arguments. Formulating correct arguments became an everyday and actual requirement. Hence the birth of logic in the hands of Socrates, and its final shaping as a

discipline by Aristotle. Hence also, the beginning of the deductive method in mathematics, approaching the modern notion of proof. Arguments became the basis for mathematical conviction. The concept of the foundations of correct arguments, of axioms and postulates, arose . That which could be obtained through "legitimate arguments" from the initial statements, considered to be valid, was considered to be convincing (and consequently, proved).

Finally, *India.* We intend to refer to some geometrical figures, taken from mediaeval Indian texts, but that does not mean that these figures did not appear in ancient India. In general, the task of dating of Indian mathematical notions gives rise to considerable difficulties, as some texts may be expositions of some other earlier texts. On the other hand , it is not so essential either: mediaeval Egypt and Greece had nothing in common with ancient Egypt and Greece, but mediaeval India remained the custodian of the intellectual heritage of ancient India. An essential trait of this tradition was (and is) the conferring of the status of highest authenticity to the inner light. Immediate inner illumination was considered to be the basic source of knowledge and it had indisputable persuasive power. That which was thus known was considered to be proved. In order to convince others of it, those others must be brought to such a state , that they themselves experience the inner illumination. That is why, a geometrical proof had two parts : a diagram, and below it the inscription — " See !".

We find the examples of such diagrams with the inscriptions "See !" in some texts dating back to the 12th-16th centuries [9, p. 76 and 154]. We reproduce below one of these diagrams; it has also been reproduced in : [15,p.75]. We are of the opinion that it deserves to be included in any modern, secondary school level, text book of geometry : it shows, more graphically than the modern proofs, that the area of a circle is equal to the area of a rectangle , the two sides of which are respectively equal to half of the circumference and half of the diameter of the given circle. See figure 5 below.

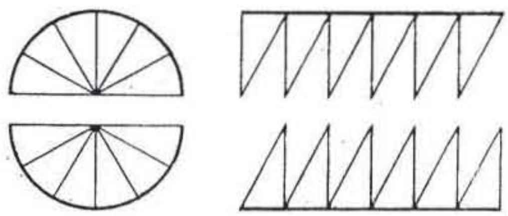


Fig 5.

The present author is aware of the fact that his views on the Indian proofs are different from those of an authority in the field of history of mathematics, like A . P. Yushkevich, who wrote : " The laconism of inferences in the Indian works on mathematics or the presence in them of diagrams together with only the inscription "See !", should not be viewed as the manifestation of some special approach to the problem of proof or of some special movement of thought "[9,p. 155]. We are of the opinion that they should be so viewed. Or else, why do we not come across this kind of "See !" anywhere else ? Why only in India ?

S.S. Demidov has put forward valuable considerations on the evolution of the concept of mathematical proof in [15], where, in particular he states that, "in the final count, the proof

(giving power) of mathematical arguments is their persuasive power. What appeared to us to be convincing yesterday, does not appear to be so to- day".

The definition of proofs as convincing / persuasive texts makes the concept of proof very subjective (for some, a text is convincing, for others — not). We do not consider it to be a deficiency of the definition. Such is the state of affairs. Perhaps, the use of the word " makes " above has been unfortunate. Our definition *does not make* the concept of proof subjective , it *only reflects* the subjective character of this concept . Even more interesting is the problem (we are very far from solving it), as to why, nevertheless, the concept of proof has an universal-cultural character in the sense, that within the limits of one and the same culture, though there occur disputes about whether or not this or that statement is true — such disputes are comparatively rare.

While speaking of such disputes, we do not have in view the disagreements among the representatives of various logical trends in mathematics, for example, those among the representatives of the ordinary, classical mathematics and the representatives of the intuitionistic (constructive) mathematics. The latter do not recognise many statements of ordinary mathematics as proved (and, on the contrary, consider them to be untrue). It may be said that the intuitionists and constructivists belong to different mathematical cultures and even the most customary words (like, say, "exists") has a different meaning for them (evidently, the intuitionists and the constructivists think that the representatives of traditional mathematics put different meanings into words , and it is they — the intuitionists — who use these words in their only correct meaning). That is why the intuitionists consider many proofs of traditional mathematics to be invalid.

Here we are talking of something else — not about changes in the semantics of terms, leading to the changes in the truth values of statements, but about the fact that a proof may turn out to be not understood and that is why not convincing (and once not convincing — it is not a proof at all). Modern mathematics has a complex structure, which has almost stopped to be visible. The proofs of some of the theorems turn out to be so cumbersome, that in order to be able to verify them one must have an extraordinarily big desire, patience and time, to say nothing of the fact that one must have special knowledge — for a number of theorems, not only the invention of their proofs , but even the verification of these proofs appear to be accessible, only to a narrow circle of refined specialists.

Sometimes the volume of the proof of this or that theorem becomes an object of interest. Here,we often find that some theorems established earlier — which are no more required to be proved — are permitted to be used as ready-made formulations in a proof. Will such an argument be a proof — i.e., a convincing text — for some one who does not know the proofs of these theorems " established earlier "? We do not intend to give an univocal answer to this question. We would like to mention further , that the very word " earlier " introduces an additional subjective "relativistic" moment (two almost simultaneously proved theorems may be chronologically differently ordered by different observers). If any reference to any theorem whatsoever proved earlier,is forbidden in a proof and if one is required to go back directly to the definitions and primary, undefined concepts (which we have discussed in our first reflection), then such a complete proof, may, in a number of instances, stretch into thousands

of pages of mathematical text (and may even be more difficult to perceive, than a proof based upon clearly formulated — though not known to the reader — facts).

The study of difficult mathematical proofs may be compared with a mountain-climber's ascent to the peak. The sea-level corrresponds to the initial concepts. Ascent from the sea-level may take months, and its mathematical analog (understanding a proof) may take years. In both the cases, there are many intermediate stops. First, you go to the common high-altitude base camp, here the climbers going to the various neighbouring peaks gather. This stage corresponds to the stage of serious mathematical preparation, sufficient for acquiring an understanding of the more special themes. Then begins the assault to the chosen peak ; here again, we have intermediate camps and stops. For mathematics, the corresponding theories and theorems play the role of these camps and stops. Just as a mountaineer may make a limited number of ascents in his life time, so may a mathematician — get to know only a limited number of proofs.

The following common trait of the mountaineer and the mathematician is an important one — there is some kind of conventionality involved in the choice of the point of departure. The ascent proper does not begin at the sea-level , but from a point, where the professional mountaineers may be able to gather without difficulty, but it may be a matter of great difficulty for an ordinary person, if s/he wants to arrive there. The proof proper begins from an analogous point : this point is situated at some general-cultural (we have in view the mathematical culture) level. However, at the present stage of mathematics the generality of the prefix "general- " is being lowered continuously, and now many proofs begin form a point, accessible only to the narrow specialists. A second common trait consists of a break up into stages, the presence of sufficient number of intermediate stops (camps).

Where from does one get the conviction in mathematics, that the proved theorems — the proof of which one never gets to know — are indeed proved , i.e., have proofs ? Evidently such conviction is based on trust alone . Seen from outside, such a situation should not appear to be very strange. Indeed how many of the readers of these lines have seen the Easter Island ? For those who have not, the conviction that this island exists, is also based , in the final analysis, on trust. But if a modern proof is based upon trust in authority, then how is it, in principle, different from the ancient Egyptian proof ?

It is not a simple question. Perhaps, the answer to it lies in the fact that proofs are gradually moving over from the ranks of the phenomena of individual experience to those of the phenomena of collective experience. Pushing the collective to the fore is in general characteristic of the history of civilization. It is well known (and widely discussed) that with the development of human society, there arises division and cooperation of labour and, this gets strengthened steadily. Only in the deep antiquity could man himself produce all that he needed: now everybody is required to use the results of the labour of others. It is known (though less discussed) that division and cooperation of scientific knowledge takes place simultaneously. It is difficult to say when — perhaps in the middle ages — one could have found individual scholars, capable of grasping the totality of the knowledge of his time. Now everybody must, this way or that , use the knowledge of others. The situation with proofs is an analogous one : the activities in the sphere of production and consumption of proofs have become as much an object of division and cooperation of labour, as are the activities in the sphere of production

and consumption of knowledge. The very concept of conviction has begun to lose its individualized nuance and, is more and more assuming the character of "collective conviction". Evidenty, we must gradually learn to speak of the conviction, of not a separate individual but, of some scientific collective. Here , collective conviction does not in any way signify that it is equal to the "immediate conviction " of each of the individual members of the collective. A collective does not function as a simple totality of its members, but as a single whole . The idea of collective conviction indicates the fact, that for every component part of a proof we have a member of the collective "answerable for it ", who is immediately convinced , namely, about that part (and other members of the collective rely on this member on that given question).

The age of informatics is introducing its own correction, also to the notion of proof. For instance, there arise situations when a proof involves sorting out of such a large number of variants, that it becomes difficult for a human being to do the sorting, but a computer can do it. Let us assume that a computer sorted out all the required variants , and the sorting led to the necessary results. Can we then say, that we have obtained a proof ? And what if the machine "malfunctioned"? (But a person also makes mistakes !) That apart , a guarantee is essential to the effect that the programme was right; correctness of programmes can be ascertained only with special proofs , and the theory of such proofs constitutes a special division of the theory of programming.

In reality, computers were used to solve the four colour problem. [For a formulation of this problem, see: the article "The four colour problem ", in the 3rd edition of the Great Soviet Encyclopedia.] In terms of simplicity of formulation, this problem, consisting of a proof of the four colour hypothesis, is hardly inferior to Fermat's problem (consisting of a proof of Fermat's hypothesis), but in terms of the naturalness of the statement (and the applied significance) it is superior to Fermat's problem . The solution of this problem was announced by Appel and Haken in 1976 [17] and set forth in 1977 [18 and 19]. This solution is based upon a reduction of the solution into a large number of particular instances, the study of which was entrusted to computers. The computers verified all of them, and thereby it was proved that every map is four colourable, as per requirement.

Appel and Haken themselves said about their proof [20]: "The proof involved an unprecedented use of the computers. The calculations used in the proof made it longer than what is traditionally considered to be permissible. In fact, the validity of the proposed proof can never be verified without the help of computers. What is more, some of the decisive ideas of this proof materialized through computer experiments. It is posssible , of course, that one fine morning there would appear a short proof of the four colour theorem ... At the same time, it is also conceivabale that such a short proof is not at all possible. In the latter case, new and interesting types of theorems emerge, for whom traditional types of proofs do not exist ".

Of late , however, the validity of the proof provided by Appel and Haken came to be doubted. The doubt is not about the computer-use part of it, but about the pre-computer, theoretical part — wherein it is sought to be established that the entire problem is really reducible to a consideration of the particular instances.

C o m m e n t a r y . Let us describe the situation involving the proof of Appel and Haken in somewhat greater detail. The basic idea of its authors is connected with the following notions.

First of all, the authors go over from the colouring of the regions in a map to the colouring of the apexes in a planar graph, such that it is a triangulation. Further, they call any sub-graph, forming a cycle, and the interior of that cycle — a configuration. If it can be proved by some standard method, that a configuration can not be immersed in a minimal counter-example of the four colour hypothesis, then it is called — *reducible.* If each planar triangulation contains one of the configuarations of a set as a sub-graph, then that set of configurations is called *unavoidable*. From these definitions it easily follows that for obtaining a (positive) solution of the four colour problem it is enough to produce an unavoidable set of reducible configurations. The authors of the proof produced 1834 explicit, reducible configurations, forming the unavoidable set [19,pp.505-567]. In each of these configurations, the length of a cycle was 14 or less. Computers were used both for finding the unavoidable set and for proving the reducibility of its terms.

If in the first case (construction of the set) the computers were drawn into a helping role , since the very proof of unavoidability of the set obtained (now it is not important, how that was done) is not based on computer-calculations, then in the second case (of verification of reducibility) the use of computers happen to be an essential component of the proof , and each configuration needs some 10 minutes of computer time for such verification. While evaluating the proof of Appel and Haken, some reviewers indicated [23] that the authors of the proof needed four years and 1200 computer hours for its construction and, that the text of the proof takes 139 pages, including 99 pages of drawings, the average size of more than 30 of these drawings being one page. The reviewers commented that the "essentially search type character of the proof makes its verification difficult (according to Appel the verification of all the details requires 300 computer hours)". Evidently, these 300 hours are required for the verification of reducibility. However , as we have already mentioned, the non-computer part of the proof — involving the verification of the unavoidability of the set of configurations produced — gives rise to doubt. The fact remains that the text of the proof [18 and 19] does not present this verification directly and exhaustively. We have been informed through a foot-note on p.460 that the details of the proof of unavoidability of the presented set (to be more precise, details of the proof of the so-called spacing out theorem, which provides the basis for this unavoidability), are contained in the microfiches, supplied as a special supplement to the journal. However, the present author could not go through that supplement.

It seems that with the development of mathematics (and with the appearance of ever more complex and long proofs) the proofs are losing an important trait — that of being convincing. One fails to understand , then what remains of the proof : conviction /persuasiveness enters into the very definition of proofs ! That apart , with the growth in the complexity of proofs, their element of subjectivity grows too. Of course, a f o r m a l proof is objective. But, firstly, not the judgements themselves, rather their expressions, their representations in formal languages, that have formal proofs. Secondly, though the verification of the statement, that a given text is a formal proof, is accomplished algorithmically, it may give rise to considerable difficulties, in the case of a voluminous text.

Large proofs begin to live by some macroscopic rules. Just as the concept of natural number gets diffused in the case of the "large" numbers (once more we refer the reader to P.K. Rashevsky's article) [16], so does our notion of a proof; it gets diffused , when the volume of a proof becomes inordinately large.

It so happenes, that though all proofs should, by definition, be convincing, some proofs are more convincing than the others, i.e., as though, some happen to be, to a greater extent proofs,

than are the others. There emerges something like a gradation of proofs according to the degrees of demonstrability — of course, such an idea fundamentally contradicts our primary notions about the identical indisputability of all proofs. But mathematical truths do permit a gradation of that kind. Each of the three following statements — "$2 \cdot 2 = 4$", "$17^{14} > 31^{11}$" and "$300! > 100^{300}$" — are true. However, we say : "True as $2 \cdot 2 = 4$", and do not say : "True as $17^{14} > 31^{11}$" or "True as $300! > 100^{300}$".

7. Can mathematics be made understandable ?

Why so many people do not understand mathematics? The great Poincare was disturbed by this problem and, he wrote: "How to explain, why many an intellect refuses to understand mathematics? Is it not paradoxical? Indeed.... here we have a problem , that does not lend itself to an easy solution; all those who wish to devote themselves to teaching must take up this problem" [2,p.353] .

Most probably both the sides involved "are to blame". Non-mathematicians are to blame: a bad education has trained them into a non-understanding of and even into taking a hostile attitude towards mathematics (as Poincare has noted " often the intellect of those people who are in need of guide lines, is very lazy to seek them out") [2,p.354]. Mathematicians are to be blamed: they do not wish to waste their strength, explaining their mathematics to the uninitiated (and how many people are astonished to find , that there still remains something to be discovered in mathematics !). Of course, in mathematics there would always remain numerous details inaccessible to the non-professionals (and even to the professionals, of a different field of mathematics). But such is the case everywhere — for example, in chess, even the other grand masters do not understand many a move, when Karpov and Kasparov battle against one another. At the same time, a very large part of mathematics, larger than what is usually thought to be the case, may be explained to a wider circle of well-meaning listeners and readers — of course, not in detail, but at the level of the herart of the matter. Clearly, this would require that the mathematicians engage themselves single-mindedly in this new direction of activity. Perhaps, thereby they would be discharging their moral duty to the humankind.

"But in order to help those who do not understand, first of all, we must know what restrains them" [2,p.345]. It appears, that the complex logical structure of mathematical definitions and statements, in which the logical connectives and the existential and universal quantifiers take turns, happens to be the hindrance in many cases. Every teacher of mathematical analysis knows the difficulty that arises in the course of parallel assimilation of the concept of limiting point of a sequence — the definition of which has the structure

$$\forall \varepsilon \forall k \exists n (A \wedge B),$$

and the concept of limit of a sequence — the definition of which has the structure

$$\forall \varepsilon \exists n \forall k (A \Rightarrow B).$$

However, are these psychological difficulties encountered by the learners, while assimilating these concepts — difficulties pertaining to the heart of the matter or, are these difficulties of linguistic expression? I do not have any final answer to this question. It is connected with an even deeper question: is it possible to separate mathematics from its linguistic formulation? In other words, does mathematics abide exclusively in the mathematical texts or does mathematics have some other essence, differnt from the texts — and the texts serve only as this or that (and

perhaps not always felicitous) mode of expression for that essence. It is clear that this question, which we have called a "deeper" question, is appplicable not only to mathematics, but also to any other discipline *. According to a formulation of Engels, mathematics is different from the other disciplines in so far as it is "an abstract science, dealing with intellectual constructs" [1,p.529]. [These intellectual constructs can hardly be understood by the human intellect, if they are not based on ordinary human logic, and consequently — on reality, from the operations with which , this logic has come into being.]

Like all rational concepts, the mathematical concepts too exist in the form of notions, not necessarily connected with texts. The linguistic texts defining these concepts should be recognized as important, but not as the only, means for their assimilation.

It appears, that now we have at our disposal more adequate means of introducing the concepts of limit and limiting point of a sequence, to those learners (who do not have special "mathematical capabilities" — that is, according to modern understanding, to those who do not have a high capability of assimilating, namely, linguistic formulations). Let us imagine a screen, on which we may draw the trajectory of the movement of a point, unboundedly approaching some other stationary point, which is the limit. This has to be repeated a number of times, with changes in the position of the limit (so that the false impression is not created, that every sequence has one and the same limit), as well as in the mode of approach of the moving point to the limit (so that, in particular, the false impression is not created, that the distance between the moving point and its limit changes monotonically).It is possible to present an analogous graphic illustration of the concept of limiting point: when, though the trajectory unboundedly approaches that point at times — at others, it moves away from it by a definite distance. It appears very likely, that any viewer of such pictures would form a correct notion both of the limit and the limiting point.

One is led to believe that with the introduction of computers, teaching will proceed along the path of visualization of concepts, traditionally considered to be entirely abstract.

Had the theme under consideration been one of pedagogical significance alone, then we would not have dwelt upon it so elaborately in an essay of philosophical character. However, this theme exceeds the bounds of pedagogics and, closes up to the question of ontological nature of mathematical concepts. Like all other rational theoretical question, this question too has an applied significance — in the given case, in the order of reverse connection, it is pedagogical. Indeed, if a mathematical concept has an essence, different from its embodiment in a linguistic definition or formula, then one can hope for a better understanding of that essence, by demonstrating its various manifestations (and not only its formulation).

In order to adduce a proof, we shall consider a fresh example. On pp.71-72 of a recently published text book [24], there is a formula that defines a mathematical concept — the so-called Clark's cone. Having formulated its definition the authors wrote : "However, at the first glance, it is neither possible to understand the properties of Clark's cone, nor the meaning of its formal definition itself". And further on, they have at first put forward some heuristic arguments explainning Clark's cone, and then translated these arguments in the language of non-standard analysis. Here one gets the idea that as though the concept of Clark's cone exists all by itself;

* Enter Jacques Derrida and post-structuralism in Mathematics?—Ed.

its definition in the form of a formula is only one of the means (and not the most felicitous means) of comprehending this concept, but descriptions like the "results of examination of the set through a microscope" [24.p.86], are useful for a better understanding of it.

Independently of the fact, whether or not such is indeed the case, we may put forward the following fruitful working hypothesis: a truely profound mathematical concept or mathematical statement must in essence be simple. And then there is a hope, that it will be understandable (or better still, understood): it is easy to get used to that which is simple, and we do not know any interpretation of "to understand", other than "to get used to".

Literature

1. *Marks K., Engels F., Coch. 2-e izd. T.20.*
2. *Poincare H., O nauke. M., 1983.*
3. *Hilbert D., The Foundations of Geometry. Chicago 1902 [Russ. tr.1948].*
4. *Neugebauer O., Vorlesungen über Geschichte der antiken mathematische Wissenschaften. Erster Band: Vorgriechische Mathematik. Berlin, 1934 [Russ. Tr. 1937].*
5. Tolkovy Slovar Russkovo Yazyka. M. 1938. T. 2.
6. *Bourbaki N., Theory of Sets. Mass., 1968. [Russ. Tr. 1965]*
7. *Church A., Introduction to Matematical Logic. Princeton, 1956 [Russ. Tr. 1960].*
8. *Hornby A.S., Parnwell E.C., An English-reader's Dictionary. Oxford, 1959.*
9. *Yushkevich A.P., Istoriya mathematiki v sredine veka. M., 1961.*
10. *Pototsky M.V., O pedagogicheskikh osnovakh obucheniya matematike. M., 1963.*
11. *Gorsky D., Opredeline// Filos. Entsiklopedia. M., 1967. T. 4. c. 150-152.*
12. *Uspensky V.A. Predislovie// Matematika v sovremennom mire. M., 1967.*
13. *Bozhich S.P. O sposobakh istinnostnoi otsenki estestvennonauchnovo vyskazyvaniya// Logika i empiricheskoe poznanie. M., 1982.*
14. Izomorfizm // Bolsh. Sov. Ents. 3-e izd. M., 1972. T. 10.
15. *Demidov S.S., K istorii aksiomaticheskovo metoda // Istoriya i metodologiya estestvennykh nauk: Matematika. Mekhanika. M., 1973. v yp. 14.*
16. *Rashevsky P.K., O dogmate naturalnovo ryda// Uspekh. mat. nauk. 1973. T. 28. Vyp. 4(172).*
17. *Appel K., Haken W., Every Planar Map is Four Colorable// Bull. Amer. Math. Soc. 1976. Vol. 82, N5.*
18. *Appel K., Haken W., Every Planar Maps is Four Colorable. Pt.I: Discharging // Ill. J. Math. 1977. Vol. 21, N3.*
19. *Appel K., Haken W., Koch J., Every Planar Map is Four Colorable. Pt. II: Reducibility // ibid.*
20. *Appel K., Haken W., The Solution of the Four-Color-Map Problem// Scientific American. 1977. Vol. 237, N4.*
21. *Uspensky V. A., Teorema Gödelya o nepolnote. M., 1982.*
22. *Plisko V.E., Teorema// Mat. Ents. M., 1985. T. 5.*

23. *Kozyrev V.P., Yushmanov S.V., Teoriya grafov: (Algoritm., algebraich., i metr. probl.)// Teoriya veroyatnosti: Mat. statistika. Teoret. kibernetika. M., 1985. T. 23.*

24. *Kusraev A.G., Kutateladze S.S., Subdifferentsialy i ikh primeneniya: Ucheb. Posobie. Novosibirsk, 1985.*

25. Uroki otkryvaet beseda s matematikom L. Pontryaginym : Interviyu akad. L. S. Pontryagina "Uchit. gaz."//*Uchit. gaz.* 1985, 23 maiya.

Source : Zakonomernosti razvitya sovremennoi matematiki. "Nauka". M., 1987. s 106-155.

About the author : Uspensky, Vladimir Andreivich (1930-). Mathematician and logician. Fields of specialization: theory of algorithms and mathematical linguistics.

Other works:

1. O ponyatii algoritmicheskoi svodimosti (1953);
2. Teorema Gödelya i teoria algoritmov (1953);
3. Ob algoritmicheskoi svodimosti (1956);
4. Ponyatie programmy i vychislimyie operatory(1956);
5. Uporyadochennye i chistichno uporyadochennye mnozhestva (1956);
6. Neskoilka zamechanii o perechislimykh mnozhestvakh (1957);
7. K opredeleniyu chasti rechi v teoretiko-mnozhestvennoi sisteme yazyka (1957);
8. K opredeleniyu padezha po A.N. Kolmogorovu(1957).

EMERGENCE AND DEVELOPMENT OF THE CONCEPT OF CONSTRUCTIVISABILITY IN MATHEMATICS

NIKOLAI NIKOLAEVICH NEPEIVODA

Constructivisability of a mathematical theory signifies the possibility of isolating the constructions of objects from their existence proofs.

Pre-Greek empirical mathematics was constructive by its very nature. It was preoccupied, namely, with the means of construction of objects, and gave empirical recipes in certain situations. Mathematical reasoning was reduced to one or, in the extreme cases, to several applications of such recipes, and the only descriptive element in it involved judging whether or not the problem at hand or, part thereof, belongs to a certain class. This descriptive element was most often reduced to an appeal to immediate obviousness.

Construction of the object being sought was the only method of proof in the Indian and Chinese mathematicses, and this construction was able to take the place of arguments (we recall the famous Indian diagrams with the word "see"). Arguments could only help in construction, they did not have any independent significance [1].

The concept of proof came to occupy a proper place in Greek mathematics. Classical logic was used with all its might. It has been established in the 20th century, that this logic was suitable, in the first place, for describing the static universe of ideal concepts, and not for carrying out intellectual constructions. Though Aristotle did highlight the special logical status of the rule of contraries : "... one of these [direct proof] proceeds from the previous [knowledge], and the other [from the opposite] from the subsequent" [2, p. 307] — this remark, which astonishingly exactly reflects the semantics of the rule of contraries in Kripke's models, did not exert any influence upon strict mathematical arguments.

Nevertheless the use of classical logic — an instrument, oriented towards descriptive and not constructive applications — did not lead Greek mathematics to non-constructive methods and theorems. As before, existence proofs included (as a rule, geometrical) constructions. Arguments from the contraries were used only to substantiate the constructions already carried out, in the main, for proving the equality or inequality of certain magnitudes.

The deeper reasons behind this phenomenon were revealed only in the last few decades. It is intimately connected with the hold-up of the Greeks in front of the concept of real number, from the point of view of traditional mathematical paradigm — with their strange antipathy to the explicit use of numbers in general, in strict mathematical arguments. Vexations regarding the specificities of Hellenic mathematics — which are indeed not quite understandable from the classical point of view — have been expressed more than once. In particular, the question arises : why the real numbers were used in a masked manner, as proportions, and why acquaintance with incommensurability did not come in its way, and yet in geometry, the natural numbers were avoided in all possible ways, though, one would think, that these are sufficiently intuitively reliable objects? Why, the Hellenic arithmetic remained something like a handicraft or an art,. never entering into the sphere of operation of "pure mathematics", save in the case of a few theorems like the one about the infinite set of prime numbers? What prevented the Greeks from formulating and utilizing such a powerful principle of conducting arguments, as the mathematical intuition ?

It has been proved, namely, that the classical geometry and the elementary theory of real numbers are complete and solvable ; see, for example, [3]. Thus, for every concrete, closed statement in the language of these theories, it is provable in them, that it is either A, or $\neg A$. Consequently, classical logic can not lead us to the non-constructivisability of the theorems proved either in geometry, or in geometry supplemented with algebraic operations on the real numbers, but without the explicit mention of the integers as a set. In any classical proof of these theorems one may mark out the construction and its substantiation, which may be carried out, in particular, also by the method of "indirect proof".

This fact once more confirms the depth of the intuition of the Greeks, which was based upon purely aesthetic and methodological considerations, but which permitted them to stop, namely there, where the rupture between argument and construction, between descriptive and constructive knowledge, became important. Such an exact halt was conducive to the fact that, the distinction between what was constructive and what was descriptive was not realized and, was correspondingly erased out of the world outlook of mathematicians. Perhaps it gave an indirect push to the courageous introduction of numbers and their functions in the mathematics of the modern times : mathematicians were still unaware of the danger of a rupture between the proof and the construction, it was erroneously accepted that [the verbs] "to prove" and "to construct" were always mutually concordant. Consequently, as before, mathematicians assumed — now, without any foundation, simply due to inertia — that a strictly proved statement provided the means for the construction of those objects, whose existence has been affirmed. When the construction was explicitly indicated in a poroof, then that was, of course, rated somewhat higher, but the pride of place was reserved for the other factors, in the first place — for the not explicitly formulated, and that is why constantly implicitly changed, aesthetic ones.

Prior to the formulation of the axiom of choice by G. Cantor and E. Zermelo, mathematicians did not realize that even after the explicit introduction of the totality of natural numbers together with the principle of mathematical induction, there would appear non-constructive theorems of existence, which would not provide the construction sought — even in principle. The axiom of choice is demonstratively ineffective. It states that, it is possible to construct a function, by choosing its elements from among each of the members of the family of non-empty sets, without saying anything about the method of carrying out this choice. The shock generated by the axiom of choice and by the paradoxes of the theory of sets — which appeared practically at the same time, forced the realization that a very large part of mathematics of the period ending in the 19th century was indeed non-constructive. The axiom of choice was magnificently inscribed upon the entirety of the hitherto formed paradigm of classical mathematics.

It should be mentioned here, that even in the 19th century attempts were made to construct some sections of mathematics upon a more constructive foundation — in particular by R. Grassmann [4] and E. Schröder [5] — but these attempts remained on the sidelines, away from the main road, and were forgotten.

Thus, the "crisis of the foundations of mathematics" sharply posed the question about the nature of mathematical constructions and about the interrelationship of mathematical

objects and reality. And this gave rise to the necessity of a more exact and, in any case, a more explicit characterization of the class of constructive methods and, if possible, of eliminating the explicitly non-constructive ones. While examining the problem of constructivisability, it is possible to mark out three major trends: pseudo-classical, non-classical and significative.

The aim of the pseudo-classical trend is to single out the constructive sub-languages of the classical theories, wherein the classical logic and the customary concept of truth are left untouched. In modern mathematics this approach begins with A.Poincare — who tried to isolate those results of classical mathematics, which were obtained without the help of the axiom of choice, as he considered them to be more reliable and, with D. Hilbert — who had a more radical programme. For a detailed analysis of Hilbert's programme from a point of view that corresponds to the present state of the investigations in the foundations of mathematics, see : *Ershov. Yu. L.* and *Samokhvalov K.F.*, O novom podkhode k methodologii matematiki // Zakonmernosti razvitiya sovremennoi matematiki ("Nauka", M., 1987), pp. 85-106. Here we shall limit ourselves only to the remark that D. Hilbert unequivocally declared that the majority of mathematical statements do not have any real meaning and, that mathematics is required to give correct results only in respect of a set of comparatively simple real statements.

In the non-classical trend the concept of effective method is considered to be of paramount importance — mathematics is viewed as the science of effective (intellectual) constructions and logic adapts itself to the methods of such constructions, and gets so modified, as to wittingly guarantee the constructivisability of the constructions.

L.E.J. Brouwer was the first to point out that while aiming at attaining constructivisability one must not blindly follow that logic, which is tied to the tradition [6]. The roots of the non-constructive structures, are often not so much mathematical, as logical. For example, in any recursively axiomatizable non-contradictory classical theory containing arithmetic, it is possible — basing oneself upon a theorem of Gödel — to construct a statement of the form $\exists x \in N A(x)$, such that it is not possible to construct even one such number n, that $A(n)$ is provable, but, nevertheless, $\exists x \in N A(x)$ is provable. Indeed, for this it is enough to take the statement A — which is unsolvable in the given theory, and to construct the formula

$$\exists x ((x = 0 \,\&\, A) \vee (x = 1 \,\&\, \neg A)). \tag{1}$$

Brouwer showed, in particular that, the following two logical principles are most open to criticism from the point of view of constructivisability : the law of excluded middle $A \vee \neg A$ and the method of indirect proof $\neg\neg A \Rightarrow A$. Indeed in the constructive substantiation of the law of excluded middle it is demanded that a general method be constructed in respect of every problem, for establishing whether or not a given statement is true, and in the majority of cases such a method does not exist. Thus, the law of excluded middle may be called "the principle of omniscience", and it may be applied only in that situation, where both the language and the interpretation are deliberately so selected as to exclude the possibility of emergence of unsolvable problems.

Analogously, the principle $\neg\neg A \Rightarrow A$ signifies, that there exists a method for transition from the formulation of a solvable problem to its solution, i.e. the existence of the so-called "universal problem solver". This too narrows down the sphere of its constructive applicability.

Brouwer showed, that in principle it is possible to develop mathematics shunning the non-constructive principles, in particular — the principle of omniscience and that of the universal problem solver. The idea of logic as a calculus of problems, and that of the logical rules as transformers of problems into solutions, was made more exact by A.N. Kolmogorov [7]. Of late it has been realized that an enormous advantage of Kolmogorov's interpretation lies in the fact that it has not been specified through to the end, and, thus, it is rather an outline that can be modified. But to begin with this advantage was taken for a deficiency ; both S.C. Kleene and A.A. Markov derived the constructivist (intuitionist) logic from an exact definition of algorithm and of the concept of natural mumber. They showed that the principles against which Brouwer raised objections, are ineffective after a natural algorithm is given for singling out the constructive task from among arithmetical formulae, and an interpretation is provided for the transformations operative in this constructive task, as algorithms (partially — recursive functions). N.A. Shanin carried this construction through to its logical end [8] and, he provided an explicit algorithm for singling out the constructive task, having showed that after this substantiation of the construction carried out, one may go ahead with the methods of classical mathematics.

The third — significative — trend views mathematics as a science of formalisms and of the modes of their transformations. E. Schröder should be considered to be the spiritual father of this trend in many of its aspects, though he did not formulate an explicit "manifesto"of significative mathematics.

Now we shall dwell upon the paths of development, the mutual interactions and, the future perspectives of these three trends.

Gödel's incompleteness theorem turned out to be a turning point in the fate of the pseudo-classical trend. Naive hopes to the effect that it will be possible to get away without any serious reexamination of the paradigm of classical mathematics, having simply "sanctified" it with the Hilbertian incantations (and then one may happily forget about them) — were not vindicated. It became clear that the pseudo-classical trend too demands a serious reconstruction of the entire system of mathematical concepts, and that is why, from the point of view of psychological protection, the easiest thing to do was to simply interpret Gödel's [incompleteness] theorem as the collapse of Hilbert's programme as a whole — to be able to get along with what one was doing, this time, openly refusing to bother about the foundations. A quite frequent methodological error crops up in the interrelationships among the theoreticians and the "practitioners" : the "practitioners" are inclined to demand that the theoreticians should substantiate their [i.e., practitioners'] positions and activities, but only a "theoretician" on the verge of becoming a charlatan can provide such substantiation, in so far as in practice there is always a mix up of the rational, extremely exact activities, not only with the non-optimal, superfluous moves, but also with the plainly bad ones, dictated by tradition. That is why a true theory cannot substantiate practice, but must reconstruct it.

Nevertheless, namely, in this slack period, principled results were obtained by P.S. Novikov [9] : he established that from the classical proof of a formula of the form $\exists x A(x)$ in arithmetic — where $A(x)$ is algorithmically solvable — it is possible to obtain the construction of an n, such that $A(n)$.

During the last few years, the demands of theoretical programming, informatics and those of the so-called "artificial intelligence" have forced the logicians to return to the pseudo-classical trend. In particular, the development of the programming languages like PROLOG [10], has put forward the task of isolating those sub-systems of the classical logic, which retain constructivisability in some sense — as one of the most important tasks in this field. It appeared, in particular, that the system of Horn's formulae of the form

$$\forall x_1, \cdots, x_n (P_1 \& \cdots \& P_k \Rightarrow Q), \tag{2}$$

upon which PROLOG is based, possesses such property.

The classical formulae, which provide an opportunity to describe and elicit constructivist consturctions, have been more systematically described in the latest works of the Novosibirsk school : [11], [12]. Here we find the theoretical foundations for all possible systems of logical programming, based upon the classical logic. In yet another work [13] semantic conditions have been introduced upon the constructivisation of classical theories, without imposing any limit upon the class of formulae used. It appeared that for the fully constructive models (where any set defined by a formula of our label (signature) is recursively solvable), the classical logic is completely constructively interpretable.

The development of constructivist logic has forced us to ear-mark yet another component in the definition of a calculus, besides the axioms and the rules of deduction : it is the global structure of the inferences ; and this has opened up yet another opportunity for the development of the pseudo-classical trend. Even the intuitionist logic can be interpreted as classical logic with a limited global structure of natural deduction ; it has been established [14] that one can obtain the classical logic from intuitionist logic by adding to it one global, structural rule of inference : the rule of accepted unexpectedness.

But all these possibilities, opened up for the use of the pseudo-classical trend do not alter the basic conclusion of Brouwer : while using a logic one must carefully investigate the class of problems and the class of implied interpretations, otherwise reasoning and construction would inevitably drift apart. The ear-marked sub-classes of formulae, theories and inferences are based on such investigations in every case.

The pseudo-classical trend has unexpectedly turned up on the highway of development of classical mathematics itself. Though it has not been explicitly recognized, category theory may be that instrument, wherin again, as in the Hellenic mathematics, the constructive and the descriptive aspects have merged into one — where, proof guarantees construction. But this has now happened owing to a transition to a new level of abstraction, which has again permitted the banning of any explicit reference to the numbers. Further, the theory of categories leads to the necessity of investigating its own inner logic, which is intuitionst in the spaces (toposes) and coherent in the more general case [15]. Thus, here also, the pseudo-classical trend closes in on the non-classical trend.

Now, the non-classical trend has divided itself into two branches : intuitionism and constructivism.

In intuitionism we essentially base ourselves upon the incompleteness of our knowledge. Namely, we do not intend to provide any precise and final definition of the class of effective constructions[16]. What is more, in intuitionism we try to use this indeterminateness, this ignorance, as a positive factor. For example, from the substantiation of the principle of continuity in [16] :

$$\forall\alpha\,\exists n\,A(\alpha, n) \Rightarrow \forall\alpha\,\exists k\forall\beta\,(\forall l\,(l<k \Rightarrow \alpha(l)=\beta(l))\;\&\;A(\beta, n) \Rightarrow A(\alpha, n)) \tag{3}$$

it is evident, that this principle signifies the absence of any knowledge of the global rules, which would indicate the behaviour of the Brouwerian "sequences of choice" or of the "sequences that have become free". What is more, later on a conception of "lawless sequences" has been worked out, wherein, in general, all accessible information constitues an initial block [17]. Thus, in intuitionism an attempt is made to demonstrate that the knowledge of ignorance happens to be the most valuable form of knowledge. Intuitionism is thus sharply at variance with the entire paradigm of classical mathematics.

Constructivism tries to unite constructivisability with maximum retention of the classical mathematical paradigm. To some extent constructivism is as Platonist, as the classical mathematics. The class of objects under consideration and the methods of their transformation (at this point we have an essential difference with classical mathematics — where one does not even think of the methods of transformation) are formulated precisely, basing the formulations upon an exact concept of algorithm. Knowledge is interpreted as a normal state, and ignorance — as an anomaly, which is inevitably present, but which must be overcome at all cost. Such an interpretation permitted A.A. Markov [18] to clearly ear-mark the system of initial abstractions, which are foundational to constructivist mathematics. This interpretation predetermined the journey of constructivist mathematics to a dead end in the narrow constructivism of N.A. Shanin [19], where an attempt has been made to totally ban ideal sentences from mathematics.

E. Bishop tried to occupy an intermediate position [20], when he tried to get away from an exact fixation of the class of effective methods, as well as from basing oneself upon ignorance. But when his conception was made more precise — see, in particular P. Martin-Leof's book [21] — it was found, that Bishop's conception lies completely within the frame-work of constuctivism. Martin-Leof was the first to make use of the circumstance — though it is true that he did not formulate it explicitly — that the giving of the consrtuctive objects and of an exact description of the methods of transformation of the objects, still does not fully determine the methods of transformation of methods, and that such constructive functionals of the higher type, metaalgorithms, can be varied completely— without touching the algorithms themselves.

During the last few years, the demands of application — in particular, of informatics, — has stimulated a Renaissance of Constructivism, even in our country; but this time it is a broad constructivism, which investigates the most diverse classes of methods and, correspondingly, the most divergent constructivist theories and even the constructivist logics. The very concept of constructivist logic is little by little tearing itself away from its unjustified ties with a

single class of problems and effective methods and is becoming a relative concept, which may be varied, depending upon the descriptive language, in which transformable objects are described and problems are posed, and the class of programmes, or methods of transforming concrete and abstract objects are put forward. Viewed thus, intuitionist logic itself appears as the most classical of the constructivist logics — this logic is constuctivist in that situation, where we are faced with the task of pure functional programming [22]. In other words, a constructivist use of intuitionist logic is possible, when we are not very much constrained in terms of time and other resources, when our computations only add new data, new knowledge, over and above what we already have and, when it is possible to use the complex structures of data and the functionals of higher type.

The use of independent constructivist logics, and not of some fragments of classical logic, is expedient, when thereby the expressive power of language is sharply raised, and when the constructivist description is by far shorter and clearer than the corresponding classical one. For example, in [23] a case of pure implicative logic has been considered, wherein the formulae have been constructed with the help of a single logical connective — implication — from amongst the propositional letters, and a translation of the problem described into the traditional languages requires all the finite type of λ–terms, predicates upon them, and quantifiers.

Practically speaking, the third — significative — trend is also highly topical. Formal calculi and transformations of formalisms are widely investigated in mathematical linguistics and theoretical programming. But the methodological and metamathematical aspects of the significative point of view have not been sufficiently analysed . In this connecition see the works of P. Lorenzen [24] and S. Yu. Maslov : [25], [26].

Let us attempt a few conclusions. It is possible that the arrival of the pseudo-classical, "Hellenic" stage — wherein the constructive and the descriptive aspects merge into one — is indicative of the maturity and conceptual unity, of the conceptual completeness, of the system of mathematical concepts. Here, explicit mention of numbers are banned from the theories.

The desire to "carry the results right upto the numbers", to have explicit theories of computational methods and, generally, of the methods of practical constructions, leads to constructivism — in one form or the other. The different problem-oriented constructivist theories must give rise to an unified pseudo-classical theory, describing the effective structures of a given class.

The intuitionist theories are more abstract and they too can give rise to an entire family of more concrete constructivist theories; but they are operative in a different situation : when it is not expedient to assume the completeness of a system of concepts or, when the system of concepts is explicitily non-formalizable and when there is no point in striving at completeness.

Finally, it is necessary to conduct serious and deep-going investigations, to ascertain the emerging shape of significative mathematics.

Literature

1. *Struik. D. J.*, A Concise History of Mathematics. N. Y., 1967 [Russ. Tr. M., 1984].
2. *Aristotle*, Works, Russ. ed., M., 1978, T. 2.
3. *Tarski A.*, A Decision Method for Elementary Algebra and Geometry. Berkley; L.A., 1951.
4. *Biryukov B.V., Biryukova L.G.*, "Uchenie o formakh (velichinakh)" Germana i Roberta Grassmanov kak predvoskhischenie knonstruktivnovo napravleniya v matematike. I. // Kibernetika i logicheskaya fomalizatsiya : Aspekty istorii i meotdologii. M., 1982.
5. *Schröder E.*, Vorlesungen über die Algebra der Logik (exakte Logik). Leipzig, 1890-1905.
6. *Brouwer L.E.J.*, De onbetrouwbaarheid der logische principes. Bd.2.
7. *Kolmogorov A.N.*, Zur Deutung der intuitionistischen Logik // *Math. Zeitschr*, 1932. Bd. 35.
8. *Shanin N.A.*, O konstruktivnom ponimanii matematicheskikh suzhdenii// Tr. MIAN SSSR, 1958. T. 52.
9. *Novikov P.S.*, On the Consistency of Certain Logical Calculus// *Mat. sb.* 1943. T. 12(54).
10. Sistema "PROLOG - ES". Kiev, 1979.
11. *Ershov Yu. L.*, Printisip Σ- perechisleniya// DAN SSSR. 1983. T. 270, No. 5.
12. *Goncharov S.S., Sviridenko D.I.*, Σ-programmirovanie // Vychislitelnye sistemy. Novosibirsk, 1985. Vyp. 107.
13. *Voronkov A.A.*, Konstruktivnaya semantika dlya Teorii modelei// Vsesoyuz. Konf. po prikl. logike: Tez. dokl. Novosibirsk, 1985.
14. *Nepeivoda N.N.*, Pravilo neozhindannosti i strukturnye *go to* // Semiotika i informatika. 1984. Vyp. 23.
15. *Goldblatt V.*, Toposy. Kategornyi analiz logiki. M., 1983.
16. *Heyting A.*, Intuitionism. Amsterdam, 1956. [Russ. Tr. M., 1965].
17. *Dragalin A.G.*, Matematicheskii intuitsionizm: Vvedenie v teoriyu dokazatelstv. M., 1979.
18. *Markov A.A.*, Teoriya algorifmov// Tr. MIAN SSSR. 1954. T. 42.
19. *Shanin N.A.*, Roil poniyatiya algoritma v semantike arifmeticheskikh yazykov//Algoritmy v sovremennoi matematike i eo prilozheniyakh. Novosibirsk, 1982. Ch. 2.
20. *Bishop E.*, Foundations of Constructive Analysis. N.Y., 1967.
21. *Martin-Leof P.*, Ocherki po konstruktivnoi matematike. M., 1975.
22. *Nepeivoda N.N.*, Logicheskii podkhod k programmirovaniyu// Algoritmy v sovremennoi matematike i ee prilozheniyakh. Ch. 2. Novosibirsk, 1982.
23. *Nepeivoda N.N.*, Konstruktivnye logiki//Neklassicheskie logiki. M., 1982.
24. *Lorenzen P.*, Einführung in die operative logik und Mathematik. B., 1954.
25. *Maslov S.Yu.*, Mutatsionnye ischisleniya: Zap. nauch. seminarov LOMI .L., 1975. T. 49.
26. *Maslov S.Yu.*, Deduktivnye sistemy i ikh ekonomicheskie primeneniya. M. Nauch. sovet po kompleks. probl. "Kibernetika" AN SSSR, 1983. Prepr.

Source : Zakonomernosti razvitiya sovremennoi matematiki. " Nauka", M., 1987, s. 219-229.

Author : Nepeivoda, Nikolai Nikolaevich, Cand. (sc).

ABOUT THE EDITORS

1. Editor of K. Marks, *Matematicheskie Rukopisi* ("Nauka", M., 1968), *Sofya Aleksandrovna Yanovskaya* (1896-1966), one of the pioneers in the study and teaching of mathematical logic in the erstwhile USSR. She translated and edited the works of Hilbert, Ackermann, Tarski, Goodstein, Kleene and Church in the field of logic. From 1931 (till death) she also shouldered the responsibility of editing the mathematical manuscripts of Karl Marx. In 1935 she obtained her doctoral degree and, subsequently became a professor of mathematical logic in the Mechanics-and-Mathematics department of Moscow State University.

Her published books and papers include :

O tak nazyvaemykh opredeleniyakh cherez abstraktsiyu//Statei po filosofii matematiki (1936); Osnovaniya matematiki i matemticheskaya logika// Matematika v SSSR za tridtsat let (1949) ; Peredovye idei N.I. Lobachevskovo — orudie boirby protiv idealizma v matematike (M.-L.,1950); Iz istorii aksiomaticheskovo metoda// Trudy 3-vo Fsesoyuznovo mathematicheskovo siezda (T.2, 1956); Iz istoril aksiomatiki // Istoriko-matematicheskie issledovaniya (Vyp.II,1958); O nekotorykh chertakh razvitiya matematicheskoi logiki i otnosheniyu eo k tekhnicheskim prelozheniyam // Primenenie logiki v nauke i tekhnike (1960); Problemy vvedeniya i isklyucheniya abstraktsii bolee vysokikh (chem pervyi) poryadkov// The Foundations of Statements and Decisions (Warszawa, 1965,) [this paper was read in Septmber 1961]; O filosofskikh voprosakh matematicheskoi logiki// Problemy Logiki (1963); O matematicheskoi strogosti// *Voprosy Filosofii*, 1966, No.3; Metodologicheskie problemy nauki (1972) [contains her articles included in the "Filosofskaya Entsiklopediya" : Logika klassov, Logika kombinatornaya, Ischislenie, Logika vyskazyvanii, Logitsizm i dr.]

2. Editor of the special supplement *Marx and Mathematics* and translator of K. Marks, *Matematicheskie Rukopisi* (M., 1968), *Pradip Baksi* (1948-), M.A. (Philosophy & Russian).

His published books and papers include :

Popper on Dialectic// *Marxist Miscellany*, New Delhi, 6, 1976, pp.50-55 ; Markser Gonit Vishayak Pandulipi Prasange // *Gonit*, Calcutta, 1983, 2(2), pp. 51-58; Marksbad, Gonit o Tarkashastra (C., 1985, 106 pp.) [a collection of three essays: Dvandataftva O Gonit, Karl Markser Gonit Vishayak Pandulipi O Tar Tatparya and, Dvandvamulak Tarkashastra O Tarkashastrer Dvandvikata] ; Marksbad O Bijnansamuher Dvandvikata (Tr. and Ed.) (C.,1986,174pp.) [contains Bengali translations of: (1) Novye materialy o K.Markse // *Voprosy Filosofii*, No.5, 1983, pp. 100-126 (Roland Daniels-Karl Marx correspondence, 8 February-1 June 1851, on Daniels' manuscript — Mikrokosmos : Entwurf einer physiologischen Anthropologie (1850) and, other issues of mutual interest); (2) Polveka raboty nad tekstami i zamyslami F.Engelsa — B.M.Kedrov and, Bibliografiya osnovnykh nauchnykh trudov B.M.Kedrova // Filosofiya i estestvoznaniya (M., 1974) (a description of fifty years of investigations on and around Frederik Engels' work on the sciences by B.M.Kedrov and, a list of Kedrov's principal works) ; and (3) Evolution of Science: The Cultural-Historical Aspect — P.P. Gaidenko // *Social Sciences*, M., 1981, vol. XIII, No. 2, pp. 131-144] ; Karl Markser Prakriti Bijnan Charcha O Bijnan Bhabona // *Mulyayan*, C., October 1988 (pp.157-174) and May 1989 (pp.74-80); Scientific and Technological Revolution, Philosophy and Marxism // *Party Life*, N.D., August (pp.4-12) and September (pp.19-27), 1990; India, Marxism and the World To-day// *Party Life*, N.D., October (pp. 1-10) and November (pp.13-20), 1991.

C O R R I G E N D A

ABBREVIATIONS : p. page: l. line, f-n. foot note, r. read, f. for.

p.1, l.12, r. *Marksizm*, f. *Markiszm*.

p.8, l.26, r. from x but , f. from x_1 but .

p.14, l. 28, r. Social, f. Soial.

p. 36, l. 19, r. $\frac{dy}{dx}$,f. $\frac{dy}{dy}$.

p.49, l. 21, r. $\frac{d^2u}{dx^2}$, f. $\frac{d^2y}{dx^2}$.

p.60,l.25,r. $[=f'(x)]$, f. $=f'(x)]$.

p. 63, l. 20, r. the, f. that.

p. 73, f-n., r. in, f. is.

p. 85, l. 23, r differential, f. differenctial.

p. 93, l. 11, r. *form*, f. *from*.

p. 94, l. 6, r. practice, f. parctice.

p. 94, l. 10, r. generally, f. genrally.

p. 94, l. 14, r. basis, f. bais.

p. 94, l. 15, r. binomial, f binomimal.

p. 94, l. 18, r. Newtonian, f. Newonian.

p. 98, l. 3, r. equivalent, f. equvalent.

p. 111, l. 5, r. Chios, f. chios.

p. 123, l. 33, r. manuscript, f. manuscipt.

p. 136, l. 21, r. $-\frac{1}{2 \cdot 3} f'''(x)\, h^3$, f. $-\frac{1}{2 \cdot 3} f''(x)\, h^3$.

p. 137, l.25, r. Marksizma, f. marxisma.

p. 177, l. 16, r. *from*, f. *form*.

p. 180, l. 6, r. that, f. t at.

p. 180, l. 29, r. $ma^{m-1}y$, f. $ma^m\ y$.

p.184, l.11, r.-67, f.-27.

p. 199, l. 8, r.

$$f(m)^m = \underbrace{f(m) \cdot f(m) \cdots f(m)}_{m\ times} = \underbrace{f(m + m + \cdots + m)}_{m\ times} = f(m^2),\ \text{f}.$$

$$f(m)^m = \underbrace{f(m) \cdot f(m) \cdots f(m)}_{m\ times} = \underbrace{f(m + m + \cdots + m)}_{m\ times}$$

p. 202, l. 12, r. what has been, f. what been.

p. 227, l. 27, r. functions, f. function.

p. 237, l. 12, r. again pages, f. again a page.

p. 237, l. 28, r. 19-23, f. 19-22.

p. 275, l. 25, r. $(B)_{1)}$, $(B)_{2)}$, f. $(B)_{1)}$. $(B)_{2)}$.

p. 285, l. 19, r. broken form, so also with, f. generalised from, with.

p. 295, l. 21, r. Cx^2, f. cx^2 .

p. 308, l. 29, r. equalities, f. equlities.

p. 320, l. 32, r. investigating, f. investigations on.

p. 326, l. 19, r. translation, f. transtation.

p. 341 l. 7, r. owing to , f. for.

p. 344, l. 28, r. differentials, f. dif rentials.

p. 379, l. 1 (col. 2), r. ischislienie, f. ischeslienie.

p. 379, l. 4 (col. 2), r. Lausanne, f. Laussane.

p. 380, l. 22 (col. 2), r. estestvoznanie, f. estestvoznania.

p. 386, l. 21, r. graphic, f. graaphic.

p. 386, l. 22, r. introduced, f. ,intoroduced.

p. 391, l. 1, r. Engels, f. Engles.

p. 399, l. 18, r. it, f. them.

p. 400, l. 8, r. by, f. of.

p. 400, l. 28, r. (No., f. No.

p. 412, l. 13, r. *introduced*, f. *introdueed.*

p. 417, l. 11, r. $f''(x) \cdot [\Phi'(t)]^2 \cdot \Delta t^2$, f. $f'(x) \cdot [\Phi'(t)]^2 \cdot \Delta t^2$.

p. 425, l. 32, r. *Kisileva* , f. *Kiseleva.*

p. 425, l. 36, r. *Rybnikov*, f. *Rybikov*.

p. 427, l. 33, r. endlessly, f. endlessl.

p. 427, l. 34, r. dearest, f. dearst.

p. 438, l. 1, r. *Tattvachintamani,* f. *Tattva Chintamani.*

p. 446, l. 1, r. Engels, f. Engles.

p. 446, l. 10, r. Nauchnykh, f. Naychnykh.

p. 453, l. 41-42, r. 1928, books since 1948 and since..., f. 1948, and since... .

p. 460, l. 24, r. C. Thruesdell, f. K. Trusdell.

p. 466, l. 23-24, r. soon it was extended to the study of the so-called Arab Diophantus by R. Rashid (1974-) and, independently by J. Sesiano, and then, f. (the same words in different order).

p. 470, l. 21, r. happens, f. happen.

p. 478, l. 1, r. been, f. ben.

p. 496, para 3 should read as :

Abstract structures can be successfully used for constructing mathematical models; we may especially use those among them, which aim at revealing not only the numerical(metric) dependencies among magnitudes, but also the relations of a non-metric character.The study of such non-metric relations is of considerable significance for those sciences, where owing to the complexity of the object under investigation, and sometimes also owing to the unelaborated stage of a theory, it is impossible to present the results numerically.That is why, there one is often required to turn to the abstract structures of order. In their investigations about the different types of relations obtainable among individuals and groups in social collectives, psychologists and sociologists have begun to apply the oriented theory of graphs, which constitutes the simplest form of algebraic category.

p. 496, para 4, l. 2, r. this line of mathematisation of, f. th s line of mathematdsat on f.

p. 496, para 4, l.3, r. In fact, f. n fact.

p. 506, l. 7, r. Kisilev or, f. Kisvpfx. psr.

p. 506, l. 11, r. what are the, f. what re the.

p. 519, l. 32, r. $\{0, '\}$, f. $\{0, \}$.

p. 520, l. 5, r. above, f. abo e.

p. 523, l. 13-14, r. $0'' \cdots '$, f. 0"··· ′.

p. 546, l. 8, r. $\ldots k\ \forall \beta (\ \forall\ l(l<k\ldots$, f. $\ldots. k\ \forall \beta (V l(l<k \ldots$